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ON A GENERAL SIMILARITY BOUNDARY LAYER EQUATION

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Abstract. In this paper we are concerned with the solutions of the differential equation $f''' + ff'' + g(f') = 0$ on $[0, \infty)$, satisfying the boundary conditions $f(0) = \alpha$, $f'(0) = \beta \geq 0$, $f'(\infty) = \lambda$, and where g is a given continuous function. This general boundary value problem includes the Falkner-Skan case, and can be applied, for example, to free or mixed convection in porous medium, or flow adjacent to stretching walls in the context of boundary layer approximation. Under some assumptions on the function g, we prove existence and uniqueness of a concave or a convex solution. We also give some results about nonexistence and asymptotic behaviour of the solution.

1. Introduction

We consider the following third order non-linear autonomous differential equation

$$f''' + ff'' + g(f') = 0 \tag{1}$$

with the boundary conditions

$$f(0) = \alpha, \tag{2}$$

$$f'(0) = \beta, \tag{3}$$

$$f'(\infty) = \lambda \tag{4}$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$ and $f'(\infty) := \lim_{t\to\infty} f'(t)$. We also assume that the given function g is locally Lipschitz on a interval J containing β and λ.

In the literature, the problem (1)–(4) with suitable g and λ arises in many fields of an application such as free or mixed convection in a fluid saturated porous medium near a semi or double infinite wall in the framework of boundary layer approximation, high frequency excitation of liquid metal, stretching walls,. . .

The problems of free convection, stretching walls and high frequency excitation of liquid metal corresponds to the function g given by $g(x) = \frac{2m}{m+1}(-x)x$ and to $\lambda = 0$. There are many papers in connection with those such as [**2**], [**3**], [**7**], [**15**], [**16**], [**17**], [**19**], [**20**], [**26**], [**30**], [**32**], [**33**], [**34**], [**35**] for the physical point of view or numerical computations, [**5**], [**6**], [**9**], [**10**], [**12**], [**14**], [**22**], [**23**], [**25**], [**37**], [**38**] for the mathematical analysis and [**11**] for a survey.

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The Falkner-Skan equation, arising in the study of two dimensional flow of a slightly viscous incompressible fluid past a wedge of angle πm under the assumptions of boundary-layer theory, is obtained for $g(x) = m(1-x)(1+x)$ and $\lambda = 1$. This famous equation has been widely studied, see for example [**18**], [**21**], [**27**], [**28**], [**29**] and the references therein for a survey, and [**31**], [**40**], [**41**], [**42**] for more recent investigations.

The mixed convection case corresponds to $g(x) = \frac{2m}{m+1}(1-x)x$ and $\lambda = 1$. This problems has appeared recently in [**1**] and [**36**]. Some first theoretical results about this equation can be found in [**13**] and [**24**].

The Blasius problem, corresponding to $g = 0$, is a particular case of all previous situations and the first historic case in which an equation of the form (1) appears. This well-known problem, that arises in [**8**] at the begining of the previous century, has been studied in a lot of papers. For more details, we refer to [**4**], [**18**] and [**27**] and the references therein.

Finally, let us notice that a first generalization of some of the previous equations can be found in [**39**]. The author considers the problem (1)–(4) with $\alpha \geq 0$, $\lambda \geq \beta > 0$ and functions g such that $g(x) = \hat{g}(x^2)$ where $\hat{g}$ is assumed to be positive and monotone decreasing on $[\beta, \lambda)$ and $\hat{g}(\lambda^2) = 0$. Under these hypotheses, he proves that there exists one and only one convex solution of this problem.

Remark 1. Let $a \neq 0$, and consider the differential equation

$$u''' + auu'' + h(u') = 0$$

where h is a given function. By setting

$$f(t) = \frac{a}{\sqrt{|a|}}\, u\left(\frac{t}{\sqrt{|a|}}\right)$$

the equation in u becomes to $f''' + ff'' + g(f') = 0$ with $g(x) = \frac{1}{a}h(x)$ if $a > 0$ and $g(x) = \frac{1}{a}h(-x)$ if $a < 0$.

2. Preliminary results

First of all, let us remark that if f satisfies the equation (1) on an interval I and if we denote by F any anti-derivative of f on I, then we have

$$\left(f''e^F\right)' = -g(f')e^F. \tag{5}$$

This, in particular implies that the concavity of f is related to changes of the sign of g.

We will need some Lemmas concerning the solutions of (1).

Lemma 1. *If $g(\mu) = 0$ and if f is a solution of* (1) *on an interval I such that there exists a point $t_0 \in I$ verifying $f''(t_0) = 0$ and $f'(t_0) = \mu$, then $f''(t) = 0$ for every $t \in I$.*

Proof. Let f be a solution of (1) on I such that $f''(t_0) = 0$ and $f'(t_0) = \mu$ for some $t_0 \in I$. Since the function $r(t) = \mu(t-t_0) + f(t_0)$ is a solution of (1) such that $r(t_0) = f(t_0)$, $r'(t_0) = f'(t_0)$ and $r''(t_0) = f''(t_0)$, we get $r = f$ and $f'' \equiv 0$. □

Lemma 2. *Let f be a solution of* (1) *on some interval* $[t_0, \infty)$*, such that* $f'(t) \to l \in \mathbb{R}$ *as* $t \to \infty$*. If moreover f is of a constant sign at infinity, then we have*

$$\lim_{t\to\infty} f''(t) = 0.$$

Proof. First of all, let us remark that since $f'(t)$ has a finite limit as $t \to \infty$, then

$$\liminf_{t\to\infty} f''(t)^2 = 0. \tag{6}$$

Multiplying (1) by f'' and integrating on $[t_0, t]$, we get

$$\frac{1}{2}f''(t)^2 - \frac{1}{2}f''(t_0)^2 + \int_{t_0}^{t} f(s)f''(s)^2 \mathrm{d}s + G(f'(t)) - G(f'(t_0)) = 0, \tag{7}$$

where we denoted by G any anti-derivative of g. As f is of a constant sign at infinity, it follows that the integral in (7), and thus $f''(t)^2$ too, have limits as $t \to \infty$. From (6) we get the result. □

Remark 2. If $l \neq 0$ we have $f(t) \sim lt$ as $t \to \infty$ and f is of a constant sign at infinity. If $l = 0$ and if f is either concave or convex at infinity, then again f is of a constant sign at infinity. From (5) it is the case, for example, if g is of a constant sign in a neighbourhood of 0.

The following Lemma shows that to expect a solution of (1)–(4), we must assume that the function g vanishes at the point λ.

Lemma 3. *Let f be a solution of* (1) *on a interval* $[t_0, \infty)$*, such that* $f'(t) \to l \in \mathbb{R}$ *as* $t \to \infty$*. Then* $g(l) = 0$*.*

Proof. Let us suppose that $2c = -g(l) > 0$. There exists $t_1 > t_0$ such that $-g(f'(t)) > c$ for $t > t_1$ and from (5) we have

$$\left(f''\,\mathrm{e}^F\right)' > c\,\mathrm{e}^F$$

on $[t_1, \infty)$. This means that f'' cannot vanish more than once and thus f is concave or convex at infinity.

- Assume now that f is bounded. Then, it follows from (1) and Lemma 2 that $f'''(t) \to 2c$ as $t \to \infty$ and we have a contradiction.
- Assume next that f is unbounded. Then $|f(t)| \to \infty$ as $t \to \infty$ and thus there exists $t_2 > t_1$ such that $f'/f > 0$ on $[t_2, \infty)$. From (1) we get

$$\frac{f'''f'}{f} + f''f' > c\frac{f'}{f} \quad \text{on} \quad [t_2, \infty). \tag{8}$$

Integrating we obtain

$$\int_{t_2}^{t} \frac{f'''(s)f'(s)}{f(s)}\mathrm{d}s + \frac{1}{2}f'(t)^2 - \frac{1}{2}f'(t_2)^2 \geq c\left(\ln|f(t)| - \ln|f(t_2)|\right).$$

It follows that

$$\int_{t_2}^{\infty} \frac{f'''(s)f'(s)}{f(s)}\mathrm{d}s = \infty. \tag{9}$$

But, we have

$$\int_{t_2}^{t} \frac{f'''(s)f'(s)}{f(s)}\mathrm{d}s = \frac{f''(t)f'(t)}{f(t)} - \frac{f''(t_2)f'(t_2)}{f(t_2)} - \int_{t_2}^{t} \frac{f''(s)^2}{f(s)}\mathrm{d}s + \int_{t_2}^{t} \frac{f''(s)f'(s)^2}{f(s)^2}\mathrm{d}s$$

which leads to a contradiction with (9), since the integrals on the right hand side have finite limits as $t \to \infty$.

For $c < 0$, same arguments give also a contradiction. The proof is now complete.

□

Remark 3. If $l \neq 0$ we can have a much simpler proof. Indeed, in this case we have $f'(t) \sim l$ and $f(t) \sim lt$ as $t \to \infty$. Integrating (1) on $[t_0, t]$ we get

$$\begin{aligned} f''(t) - f''(t_0) - f(t_0)f'(t_0) &= -f(t)f'(t) + \int_{t_0}^{t} f'(s)^2\mathrm{d}s - \int_{t_0}^{t} g(f'(s))\mathrm{d}s \\ &= -l^2t(1+o(1)) + l^2t(1+o(1)) - g(l)t(1+o(1)) \\ &= -g(l)t + o(t) \end{aligned}$$

which is a contradiction since $f''(t) \to 0$ as $t \to \infty$ by Lemma 2.

Remark 4. A solution, for which the first derivative does not have a finite limit, does exist. For example

- For any $a \in \mathbb{R}^*$ and any $b \in \mathbb{R}$ the function f defined by
$$f(t) = at^2 + bt + \frac{b^2-1}{4a}$$
is a solution of (1) with $g(x) = \frac{1}{2}(1-x^2)$ and $f'(t) \to \pm\infty$ as $t \to \infty$ (Falkner-Skan with $m = \frac{1}{2}$).
- If $g(x) = -x^2 + x + 1$, then $f(t) = \sin t$ is a solution of (1) for which f' does not have a limit at infinity.

In order to get solutions of (1)–(4) for given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$, we will consider the initial value problem

$$(10) \qquad \begin{cases} f''' + ff'' + g(f') = 0, \\ f(0) = \alpha, \\ f'(0) = \beta, \\ f''(0) = \gamma \end{cases}$$

and use a shooting technique on the parameter γ. We will denote by f_γ its solution and by $[0, T_\gamma)$ its right maximal interval of existence. Integrating (1) on $[0, t]$ for $0 < t < T_\gamma$, we obtain the useful identity

$$(11) \qquad f''_\gamma(t) = \gamma - f_\gamma(t)f'_\gamma(t) + \alpha\beta + \int_0^t \left(f'_\gamma(s)^2 - g(f'_\gamma(s))\right)\mathrm{d}s.$$

Remark 5. Looking at (11) we see that if we take $g(x) = x^2$ the integral on the right hand side vanishes. Then, integrating on $[0, t]$ we obtain

$$f'(t) + \frac{1}{2}f(t)^2 = (\gamma + \alpha\beta)t + \frac{\alpha^2}{2} + \beta$$

and choosing $\gamma = -\alpha\beta$ we have for $\beta > -\alpha^2/2$ the function

$$f(t) = \frac{1}{-\frac{1}{2d} + \frac{\alpha+d}{2\alpha d - 2d^2}\,\mathrm{e}^{\mathrm{d}t}} + d$$

with $d = \sqrt{\alpha^2 + 2\beta}$ which is a bounded solution of the problem (1)–(4) for $\lambda = 0$.

Remark 6. Let us take a look at the case $\beta = \lambda > 0$. If $g(\lambda) = 0$, then the function $f_0(t) = \lambda t + \alpha$ is a solution of (1)–(4). Without additional hypotheses on g, we cannot say anything about uniqueness. However, if we assume, for example, that $g(x) < 0$ for $x > \lambda$ and $g(x) > 0$ for $x < \lambda$, then f_0 is the unique solution of (1)–(4). Indeed, let f_γ be another solution of (1)–(4) such that $\gamma > 0$. Since $f'_\gamma(0) = f'_\gamma(\infty) = \lambda$, there exists $t_0 > 0$ such that $f'_\gamma(t_0) > \lambda$, $f''_\gamma(t_0) = 0$ and $f'''_\gamma(t_0) \leq 0$. From (1) we obtain $f'''_\gamma(t_0) = -g(f'_\gamma(t_0)) > 0$ and thus a contradiction. If $\gamma < 0$, the same approach leads again to a contradiction.

In the following, we will focus first on the concave solutions and next on the convex solutions of (1)–(4) with functions g such that $g(\lambda) = 0$. As seen in Lemma 3, this hypothesis is necessary to realize the condition $f'(t) \to \lambda$ as $t \to \infty$. In addition, we will assume that some condition on the sign of g is satisfied between β and λ in order to get existence and uniqueness of a concave solution (when $\lambda < \beta$) or a convex solution (when $\lambda > \beta$) of the problem (1)–(4). Such an assumption holds in the physical cases evoked in the introduction for the positive values of the parameter m.

When the proofs in the convex case are close to the ones of the concave case we will remove some details in order to shorten them.

3. Concave solutions

Theorem 1. *Let $\alpha \in \mathbb{R}$ and $0 \leq \lambda < \beta$. If $g(x) < 0$ for $x \in (\lambda, \beta]$ and $g(\lambda) = 0$, then the problem* (1)–(4) *admits a unique concave solution.*

Proof of existence. Let f_γ be a solution of the initial value problem (10) with $\lambda < \beta$ and $\gamma \leq 0$. As long as we have $f'_\gamma > \lambda$ and $f''_\gamma < 0$, f_γ exists. Because of Lemma 1, there are only three possibilities

(a) f''_γ becomes positive from a point such that $f'_\gamma > \lambda$,
(b) f'_γ takes the value λ at a point for which $f''_\gamma < 0$,
(c) we always have $f'_\gamma > \lambda$ and $f''_\gamma < 0$.

As $f'_0(0) = \beta > \lambda$, $f''_0(0) = 0$ and $f'''_0(0) = -g(f'_0(0)) = -g(\beta) > 0$ we have $f'_0(t) > \lambda$ and $f''_0(t) > 0$ on a interval $[0, t_0)$. By continuity it follows that f''_γ becomes positive at a point for which $f'_\gamma > \lambda$ for small values of $-\gamma$ and thus f_γ is of type (a).

On the other hand, as long as $f_\gamma''(t) < 0$ and $f_\gamma'(t) \geq \lambda$, we have $f_\gamma(t) \geq \alpha$, $f_\gamma'(t) \leq \beta$ and, using (11) we obtain

$$\begin{aligned} f_\gamma''(t) &\leq \gamma + |\alpha|\beta + \alpha\beta + \beta^2 t - \int_0^t g(f_\gamma'(\xi))\mathrm{d}\xi \\ &\leq \gamma + |\alpha|\beta + \alpha\beta + (\beta^2 + C)t \end{aligned}$$

where $C = \max\{-g(x)\ ;\ x \in [\lambda, \beta]\} > 0$. Integrating once again we have

$$\lambda \leq f_\gamma'(t) \leq \frac{\beta^2 + C}{2} t^2 + (\gamma + \alpha\beta + |\alpha|\beta)t + \beta := P_\gamma(t).$$

Hence, for $-\gamma$ large enough, the equation $P_\gamma(t) = \lambda$ has two positive roots $t_0 < t_1$, and therefore we have $f_\gamma'(t_0) = \lambda$ and $f_\gamma''(t) < 0$ for $t \leq t_0$, and f_γ is of type (b).

Defining $A = \{\gamma < 0\,;\, f_\gamma \text{ is of type (a)}\}$ and $B = \{\gamma < 0\,;\, f_\gamma \text{ is of type (b)}\}$ we have $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$. Both A and B are open sets, so there exists a $\gamma_* < 0$ such that the solution f_{γ_*} of (10) is of type (c) and is defined on the whole interval $[0, \infty)$. For this solution we have $f_{\gamma_*}' > \lambda$ and $f_{\gamma_*}'' < 0$ which implies that $f_{\gamma_*}' \to l \in [\lambda, \beta)$ as $t \to \infty$. From Lemma 3 and the fact that $g < 0$ on $(\lambda, \beta]$ we get $l = \lambda$. □

Proof of uniqueness. Let f be a concave solution of (1)–(4). As f' is positive and strictly decreasing, we can define a function $v : (\lambda^2, \beta^2] \to [\alpha, \infty)$ such that

$$\forall t \geq 0, \quad v(f'(t)^2) = f(t).$$

Setting $y = f'(t)^2$ leads to

$$f(t) = v(y), \quad f''(t) = \frac{1}{2v'(y)} \quad \text{and} \quad f'''(t) = -\frac{v''(y)\sqrt{y}}{2v'(y)^3}. \tag{12}$$

Then, using (1) we obtain

$$\forall y \in (\lambda^2, \beta^2], \quad v''(y) = \frac{v(y)v'(y)^2}{\sqrt{y}} + \frac{2v'(y)^3 g(\sqrt{y})}{\sqrt{y}} \tag{13}$$

with

$$v(\beta^2) = v(f'(0)^2) = \alpha \quad \text{and} \quad v'(\beta^2) = \frac{1}{2\gamma} < 0.$$

Suppose now that there are two concave solutions f_1 and f_2 of (1)–(4) with $f_i''(0) = \gamma_i < 0$, $i \in \{1, 2\}$ and $\gamma_1 > \gamma_2$. They give v_1, v_2 solutions of the equation (13) defined on $(\lambda^2, \beta^2]$ such that, if $w = v_1 - v_2$, we have

$$w(\beta^2) = 0 \quad \text{and} \quad w'(\beta^2) = \frac{1}{2\gamma_1} - \frac{1}{2\gamma_2} < 0.$$

If w' vanishes, there exists an x in $(\lambda^2, \beta^2]$ such that $w'(x) = 0$, $w''(x) \leq 0$ and $w(x) > 0$. But from (13) we then obtain

$$w''(x) = \frac{v_1'(x)^2}{\sqrt{x}} w(x) > 0$$

and this is a contradiction. Therefore, $w' < 0$ and $w > 0$ on $(\lambda^2, \beta^2]$. Set now $V_i = \frac{1}{v_i'}$ for $i \in \{1, 2\}$ and $W = V_1 - V_2$ to obtain

$$W'(y) = -\frac{w(y)}{\sqrt{y}} - \frac{2w'(y)g(\sqrt{y})}{\sqrt{y}} < 0.$$

But, using (12) we have $V_i(f_i'(t)^2) = 2f_i''(t)$ and thanks to Lemma 2 we get $W(y) \to 0$ as $y \to \lambda^2$. Since W is decreasing and $W(\beta^2) = 2(\gamma_1 - \gamma_2) > 0$ this is a contradiction. □

Remark 7. If, in addition to the hypotheses of Theorem 1, the function g is assumed to be non-increasing, then we can write a much simpler proof for the uniqueness result. For that, let f_1 and f_2 be two concave solutions of (1)–(4) and let $\gamma_i = f_i''(0) < 0$, $i \in \{1, 2\}$ with $\gamma_1 > \gamma_2$. Writing $u = f_1 - f_2$, we have $u'(0) = 0$, $u'(\infty) = 0$ and $u''(0) > 0$. Hence u' admits a positive local maximum at some $t_0 > 0$ such that $u'(t) > 0$ for $t \in (0, t_0]$. As u is increasing on $[0, t_0]$ and $u(0) = 0$ we have $u(t_0) > 0$. Then, from (1) and since $f_i'' < 0$ and $f_1''(t_0) = f_2''(t_0)$ we get

$$u'''(t_0) = -f_1''(t_0)u(t_0) + g(f_2'(t_0)) - g(f_1'(t_0)) > 0$$

because $f_1'(t_0) > f_2'(t_0)$, and we have a contradiction with the fact that $u'''(t_0) \le 0$.

The following Proposition gives some information about the behaviour at infinity of the concave solution of the problem (1)–(4) obtained in Theorem 1.

Proposition 1. *Let $\alpha \in \mathbb{R}$ and $0 \le \lambda < \beta$. Let us assume that $g < 0$ on $(\lambda, \beta]$ and $g(\lambda) = 0$, and let f be the concave solution of* (1)–(4)*. Then, there exists a constant μ such that $\alpha < \mu < \sqrt{\alpha^2 + 2(\beta - \lambda)}$ and*

$$\lim_{t\to\infty} \{f(t) - (\lambda t + \mu)\} = 0. \tag{14}$$

Moreover, for all $t \ge 0$, one has $\lambda t + \alpha \le f(t) \le \lambda t + \mu$.

Proof. Since f is concave, then for all $t \ge 0$ we have $f'(t) \in (\lambda, \beta]$ and the function $t \mapsto f(t) - \lambda t$ is increasing. Hence $f(t) - \lambda t \to \mu \in (\alpha, \infty]$ as $t \to \infty$. In addition, we have

$$\forall t \ge 0, \qquad f'''(t) = -f(t)f''(t) - g(f'(t)) \ge -f(t)f''(t)$$

and thus

$$\forall t \ge 0, \qquad \frac{f'''(t)}{-f''(t)} \ge f(t) \ge f(t) - \lambda t.$$

If we assume that $\mu = \infty$, it follows that

$$\lim_{t\to\infty} \frac{f'''(t)}{-f''(t)} = \infty.$$

Therefore, there exists $t_0 \ge 0$ such that $f'''(t) \ge -f''(t)$ for $t \ge t_0$. Then integrating twice and using Lemma 2 we get

$$\forall t \ge t_0, \qquad -f''(t) \ge -\lambda + f'(t)$$

and

$$\forall t \geq t_0, \qquad -f'(t) + f'(t_0) \geq -\lambda t + \lambda t_0 + f(t) - f(t_0).$$

Since the left hand side is bounded, we get a contradiction. Therefore, $\mu < \infty$ and we have

$$\forall t > 0, \qquad \lambda t + \alpha < f(t) < \lambda t + \mu.$$

Finally, let us introduce the auxiliary nonnegative function

$$u(t) = f'(t) + \frac{1}{2}(f(t) - \lambda t)^2.$$

From (14), we see that u is bounded. Moreover, we have

$$u''(t) = -g(f'(t)) - \lambda t f''(t) + (f'(t) - \lambda)^2 > 0$$

and u is convex. Therefore u is decreasing and thus

$$\beta + \frac{\alpha^2}{2} = u(0) > u(\infty) = \lambda + \frac{\mu^2}{2}.$$

This completes the proof. □

Remark 8. If $\lambda = 0$ the previous result means that the concave solution of (1)–(4) is bounded.

We have the following nonexistence result

Theorem 2. *Let $\alpha \leq 0$ and $0 \leq \lambda < \beta$. If g is differentiable and if $\forall x \in [\lambda, \beta]$,*

$$g(x) \geq x^2 - \lambda x \quad \text{and} \quad -\alpha + \max_{x\in[\lambda,\beta]} \{g(x) - x^2 + \lambda x\} > 0, \tag{15}$$

then the problem (1)–(4) *does not admit concave solutions.*

Proof. We follow an idea of [**40**]. Let $0 \leq \lambda < \beta$ and suppose that f is a concave solution of (1)–(4). As f' is strictly decreasing, we can define a negative function $v : (\lambda, \beta] \to \mathbb{R}$ such that

$$\forall t \geq 0, \quad v(f'(t)) = f''(t).$$

Setting $y = f'(t)$ we obtain

$$\begin{aligned} f''(t) = v(y), \quad & f'''(t) = v(y)v'(y) \\ \text{and} \quad & f^{(4)}(t) = v(y)^2 v''(y) + v(y)v'(y)^2. \end{aligned} \tag{16}$$

Derivating equation (1) leads to

$$f^{(4)} + f'f'' + ff''' + g'(f')f'' = 0$$

and as

$$f = \frac{-f''' - g(f')}{f''}$$

we get

$$v''(y) = -\frac{y}{v(y)} - \left(\frac{g(y)}{v(y)}\right)'. \tag{17}$$

Integrating (17) on $[z, z_1]$ with $\lambda \le z \le z_1$ leads to

$$v'(z_1) - v'(z) = -\int_z^{z_1} \frac{y}{v(y)}\mathrm{d}y - \frac{g(z_1)}{v(z_1)} + \frac{g(z)}{v(z)}. \tag{18}$$

Using the equation (1) and (16), the equality (18) becomes

$$-f(s_1) - v'(z) = -\int_z^{z_1} \frac{y}{v(y)}\mathrm{d}y + \frac{g(z)}{v(z)}$$

with s_1 such that $f'(s_1) = z_1$.

Integrating on $[\lambda, x]$ with $\lambda \le x \le z_1$, and since $v(\lambda) = f''(\infty) = 0$ by Lemma 2, we get

$$\begin{aligned} -f(s_1)(x-\lambda) - v(x) &= -\int_\lambda^x \left(\int_z^{z_1} \frac{y}{v(y)}\mathrm{d}y\right)\mathrm{d}z + \int_\lambda^x \frac{g(z)}{v(z)}\mathrm{d}z \\ &= -\left[z\int_z^{z_1} \frac{y}{v(y)}\mathrm{d}y\right]_\lambda^x + \int_\lambda^x \frac{-z^2 + g(z)}{v(z)}\mathrm{d}z \\ &= -x\int_x^{z_1} \frac{y}{v(y)}\mathrm{d}y + \lambda\int_\lambda^{z_1} \frac{z}{v(z)}\mathrm{d}z + \int_\lambda^x \frac{-z^2+g(z)}{v(z)}\mathrm{d}z \end{aligned}$$

and taking $x = z_1$ and $z_1 \to \beta$ we derive

$$-v(\beta) = \int_\lambda^\beta \frac{\lambda z - z^2 + g(z)}{v(z)}\mathrm{d}z + \alpha(\beta - \lambda). \tag{19}$$

Since $v(\beta) \le 0$, the right hand side of (19) must be nonnegative, but this cannot be the case if (15) holds. □

Remark 9. Using the previous Theorem we can recover the following nonexistence result

- for free convection (i.e. $g(x) = -\frac{2m}{m+1}x^2$ and $\lambda = 0$) there is no concave solution for $-1 < m \le -\frac{1}{3}$ when $\alpha < 0$, and $-1 < m < -\frac{1}{3}$ when $\alpha = 0$ (see [**5**], [**6**], [**9**] and [**14**]),

and obtain the new results

- for the Falkner-Skan case (i.e. $g(x) = m(1-x)(1+x)$ and $\lambda = 1$) there is no concave solution with $\alpha \le 0$ and $\beta > 1$ for $m \le -\frac{\beta}{1+\beta}$,
- for mixed convection (i.e. $g(x) = \frac{2m}{m+1}x(1-x)$ and $\lambda = 1$) there is no concave solution with $\alpha \le 0$ and $\beta > 1$ for $-1 < m < -\frac{1}{3}$.

4. Convex solutions

Theorem 3. *Let* $\alpha \in \mathbb{R}$ *and* $0 \le \beta < \lambda$. *If* $g(x) > 0$ *for* $x \in [\beta, \lambda)$ *and* $g(\lambda) = 0$, *then the problem* (1)–(4) *admits a unique convex solution.*

Proof of existence. Let f_γ be a solution of the initial value problem (10) with $0 \le \beta < \lambda$ and $\gamma \ge 0$. We notice that f_γ exists as long as we have $f_\gamma'' > 0$ and $f_\gamma' < \lambda$. From Lemma 1, f_γ'' cannot vanish at a point where $f_\gamma' = \lambda$ and it follows that there are only three possibilities

(a) f_γ'' becomes negative from a point such that $f_\gamma' < \lambda$,
(b) f_γ' takes the value λ at a point for which $f_\gamma'' > 0$,
(c) we always have $\beta \le f_\gamma' < \lambda$ and $f_\gamma'' > 0$.

As $f_0'(0) = \beta < \lambda$, $f_0''(0) = 0$ and $f_0'''(0) = -g(\beta) < 0$, we have f_0 is of type (a), and by continuity it must be so for f_γ with $\gamma > 0$ small enough.

On the other hand, as long as $f_\gamma''(t) > 0$ and $f_\gamma'(t) \le \lambda$, we have $f_\gamma(t) \le \lambda t + \alpha$, and (11) leads to

$$\begin{aligned} f_\gamma''(t) &\ge \gamma - f_\gamma(t) f_\gamma'(t) + \alpha\beta - \int_0^t g(f_\gamma'(\xi))\mathrm{d}\xi \\ &\ge \gamma - (\lambda t + |\alpha|)\lambda + \alpha\beta - \int_0^t g(f_\gamma'(\xi))\mathrm{d}\xi \\ &\ge \gamma - |\alpha|\lambda + \alpha\beta - (\lambda^2 + C)t \end{aligned}$$

where $C = \max\{g(x)\ ;\ x \in [\beta, \lambda]\} > 0$ and integrating once again we have

$$\lambda \ge f_\gamma'(t) \ge -\frac{\lambda^2 + C}{2} t^2 + (\gamma - |\alpha|\lambda + \alpha\beta)t + \beta := P_\gamma(t).$$

Hence, for γ large enough, the equation $P_\gamma(t) = \lambda$ has two positive roots $t_0 < t_1$, and therefore, for such a γ, we have $f_\gamma'(t_0) = \lambda$ and $f_\gamma''(t) > 0$ for $t \le t_0$, and f_γ is of type (b).

Defining $A = \{\gamma > 0\,;\, f_\gamma \text{ is of type (a)}\}$ and $B = \{\gamma > 0\,;\, f_\gamma \text{ is of type (b)}\}$ we have $A \ne \emptyset$, $B \ne \emptyset$ and $A \cap B = \emptyset$. Both A and B are open sets, so there exists a $\gamma_* > 0$ such that the solution f_{γ_*} of (10) is of type (c) and is defined on the whole interval $[0, \infty)$. For this solution we have $0 < f_{\gamma_*}' < \lambda$ and $f_{\gamma_*}'' > 0$ which implies that $f_{\gamma_*}' \to l \in (\beta, \lambda]$ as $t \to \infty$. From Lemma 3 and the fact that $g > 0$ on $[\beta, \lambda)$ we get $l = \lambda$. □

Proof of uniqueness. Let f be a convex solution of (1)–(4). As f' and f'' are positive, we can define a function $v : [\beta^2, \lambda^2) \to [\alpha, \infty)$ such that

$$\forall t \ge 0, \quad v(f'(t)^2) = f(t).$$

We have

$$\forall y \in [\beta^2, \lambda^2), \quad v''(y) = \frac{v(y)v'(y)^2}{\sqrt{y}} + \frac{2v'(y)^3 g(\sqrt{y})}{\sqrt{y}} \tag{20}$$

and

$$v(\beta^2) = v(f'(0)^2) = \alpha \quad \text{and} \quad v'(\beta^2) = \frac{1}{2\gamma} > 0.$$

Suppose now that there are two convex solutions f_1 and f_2 of (1)–(4) with $f_i''(0) = \gamma_i > 0$, $i \in \{1, 2\}$ and $\gamma_1 > \gamma_2$. They give v_1, v_2 solutions of the equation (20) defined on $[\beta^2, \lambda^2)$ such that for $w = v_1 - v_2$, we have $w(\beta^2) = 0$ and $w'(\beta^2) < 0$. If w' vanishes, there exists an x in $[\beta^2, \lambda^2)$ such that $w'(x) = 0$, $w''(x) \ge 0$ and $w(x) < 0$. But from (20) we then obtain $w''(x) < 0$ and this is a contradiction. Therefore $w' < 0$ and $w < 0$ on $[\beta^2, \lambda^2)$. Set now $V_i = \frac{1}{v_i'}$ for $i \in \{1, 2\}$ and $W = V_1 - V_2$. We have $W > 0$ and using (20) we obtain $W'(y) > 0$. But,

$W(\beta^2) = 2(\gamma_1 - \gamma_2) > 0$ and $W(y) \to 0$ as $y \to \lambda^2$, this contradicts the fact that W is increasing. $\square$

Proposition 2. *Let $\alpha \in \mathbb{R}$ and $0 \le \beta < \lambda$. Assume that $g > 0$ on $[\beta, \lambda)$ and $g(\lambda) = 0$, and let f be the convex solution of* (1)–(4)*. Then, there exists a constant $\mu < \alpha$ such that*

$$\lim_{t\to\infty} \{f(t) - (\lambda t + \mu)\} = 0.$$

Moreover, for all $t \ge 0$, one has $\lambda t + \mu \le f(t) \le \lambda t + \alpha$.

Proof. Since f is convex, for all $t \ge 0$ we have $f'(t) \in [\beta, \lambda)$. Then the function $t \mapsto f(t) - \lambda t$ is decreasing and thus $f(t) - \lambda t \to \mu \in [-\infty, \alpha)$ as $t \to \infty$. On the other hand, we have

$$\forall t \ge 0, \quad f'''(t) = -f(t)f''(t) - g(f'(t)) \le -f(t)f''(t)$$

and since $f(t) \to \infty$ as $t \to \infty$, there exists $t_0 \ge 0$ such that $f'''(t) \le -f''(t)$ for $t \ge t_0$. Then integrating twice and using Lemma 2 we get

$$\forall t \ge t_0, \quad -f''(t) \le -\lambda + f'(t)$$

and

$$\forall t \ge t_0, \quad -f'(t) + f'(t_0) \le -\lambda t + \lambda t_0 + f(t) - f(t_0).$$

Since the left hand side is bounded, we necessarily get $\mu > -\infty$, and then we have $\lambda t + \mu \le f(t) \le \lambda t + \alpha$ for all $t \ge 0$. $\square$

Let us finish this section with the following nonexistence result.

Theorem 4. *Let $\alpha \le 0$ and $0 \le \beta < \lambda$. If g is differentiable and if $\forall x \in [\beta, \lambda]$,*

$$g(x) \le x^2 - \lambda x \quad \textit{and} \quad -\alpha + \max_{x\in[\beta,\lambda]} \{x^2 - \lambda x - g(x)\} > 0, \tag{21}$$

then the problem (1)–(4) *does not admit convex solutions*

Proof. Let $0 \le \beta < \lambda$ and suppose that f is a convex solution of (1)–(4). As f' is positive, we can define a positive function $v : [\beta, \lambda) \to \mathbb{R}$ such that

$$\forall t \ge 0, \quad v(f'(t)) = f''(t).$$

Following the method used in the proof of Theorem 2, we see that this function v satisfies (17) and

$$-v(\beta) = \int_\beta^\lambda \frac{y^2 - \lambda y - g(y)}{v(y)} \mathrm{d}y - \alpha(\lambda - \beta). \tag{22}$$

Since $v(\beta) \ge 0$, the right hand side of (22) must be nonpositive, but this cannot be the case if (21) holds. $\square$

Remark 10. Using the previous Theorem we can recover the following nonexistence result

- for the Falkner-Skan case (i.e. $g(x) = m(1-x)(1+x)$ and $\lambda = 1$) there is no convex solution for $m \le -\frac{1}{2}$ (see [**40**]),

and obtain the new result

- for mixed convection (i.e. $g(x) = \frac{2m}{m+1}x(1-x)$ and $\lambda = 1$) there is no convex solution for $-1 < m \leq -\frac{1}{3}$.

5. Conclusion

In this paper we have obtained existence, uniqueness and nonexistence results for the concave or convex solutions of a general boundary value problem arising in many fields of application under some reasonable hypotheses. All these hypotheses are verified in important physical cases in the framework of boundary layer approximations such as free or mixed convection, flow adjacent to stretching walls, high frequency excitation of liquid metal and two dimensional flow of a slightly viscous incompressible fluid past a wedge.

All our results hold for β, λ and g such that the function g vanishes at λ but does not vanish between β and λ. Of course, under the same hypotheses, solutions whose concavity changes may exist as it can be seen in [**14**] for the case of free convection and in [**29**] for the Falkner-Skan problem when the parameter m is positive.

In addition, if we assume that the sign of g is opposite of the one that we have in the Theorems 1 and 3, then we can have multiple concave or convex solutions, as it is the case for free or mixed convection and for the Falkner-Skan problem when the parameter m is negative. See for example [**9**], [**11**], [**22**], [**24**] and [**27**].

References

1. Aly E. H., Elliott L. and Ingham D. B., *Mixed convection boundary-layer flows over a vertical surface embedded in a porous medium*, Eur. J. Mech. B Fluids **22** (2003), 529–543.

2. Banks W. H. H., *Similarity solutions of the boundary layer equations for a stretching wall*, J. de Méchan. Théor. et Appl. **2** (1983), 375–392.

3. Banks W. H. H. and Zaturska M. B., *Eigensolutions in boundary layer flow adjacent to a stretching wall*, IMA J. Appl. Math. **36** (1986), 263–273.

4. Belhachmi Z., Brighi B. and Taous K., *On the concave solutions of the Blasius equation*, Acta Math. Univ. Comenianae **LXIX**(2) (2000), 199–214.

5. ______, *Solutions similaires pour un problème de couche limite en milieux poreux*, C. R. Mécanique **328** (2000), 407–410.

6. ______, *On a family of differential equations for boundary layer approximations in porous media*, Euro. Jnl of Applied Mathematics **12**(4) (2001), 513–528.

7. Belhachmi Z., Brighi B., Sac-Epee J. M. and Taous K., *Numerical simulations of free convection about a vertical flat plate embedded in a porous medium*, Computational Geosciences **7** (2003), 137–166.

8. Blasius H., *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Z. Math. Phys. **56** (1908), 1–37.

9. Brighi B., *On a similarity boundary layer equation*, Zeitschrift für Analysis und ihre Anwendungen **21**(4) (2002), 931–948.

10. Brighi B. and Hoernel J.-D., *On similarity solutions for boundary layer flows with prescribed heat flux*, Mathematical Methods in the Applied Sciences **28**(4) (2005), 479–503.

11. ______, *Recent advances on similarity solutions arising during free convection*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 63, pp. 83–92, Birkhäuser Verlag Basel/Switzerland, 2005.

12. ______, *Asymptotic behavior of the unbounded solutions of some boundary layer equation*, Archiv der Mathematik **85**(2) (2005), 161–166.

13. ______, *On the concave and convex solutions of mixed convection boundary layer approximation in a porous medium*, Appl. Math. Lett. **19**(1) (2006), 69–74.

14. Brighi B. and Sari T., *Blowing-up coordinates for a similarity boundary layer equation*, Discrete and Continuous Dynamical Systems (Serie A) **12**(5) (2005), 929–948.

15. Chaudhary M. A., Merkin J. H. and Pop I., *Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: I prescribed surface temperature*, Eur. J. Mech. B-Fluids **14** (1995), 217–237.

16. ______, *Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: II prescribed surface heat flux*, Heat and Mass Transfer **30** (1995), 341–347.

17. Cheng P., Minkowycz W. J., *Free-convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike*, J. Geophys. Res. **82**(14) (1977), 2040–2044.

18. Coppel W. A., *On a differential equation of boundary layer theory*, Phil. Trans. Roy. Soc. London, Ser A **253** (1960), 101–136.

19. Crane L. E., *Flow past a stretching plane*, Z. Angew. Math. Phys. **21** (1970), 645–647.

20. Ene E. I. and Poliševski D., *Thermal flow in porous media*, D. Reidel Publishing Company, Dordrecht, 1987.

21. Falkner V. M. and Skan S. W., *Solutions of the boundary layer equations*, Phil. Mag. **7**(12) (1931), 865–896.

22. Guedda M., *Nonuniqueness of solutions to differential equations for boundary layer approximations in porous media*, C. R. Mécanique **330** (2002), 279–283.

23. ______, *Similarity solutions of differential equations for boundary layer approximations in porous media*, Z. angew. Math. Phys. **56** (2005), 749–762.

24. ______, *Multiple solutions of mixed convection boundary-layer approximations in a porous medium*, Appl. Math. Lett. **19**(1) (2006) 63–68.

25. Guo J.-S. and Tsai J.-C., *The structure of the solutions for a third order differential equation in boundary layer theory* Japan J. Indust. Appl. Math. **22** (2005), 311–351.

26. Gupta P. S. and Gupta A. S., *Heat and mass transfer on a stretching sheet with suction or blowing*, Can. J. Chem. Eng. **55** (1977), 744–746.

27. Hartmann P., *Ordinary Differential Equations*, Wiley, New-York, 1964.

28. Hastings S. P. and Troy W. C., *Oscillating solutions of the Falkner-Skan equations for negative β*, SIAM J. Math. Anal. **18**(2) (1987), 422–429.

29. ______, *Oscillating solutions of the Falkner-Skan equations for positive β*, J. Diff. Equations **71**(1) (1988), 123–144.

30. Ingham D. B. and Brown S. N., *Flow past a suddenly heated vertical plate in a porous medium*, J. Proc. R. Soc. Lond. A **403** (1986), 51–80.

31. Ishimura N. and Ushijima T. K., *An elementary approach to the analysis of exact solutions for the Navier-Stokes stagnation flows with slips*, Arch. Math. **82** (2004), 432–441.

32. Magyari E. and Keller B., *Exact solutions for self-similar boundary-layer flows induced by permeable stretching wall*, Eur. J. Mech. B-Fluids **19** (2000), 109–122.

33. Magyari E., Pop I. and Keller B., *The 'missing' self-similar free convection boundary-layer flow over a vertical permeable surface in a porous medium*, Transport in Porous Media **46** (2002), 91–102.

34. Merkin J. H. and Zhang G., *On the similarity solutions for free convection in a saturated porous medium adjacent to impermeable horizontal surfaces*, Wärme und Stoffübertr. **25** (1990), 179–184.

35. Moffatt H. K., *High-frequency excitation of liquid metal systems*, IUTAM Symposium: Metallurgical Application of Magnetohydrodynamics, (1982) Cambridge.

36. Nazar R., Amin N. and Pop I., *Unsteady mixed convection boundary-layer flow near the stagnation point on a vertical surface in a porous medium*, Int. J. Heat Mass Transfer **47** (2004), 2681–2688.

37. Tsai J.-C., *Similarity solutions for boundary layer flows with prescribed surface temperature*, Appl. Math. Lett., **21**(1) (2008), 67–73.
38. Tsai J.-C. and Wang,C.-A. *A note on similarity solutions for boundary layer flows with prescribed heat flux*, Mathematical Methods in the Applied Sciences, **30**(12) (2007), 1453-1466.
39. Utz W. R., *Existence of solutions of a generalized Blasius equation*, J. Math. Anal. Appl. **66** (1978), 55–59.
40. Wang J., Gao W. and Zhang Z., *Singular nonlinear boundary value problems arising in boundary layer theory*, J. Math. Anal. Appl. **233** (1999), 246–256.
41. Yang G. C., *Existence of solutions to the third-order nonlinear differential equations arising in boundary layer theory*, Appl. Math. Lett. **16**(6) (2003), 827–832.
42. ———, *A note on* $f''' + ff'' + \lambda(1 - f'^2) = 0$ *with* $\lambda \in (-1/2, 0)$ *arising in boundary layer theory*, Appl. Math. Lett. **17**(11) (2004) 1261–1265.

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