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RELATIONS BETWEEN THE RECIPROCAL SUM AND THE ALTERNATING SUM FOR GENERALIZED LUCAS NUMBERS

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Abstract. We establish relations between reciprocal and alternating sums involving generalized Lucas numbers.

1. Introduction

Let p be a nonzero real number. The generalized Fibonacci and Lucas numbers are defined by

$$U_0 = 0, \qquad U_1 = 1, \qquad U_{n+2} = pU_{n+1} + U_n, \qquad (n \geq 0) \tag{1}$$

$$V_0 = 2, \qquad V_1 = p, \qquad V_{n+2} = pV_{n+1} + V_n, \qquad (n \geq 0) \tag{2}$$

respectively. Let α and β be the roots of $x^2 - px - 1 = 0$. Then we have the Binet's formulas:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{3}$$

$$V_n = \alpha^n + \beta^n. \tag{4}$$

For $p = 1$, $\{U_n\}$ and $\{V_n\}$ are the well-known Fibonacci numbers F_n and Lucas numbers L_n, respectively.

In [**2**], S. Rabinowitz discusses algorithmic aspects of certain finite reciprocal sums and propose several open problems involving Fibonacci and Lucas numbers. Melham [**1**] gave several relations between the reciprocal sum and the alternating sum for generalized Fibonacci numbers, which we subsequently extended in [**4**]. In this note we establish some relations between the reciprocal sum and the alternating sum for generalized Lucas numbers. For U_n and V_n, we have the following

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well known expansions:

$$U_{n+1} = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} p^{n-2k}, \qquad (n \geq 0), \tag{5}$$

and

$$V_n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n}{n-k} \binom{n-k}{k} p^{n-2k}, \qquad (n \geq 1). \tag{6}$$

See [**3**, (2.7),(2.8)]. From (5) and (6), since p is a nonzero real number, it is easy to obtain that $U_n > 0$ $(n \geq 1)$ and $V_n > 0$ $(n \geq 0)$ for $p > 0$. For $p < 0$, if n is an odd number, then we have $U_{n+1} < 0$ $(n \geq 0)$ and $V_n < 0$ $(n \geq 1)$. If n is an even number, then we have $U_{n+1} > 0$ $(n \geq 0)$ and $V_n > 0$ $(n \geq 0)$. Hence we have $U_n \neq 0$ $(n \geq 1)$ and $V_n \neq 0$ $(n \geq 0)$. Our main result is the following.

Theorem 1.1. *Let m and k be positive integers. Put*

$$\mathcal{S}_k(m) = \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} V_{n+2k} \ldots V_{n+mk}} \tag{7}$$

and

$$\mathcal{T}_k(m) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{V_n V_{n+k} V_{n+2k} \ldots V_{n+mk}}. \tag{8}$$

Then

$$\begin{aligned} \mathcal{S}_k(m) = {} & \frac{1}{1+(-1)^{(m-1)k} - V_{(m+1)k}} \\ & \cdot \Bigg[\sum_{i=1}^{k} \frac{(-1)^{(m-1)k} V_{i+(m+1)k} - V_{i+2(m+1)k}}{V_i V_{i+k} V_{i+2k} \ldots V_{i+(m+1)k}} \\ & \qquad + (p^2+4) U_{(m+1)k} U_{(m+2)k} \mathcal{T}_k(m+2) \Bigg] \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathcal{T}_k(m) = {} & \frac{1}{1+(-1)^{(m-1)k} - (-1)^k V_{(m+1)k}} \Bigg\{ \sum_{i=1}^{k} \frac{(-1)^{i-1}}{V_i V_{i+k} V_{i+2k} \ldots V_{i+(m+1)k}} \\ & \cdot \Big[(-1)^{(m-1)k} V_{i+(m+1)k} - (-1)^k V_{i+2(m+1)k} \Big] \\ & + (p^2+4) U_{(m+1)k} U_{(m+2)k} \mathcal{S}_k(m+2) \Bigg\}. \end{aligned} \tag{10}$$

When $p = 1$, by special choices of k and m, we have the following interesting results:

$$\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}} = \frac{7}{12} - 10 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2} L_{n+3}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+1}} = \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+1}L_{n+2}L_{n+3}};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+1}L_{n+2}} = \frac{3}{28} - \frac{15}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+1}L_{n+2}} = \frac{11}{168} + \frac{15}{2}\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+1}L_{n+2}L_{n+3}} = \frac{13}{924} - 15\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}L_{n+5}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+1}L_{n+2}L_{n+3}} = \frac{29}{2772} + \frac{25}{3}\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}L_{n+5}};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+2}} = \frac{139}{396} - 24\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+2}L_{n+4}L_{n+6}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+2}} = \frac{95}{396} - 242\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+2}L_{n+4}L_{n+6}};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+2}L_{n+4}} = \frac{123}{16\cdot 11\cdot 29} + \frac{199}{84\cdot 18\cdot 47} - \frac{105}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+2}L_{n+4}L_{n+6}L_{n+8}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_nL_{n+2}L_{n+4}} = \frac{123}{16\cdot 11\cdot 29} - \frac{199}{84\cdot 18\cdot 47} - \frac{105}{2}\sum_{n=1}^{\infty} \frac{1}{L_nL_{n+2}L_{n+4}L_{n+6}L_{n+8}}.$$

2. The proof of the results

Lemma 2.1. *We have*

$$V_{n+k} + (-1)^k V_{n-k} = V_nV_k; \tag{11}$$

$$V_{n+k} = V_nU_{k+1} + V_{n-1}U_k; \tag{12}$$

$$V_{n+k-1} = \frac{U_{k-1}V_{n+k} + (-1)^{k-1}V_n}{U_k}; \tag{13}$$

$$U_{2k} = U_kV_k; \tag{14}$$

$$U_{n+k} = U_nU_{k+1} + U_{n-1}U_k; \tag{15}$$

$$U_{(m+1)k}^2 - U_{mk}U_{(m+2)k} = (-1)^{mk}U_k^2; \tag{16}$$

$$V_{n+(m+1)k}^2 - V_{n+mk}V_{n+(m+2)k} = -(-1)^{n+mk}(p^2+4)U_k^2; \tag{17}$$

$$U_kV_{n-mk} = (-1)^{mk}(V_nU_{mk+k} - V_{n+k}U_{mk}); \tag{18}$$

$$U_{(m+2)k}V_{i+(m+1)k} - (-1)^kU_{(m+1)k}V_{i+mk} = U_kV_{i+2(m+1)k}; \tag{19}$$

$$U_{n+k} - (-1)^kU_{n-k} = V_nU_k; \tag{20}$$

$$\begin{aligned} V_nV_{n+(m+1)k} = & -\frac{(-1)^{(m+1)k}}{U_k}U_{(m+1)k}V_{n+(m+1)k}V_{n+(m+2)k} \\ & + \frac{(-1)^{(m+1)k}}{U_k}U_{(m+2)k}V_{n+mk}V_{n+(m+2)k} \\ & - (-1)^{n+k}(p^2+4)U_kU_{(m+2)k}; \end{aligned} \tag{21}$$

$$\begin{aligned} & V_{n+mk}V_{n+(m+2)k} - V_nV_{n+(m+1)k} \\ & = \frac{(-1)^{(m+1)k}}{U_k}U_{(m+1)k}V_{n+(m+1)k}V_{n+(m+2)k} \\ & + \left[1 - \frac{(-1)^{(m+1)k}}{U_k}U_{(m+2)k}\right]V_{n+mk}V_{n+(m+2)k} \\ & + (-1)^{n+k}(p^2+4)U_kU_{(m+2)k}; \end{aligned} \tag{22}$$

$$\begin{aligned} & V_{n+mk}V_{n+(m+2)k} + (-1)^{k-1}V_nV_{n+(m+1)k} \\ & = \frac{(-1)^{mk}}{U_k}U_{(m+1)k}V_{n+(m+1)k}V_{n+(m+2)k} \\ & + \left[1 - \frac{(-1)^{mk}}{U_k}U_{(m+2)k}\right]V_{n+mk}V_{n+(m+2)k} \\ & + (-1)^n(p^2+4)U_kU_{(m+2)k}. \end{aligned} \tag{23}$$

Proof. Use the Binet formulas (3) and (4) of U_n and V_n. □

Lemma 2.2. *We have*

$$\begin{aligned} & \sum_{n=1}^{\infty}\frac{1}{V_nV_{n+k}\dots V_{n+(m-1)k}V_{n+(m+1)k}} \\ & = \frac{U_{mk}+(-1)^{mk}U_k}{U_{(m+1)k}}\mathcal{S}_k(m) - \frac{(-1)^{mk}U_k}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}} \end{aligned} \tag{24}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{V_n V_{n+k} \dots V_{n+(m-1)k} V_{n+(m+1)k}}$$

$$= \frac{U_{mk} + (-1)^{(m+1)k} U_k}{U_{(m+1)k}} \mathcal{T}_k(m) - \frac{(-1)^{(m+1)k} U_k}{U_{(m+1)k}} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{V_i V_{i+k} \dots V_{i+mk}}. \tag{25}$$

Proof. We give only the proof of (24), the proof of (25) is similar. Using (11), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \dots V_{n+(m-1)k} V_{n+(m+1)k}} \\
&\quad = \sum_{n=1}^{\infty} \left(\frac{V_{n+(m+1)k} + (-1)^k V_{n+(m-1)k}}{V_n V_{n+k} \dots V_{n+(m-1)k} V_{n+(m+1)k}} \right) \frac{1}{V_k V_{n+mk}} \\
&\quad = \sum_{n=1}^{\infty} \left(\frac{1}{V_n V_{n+k} \dots V_{n+(m-1)k}} + \frac{(-1)^k}{V_n V_{n+k} \dots V_{n+(m-2)k} V_{n+(m+1)k}} \right) \frac{1}{V_k V_{n+mk}} \\
&\quad = \frac{1}{V_k} \mathcal{S}_k(m) + \frac{(-1)^k}{V_k} \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \dots V_{n+(m-2)k} V_{n+mk} V_{n+(m+1)k}}.
\end{aligned}$$

Using (12), (13) and (15), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \dots V_{n+(m-2)k} V_{n+mk} V_{n+(m+1)k}} \\
&\quad = \sum_{n=1}^{\infty} \left(\frac{V_{n+(m+1)k} + (-1)^k \frac{U_{2k}}{U_k} V_{n+(m-2)k}}{V_n V_{n+k} \dots V_{n+(m-2)k} V_{n+mk} V_{n+(m+1)k}} \right) \frac{U_k}{U_{3k} V_{n+(m-1)k}} \\
&\quad = \sum_{n=1}^{\infty} \left(\frac{1}{V_n V_{n+k} \dots V_{n+(m-2)k} V_{n+mk}} \right. \\
&\qquad\qquad \left. + \frac{(-1)^k \frac{U_{2k}}{U_k}}{V_n V_{n+k} \dots V_{n+(m-3)k} V_{n+mk} V_{n+(m+1)k}} \right) \cdot \frac{U_k}{U_{3k} V_{n+(m-1)k}} \\
&\quad = \frac{U_k}{U_{3k}} \mathcal{S}_k(m) \\
&\qquad + \frac{(-1)^k U_{2k}}{U_{3k}} \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \dots V_{n+(m-3)k} V_{n+(m-1)k} V_{n+mk} V_{n+(m+1)k}}.
\end{aligned}$$

Similarly, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{V_nV_{n+2k}\dots V_{n+(m+1)k}}$$

$$=\frac{U_k}{U_{(m+1)k}}\mathcal{S}_k(m)+\frac{(-1)^kU_{mk}}{U_{(m+1)k}}\sum_{n=1}^{\infty}\frac{1}{V_{n+k}\dots V_{n+(m+1)k}}$$

$$=\frac{U_k}{U_{(m+1)k}}\mathcal{S}_k(m)+\frac{(-1)^kU_{mk}}{U_{(m+1)k}}\mathcal{S}_k(m)-\frac{(-1)^kU_{mk}}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}.$$

Hence, using (16) and (17), we have

$$\sum_{n=1}^{\infty}\frac{1}{V_nV_{n+k}\dots V_{n+(m-1)k}V_{n+(m+1)k}}$$

$$=\frac{1}{V_k}\left\{1+\left[\frac{(-1)^kU_k}{U_{3k}}+\frac{(-1)^{2k}U_kU_{2k}}{U_{3k}U_{4k}}+\dots+\frac{(-1)^{mk}U_kU_{2k}}{U_{mk}U_{(m+1)k}}\right]\right.$$

$$\left.+\frac{(-1)^{mk}U_{2k}}{U_{(m+1)k}}\right\}S_k(m)-\frac{(-1)^{mk}U_{2k}}{V_kU_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}$$

$$=\frac{1}{V_k}\left[1+\frac{U_kU_{2k}}{U_k^2}\left(-\frac{U_k}{U_{2k}}+\frac{U_{2k}}{U_{3k}}-\frac{U_{2k}}{U_{3k}}+\frac{U_{3k}}{U_{4k}}-\dots-\frac{U_{(m-1)k}}{U_{mk}}+\frac{U_{mk}}{U_{(m+1)k}}\right)\right.$$

$$\left.+\frac{(-1)^{mk}U_{2k}}{U_{(m+1)k}}\right]\cdot\mathcal{S}_k(m)-\frac{(-1)^{mk}U_k}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}$$

$$=\frac{1}{V_k}\left[1-1+\frac{U_{mk}U_{2k}}{U_kU_{(m+1)k}}+\frac{(-1)^{mk}U_{2k}}{U_{(m+1)k}}\right]\mathcal{S}_k(m)$$

$$-\frac{(-1)^{mk}U_k}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}$$

$$=\frac{U_{mk}+(-1)^{mk}U_k}{U_{(m+1)k}}\mathcal{S}_k(m)-\frac{(-1)^{mk}U_k}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}.$$

□

The proof of the theorem: We give only the proof of (9), the proof of (10) is similar. Since

$$\sum_{n=1}^{\infty}\left(\frac{1}{V_nV_{n+k}\dots V_{n+(m-1)k}V_{n+(m+1)k}}-\frac{1}{V_{n+k}V_{n+2k}\dots V_{n+mk}V_{n+(m+2)k}}\right)$$

$$=\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+(m-1)k}V_{i+(m+1)k}}, \tag{26}$$

using (22) and Lemma 2.1, then the left side of (26) can be written as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{V_{n+mk}V_{n+(m+2)k}-V_nV_{n+(m+1)k}}{V_nV_{n+k}\dots V_{n+(m+1)k}V_{n+(m+2)k}}\\
&=\frac{(-1)^{(m+1)k}}{U_k}U_{(m+1)k}\mathcal{S}_k(m)-(-1)^k(p^2+4)U_kU_{(m+2)k}\mathcal{T}_k(m+2)\\
&\quad+\left[1-\frac{(-1)^{(m+1)k}}{U_k}U_{(m+2)k}\right]\sum_{n=1}^{\infty}\frac{1}{V_nV_{n+k}\dots V_{n+(m-1)k}V_{n+(m+1)k}}\\
&=\frac{(-1)^{(m+1)k}}{U_k}U_{(m+1)k}\mathcal{S}_k(m)-(-1)^k(p^2+4)U_kU_{(m+2)k}\mathcal{T}_k(m+2)\\
&\quad+\left[1-\frac{(-1)^{(m+1)k}}{U_k}U_{(m+2)k}\right]\left[\frac{U_{mk}+(-1)^{mk}U_k}{U_{(m+1)k}}\mathcal{S}_k(m)\right.\\
&\qquad\qquad\left.-\frac{(-1)^{mk}U_k}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}\right]\\
&=\frac{(-1)^k+(-1)^{mk}-(-1)^kV_{(m+1)k}}{U_{(m+1)k}}U_k\mathcal{S}_k(m)-(-1)^k(p^2+4)U_kU_{(m+2)k}\mathcal{T}_k(m+2)\\
&\quad-\frac{(-1)^{mk}U_k-(-1)^kU_{(m+2)k}}{U_{(m+1)k}}\sum_{i=1}^{k}\frac{1}{V_iV_{i+k}\dots V_{i+mk}}. \qquad (27)
\end{aligned}$$

Considering the right sides of (26) and (27), we have

$$\begin{aligned}
&\frac{(-1)^k+(-1)^{mk}-(-1)^kV_{(m+1)k}}{U_{(m+1)k}}U_k\mathcal{S}_k(m)\\
&=\sum_{i=1}^{k}\frac{\left[(-1)^{mk}U_k-(-1)^kU_{(m+2)k}\right]V_{i+(m+1)k}+U_{(m+1)k}V_{i+mk}}{U_{(m+1)k}V_iV_{i+k}\dots V_{i+(m+1)k}}\\
&\quad+(-1)^k(p^2+4)U_kU_{(m+2)k}\mathcal{T}_k(m+2)\\
&=\sum_{i=1}^{k}\frac{(-1)^{mk}V_{i+(m+1)k}-(-1)^kV_{i+2(m+1)k}}{U_{(m+1)k}V_iV_{i+k}\dots V_{i+(m+1)k}}U_k+(-1)^k(p^2+4)U_kU_{(m+2)k}\mathcal{T}_k(m+2).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{S}_k(m) = \frac{1}{1+(-1)^{(m-1)k}-V_{(m+1)k}}&\left[\sum_{i=1}^{k}\frac{(-1)^{(m-1)k}V_{i+(m+1)k}-V_{i+2(m+1)k}}{V_iV_{i+k}V_{i+2k}\dots V_{i+(m+1)k}}\right.\\
&\left.+(p^2+4)U_{(m+1)k}U_{(m+2)k}\mathcal{T}_k(m+2)\right].
\end{aligned}$$

The proof of (9) is completed. □

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