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REGULARITY OF WEAK SOLUTIONS OF DEGENERATE ELIPTIC EQUATIONS

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Abstract. In this article we establish the existence of higher order weak derivatives of weak solutions of the Dirichlet problem for a class of degenerate elliptic equations.

1. Introduction

In this paper we shall study the existence of higher order weak derivatives (see Theorem 3.9) of weak solutions of degenerate elliptic equations $Lu = g$, where L is an elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u)(x)) \tag{1.1}$$

whose coefficients a_{ij} are measurable, real-valued functions, and whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$\omega(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le v(x)|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega \subset \mathbb{R}^n$, where Ω is a bounded open set, ω and v are weight functions (that is, ω and v are locally integrable and nonnegative functions on $\mathbb{R}^n$).

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occurs as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients it is natural to look for solutions in weighted Sobolev spaces (see [**1**], [**2**], [**3**], [**4**], [**5**] and [**8**]).

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2. Definitions and basic results

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus $\omega(E) = \int_E \omega \,\mathrm{d}x$ for measurable sets $E \subset \mathbb{R}^n$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open and let ω be a weight. For $1 < p < \infty$, we define $L^p(\Omega, \omega)$, the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x) \,\mathrm{d}x \right)^{1/p} < \infty.$$

Definition 2.2. Let $1 \le p < \infty$.

(a) The weight ω belongs to the Muckenhoupt class A_p ($\omega \in A_p$) if there is a constant $\mathbf{C} = C_{p,\omega}$ (called A_p-constant) such that

$$\left(\frac{1}{|B|} \int_B \omega \,\mathrm{d}x \right) \left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)} \,\mathrm{d}x \right)^{p-1} \le \mathbf{C}, \qquad \text{when } 1 < p < \infty$$

$$\left(\frac{1}{|B|} \int_B \omega \,\mathrm{d}x \right) \left(\operatorname*{ess\,sup}_B \frac{1}{\omega} \right) \le \mathbf{C}, \qquad \text{when } p = 1,$$

for every ball $B \subset \mathbb{R}^n$, where $|B|$ is the n-dimensional Lebesgue measure of B.

(b) Let ω and v be weights. We shall say that the pair of weights (v, ω) satisfies the condition A_p, $1 \le p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|B|} \int_B v(x) \,\mathrm{d}x \right) \left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)}(x) \,\mathrm{d}x \right)^{p-1} \le C, \qquad \text{when } 1 < p < \infty,$$

$$\frac{1}{|B|} \int_B v(x) \,\mathrm{d}x \le C \operatorname*{ess\,inf}_B \omega, \qquad \text{when } p = 1,$$

for every ball B in $\mathbb{R}^n$. The smallest constant C will be called the A_p-constant for the pair (ω, v).

Remark 2.3. If $(v, \omega) \in A_p$ and $\omega \le v$ then $\omega \in A_p$ and $v \in A_p$.

Example 2.4. The function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is a weight A_p if and only if $-n < \alpha < n(p-1)$ (see [**7**, Chapter 15]).

Remark 2.5. If $\omega \in A_p$, $1 \le p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, when $p > 1$, and $1/\omega$ is locally bounded, when $p = 1$, we have $L^p(\Omega, \omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω and such that convergence in $L^p(\Omega, \omega)$ implies local convergence in $L^1(\Omega)$. If Ω is bounded, in the same way one obtains $L^p(\Omega, \omega) \subset L^1(\Omega)$. It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.6. We shall say that the pair of weights (v, ω) satisfies the condition S_p ($1 < p < \infty$) if there is a constant C (called the S_p-constant) such that

$$\int_B \Big|M(\mu\chi_B)(x)\Big|^p v(x)\,\mathrm{d}x \,\leq\, C\mu(B) < \infty,$$

for every ball B, where $[Mf](x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\,dy$ is the Hardy-Littlewood maximal function, $\mu = \omega^{-1/(p-1)}$ and $\mu(B) = \int_B \mu(x)\,\mathrm{d}x$.

Remark 2.7. If $(v,\omega)\in S_p$, $1 < p < \infty$, then $(v,\omega)\in A_p$.

Theorem 2.8 (Muckenhoupt generalized theorem)**.** *Let $1 < p < \infty$ and let (v,ω) be a pair of weights in $\mathbb{R}^n$. Then $M : L^p(\mathbb{R}^n,\omega)\to L^p(\mathbb{R}^n, v)$ is bounded*

$$\|Mf\|_{L^p(\mathbb{R}^n,v)} \,\leq\, C_M\|f\|_{L^p(\mathbb{R}^n,\omega)},$$

if and only if $(v,\omega)\in S_p$. The constant C_M is called Muckenhoupt constant and C_M depends only on n, p and the S_p-constant of (v,ω).

Proof. See Theorem 4.9, Chapter IV in [**6**]. □

Definition 2.9. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and let v,ω be weights. We define the space

$$W^{k,2}(\Omega,\omega,v) = \left\{u\in L^2(\Omega,v) : \int_\Omega \langle A\nabla u,\nabla u\rangle\,\mathrm{d}x < \infty \text{ and } D^\alpha u\in L^2(\Omega,\omega),\ 2\leq|\alpha|\leq k\right\}$$

with the norm

$$\|u\|_{W^{k,2}(\Omega,\omega,v)} = \left(\int_\Omega u^2 v\,\mathrm{d}x + \int_\Omega \langle A\nabla u,\nabla u\rangle\,\mathrm{d}x + \sum_{2\,\leq\,|\alpha|\,\leq\,k}\int_\Omega |D^\alpha u|^2\omega\,\mathrm{d}x\right)^{1/2}$$

where $A = (a_{ij})_{i,j=1,\ldots,n}$ is the coefficient matrix of the operator L.

Remark 2.10. If $(v,\omega)\in S_2$ and $\omega\,\leq\,v$ then $C^\infty(\Omega)$ is dense in $W^{k,2}(\Omega,\omega,v)$ (see [**1**, Theorem 4.7]). In this case, we define $W_0^{k,2}(\Omega,\omega,v)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,2}(\Omega,\omega,v)} = \left(\int_\Omega \langle A\nabla u,\nabla u\rangle\,\mathrm{d}x + \sum_{2\leq|\alpha|\leq k}\int_\Omega |D^\alpha u|^2\omega\,\mathrm{d}x\right)^{1/2}.$$

Note that, by (1.2), we have

$$\int_\Omega |\nabla u|^2\omega\,\mathrm{d}x \,\leq\, \int_\Omega \langle A\nabla u,\nabla u\rangle\,\mathrm{d}x \,\leq\, \int_\Omega |\nabla u|^2 v\,\mathrm{d}x.$$

Definition 2.11. We say that an element $u\in W^{1,2}(\Omega,\omega,v)$ is a weak solution of the equation $Lu = g$ if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}D_i u D_j\varphi\,\mathrm{d}x = \int_\Omega g\,\varphi\,\mathrm{d}x$$

for every $\varphi\in W_0^{1,2}(\Omega,\omega,v)$.

Remark 2.12. The existence and uniqueness result for the Dirichlet problem

$$(P)\begin{cases} Lu = g, \text{ in } \Omega \\ u - \psi \in W_0^{1,2}(\Omega, \omega, v) \end{cases}$$

where $\psi \in W^{1,2}(\Omega, \omega, v)$, can be found in [**1**, Theorem 4.9].

3. Differentiability of Weak Solutions

In this section we prove that weak solutions $u \in W^{1,2}(\Omega, \omega, v)$ of the equation $Lu = g$, with some hypotheses, are twice weakly differentiable and $D_{ij}u \in L^2(\Omega', \omega)$ (that is, $u \in W^{2,2}(\Omega', \omega, v)$, $\forall \Omega' \subset\subset \Omega$).

Definition 3.1. Let u be a function on a bounded open set $\Omega \subset \mathbb{R}^n$ and denote by e_i the unit coordinate vector in the x_i direction. We define the difference quotient of u at x in the direction e_i by

$$\Delta_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}\ , \ (0 < |h| < \text{dist} \quad (x, \partial\Omega)). \tag{3.1}$$

Lemma 3.2. *Let* $\Omega' \subset\subset \Omega$ *and* $0 < |h| < \text{dist}(\Omega', \partial\Omega)$. *If* $u, \varphi \in L^2_{\text{loc}}(\Omega, \omega)$, $\text{supp}(\varphi) \subset \Omega'$ *and* g *is a measurable function with* $|g(x)| \le C\omega(x)$, *then*

(a) $\Delta_k^h(u\varphi)(x) = u(x + he_k)\Delta_k^h\varphi(x) + \varphi(x)\Delta_k^h u(x)$, *with* $1 \le k \le n$.

(b) $\displaystyle\int_\Omega g(x)u(x)\Delta_k^{-h}\varphi(x)\,\mathrm{d}x = -\int_\Omega \varphi(x)\Delta_k^h(gu)(x)\,\mathrm{d}x.$

(c) *If* $\varphi \in C^1(\Omega)$, *then* $\Delta_k^h(D_j\varphi)(x) = D_j(\Delta_k^h\varphi)(x)$.

Proof. The proof of this lemma follows trivially from the Definition 3.1. □

Definition 3.3. Let ω be a weight in $\mathbb{R}^n$. We say that ω is uniformly A_p in each coordinate if

(a) $\omega \in A_p(\mathbb{R}^n)$;

(b) $\omega_i(t) = \omega(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ is in $A_p(\mathbb{R})$, for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, \dots, x_n$ a.e., $1 \le i \le n$, with A_p constant of ω_i bounded independently of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

Example 3.4. Let $\omega(x, y) = \omega_1(x)\omega_2(y)$, with $\omega_1(x) = |x|^{1/2}$ and $\omega_2(y) = |y|^{1/2}$. We have ω is uniformly A_2 in each coordinate.

Definition 3.5. Let v, ω be weights in $\mathbb{R}^n$.

(a) We say that (v, ω) is uniformly A_p in each coordinate if $(v, \omega) \in A_p(\mathbb{R}^n)$ and $(v_i, \omega_i) \in A_p(\mathbb{R})$ $(1 \le i \le n)$ with constant A_p of (v_i, ω_i) bounded independently of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

(b) We say that (v, ω) is uniformly S_p in each coordinate if $(v, \omega) \in S_p(\mathbb{R}^n)$ and $(v_i, \omega_i) \in S_p(\mathbb{R})$ $(1 \le i \le n)$ with S_p-constant of (v_i, ω_i) bounded independently of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

Lemma 3.6. *Let $u\in W^{1,2}(\Omega,\omega,v)$ and let (ω,v) be uniformly S_2 in each coordinate. Then for any $\Omega'\subset\subset\Omega$ and $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$, we have*

$$\|\Delta_k^h u\|_{L^2(\Omega',v)} \leq C\|D_k u\|_{L^2(\Omega,\omega)} \tag{3.2}$$

where $C=2C_M$, and C_M is the Muckenhoupt constant.

Proof. <u>*Case 1*</u>: Let us suppose initially that $u\in C^\infty(\Omega)$. We have,

$$\begin{aligned}
\Delta_k^h u(x) &= \frac{u(x+he_k)-u(x)}{h} = \frac{1}{h}\int_0^h D_k(x+\zeta e_k)\,d\zeta \\
&= \frac{1}{h}\int_0^h D_k u(x_1,\dots,x_{k-1},x_k+\zeta,x_{k+1},\dots,x_n)\,d\zeta.
\end{aligned}$$

For $1\leq k\leq n$, we define the functions

$$G_k(x)=\begin{cases} D_k u(x), & \text{if } x\in\Omega \\ 0, & \text{if } x\notin\Omega. \end{cases}$$

We have for $x\in\Omega'\subset\subset\Omega$ and h satisfying $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$,

$$\begin{aligned}
|\Delta_k^h u(x)| &\leq \frac{1}{|h|}\left|\int_0^h |D_k u(x_1,...,x_{k-1},x_k+\zeta,x_{k+1},\dots,x_n)|\,d\zeta\right| \\
&= \frac{1}{|h|}\left|\int_{x_k}^{x_k+h} |G_k(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)|\,dt\right| \\
&\leq \frac{1}{|h|}\left|\int_{x_k-h}^{x_k+h} |G_k(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)|\,dt\right| \\
&\leq 2M(G_k^{x_1,\dots,x_{k-1},x_{k+1},\dots,x_n})(\mathbf{x_k}),
\end{aligned}$$

where $G_k^{x_1,\dots,x_{k-1},x_{k+1},\dots,x_n}(\mathbf{x_k})=G_k(x_1,\dots,\mathbf{x_k},\dots,x_n)$. Consequently, using the notation $\widehat{\mathrm{d}x_k}=\mathrm{d}x_1\dots\mathrm{d}x_{k-1}\mathrm{d}x_{k+1}\dots\mathrm{d}x_n$ (where the hat indicates the term that must be omitted in the product) and by Theorem 2.8, we obtain

$$\begin{aligned}
&\int_{\Omega'}|\Delta_k^h u(x)|^2 v(x)\,\mathrm{d}x \\
&\quad\leq 2^2\int_{\Omega'}[M(G_k^{x_1,\dots,x_{k-1}x_{k+1},\dots,x_n})]^2(\mathbf{x_k})v(x_1,\dots,x_k,\dots,x_n)\,\mathrm{d}x \\
&\quad\leq 4\int_{\mathbb{R}^n}[M(G_k^{x_1,\dots,x_{k-1},x_{k+1},\dots,x_n})]^2(\mathbf{x_k})v(x_1,\dots,x_k,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_k\dots\mathrm{d}x_n \\
&\quad\leq 4\int_{\mathbb{R}^{n-1}}\left(C_M^2\int_{\mathbb{R}}|G_k^{x_1,\dots,x_{k-1},x_{k+1},\dots,x_n}(\mathbf{x_k})|^2\omega(x_1,\dots,\mathbf{x_k},\dots,x_n)\,\mathrm{d}x_k\right)\widehat{\mathrm{d}x_k} \\
&\quad= 4C_M^2\int_{\mathbb{R}^n}|G_k(x)|^2\omega(x)\,\mathrm{d}x = 4C_M^2\int_{\Omega}|D_k u(x)|^2\omega(x)\,\mathrm{d}x,
\end{aligned}$$

where C_M is independent of $x_1,\dots,x_{k-1},x_{k+1},\dots,x_n$ because (v,ω) is uniformly S_2 in each coordinate. Therefore

$$\|\Delta_k^h u\|_{L^2(\Omega',v)} \leq C\|D_k u\|_{L^2(\Omega,\omega)},\ \text{where } C=2C_M.$$

Case 2: If $u \in W^{1,2}(\Omega,\omega,v)$ then there exists a sequence $\{u_m\}$, $u_m \in C^\infty(\Omega)$, Cauchy sequence in the norm $\|\cdot\|_{W^{1,p}(\Omega,\omega,v)}$. By Remark 2.10, we have

$$u_m \to u \quad \text{in} \quad L^2(\Omega,v), \qquad \text{and} \qquad D_k u_m \to D_k u \quad \text{in} \quad L^2(\Omega,\omega).$$

Since $(v,\omega)\in S_2$ and $\omega \leq v$, we have $\omega\in A_2$ and $v\in A_2$. Consequently, by Remark 2.5, there exists a subsequence $\{u_{m_j}\}$ such that $u_{m_j} \to u$ a.e. and $D_k u_{m_j} \to D_k u$ a.e.. This implies, for $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$, that

$$\Delta_k^h u_{m_j} \to \Delta_k^h u \quad \text{a.e.}.$$

We have $\{\Delta_k^h u_{m_j}\}$ is a Cauchy sequence in $L^2(\Omega',v)$, for any $\Omega' \subset\subset \Omega$. In fact, using the first case, we have

$$\begin{aligned}
\|\Delta_k^h u_{m_r} - \Delta_k^h u_{m_s}\|_{L^2(\Omega',v)} &= \|\Delta_k^h (u_{m_r} - u_{m_s})\|_{L^2(\Omega',v)} \\
&\leq C\|D_k(u_{m_r} - u_{m_s})\|_{L^2(\Omega,\omega)} \\
&= C\|D_k u_{m_r} - D_k u_{m_s}\|_{L^2(\Omega,\omega)} \\
&\to 0, \qquad \text{as} \quad m_r, m_s \to \infty.
\end{aligned}$$

Therefore, there exists $g \in L^2(\Omega',v)$ such that $\Delta_k^h u_{m_j} \to g$ in $L^2(\Omega',v)$. Consequently, there exists a subsequence $\Delta_k^h u_{m_{j_r}} \longrightarrow g$ a.e.. We can conclude that $\Delta_k^h u = g$ a.e.. Hence

$$\Delta_k^h u_{m_j} \to \Delta_k^h u \quad \text{in} \quad L^2(\Omega',v).$$

This implies that

$$\begin{aligned}
\|\Delta_k^h u\|_{L^2(\Omega',v)} &= \lim_{m_j\to\infty} \|\Delta_k^h u_{m_j}\|_{L^2(\Omega',v)} \\
&\leq C \lim_{m_j\to\infty} \|D_k u_{m_j}\|_{L^2(\Omega,\omega)} \\
&= C\|D_k u\|_{L^2(\Omega,\omega)},
\end{aligned}$$

that is, $\|\Delta_k^h u\|_{L^2(\Omega',v)} \leq C\|D_k u\|_{L^2(\Omega,\omega)}$. □

Lemma 3.7. *Let $u\in L^p(\Omega,\omega)$, $1<p<\infty$, $\omega\in A_p$ and suppose there exists a constant C such that*

$$\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \leq C, \qquad k=1,2,\dots,n \tag{3.3}$$

for any $\Omega' \subset\subset \Omega$ and $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$ (with C independent of h). Then there exists $\vartheta_k \in L^p(\Omega,\omega)$ such that $D_k u = \vartheta_k$ in the weak sense and $\|D_k u\|_{L^p(\Omega,\omega)} \leq C$.

Proof. Since $\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \leq C$, using $L^p(\Omega,\omega)$ is reflexive ($1<p<\infty$), there exists a sequence $\{h_m\}$, $h_m \to 0$, and a function $\vartheta_k \in L^p(\Omega,\omega)$, with $\|\vartheta_k\|_{L^p(\Omega,\omega)} \leq C$, such that

$$\int_\Omega \Delta_k^{h_m} u(x)\varphi(x)\omega(x)\,\mathrm{d}x \;\to\; \int_\Omega \vartheta_k(x)\varphi(x)\omega(x)\,\mathrm{d}x \tag{3.4}$$

for all $\varphi \in L^{p'}(\Omega,\omega)$. Since $\omega\in A_p$, we have $\varphi = \psi/\omega \in L^{p'}(\Omega,\omega)$ for any $\psi\in C_0^\infty(\Omega)$. In fact,

$$\begin{aligned}\int_\Omega |\varphi|^{p'}\omega\,\mathrm{d}x &= \int_\Omega |\psi|^{p'}\omega^{-p'}\omega\,\mathrm{d}x\\ &\le C_\psi \int_\Omega \omega^{1-p'}\,\mathrm{d}x < \infty \qquad\qquad (\text{because } \omega\in A_p).\end{aligned}$$

Setting $\varphi = \psi/\omega$ in (3.4), we obtain

$$\int_\Omega \Delta_k^{h_m} u(x)\psi(x)\,\mathrm{d}x \;\to\; \int_\Omega \vartheta_k(x)\psi(x)\,\mathrm{d}x, \quad \forall\,\psi\in C_0^\infty(\Omega).$$

Now for $h_m < \operatorname{dist}(\operatorname{supp}\psi,\partial\Omega)$, we have

$$\begin{aligned}\int_\Omega \Delta_k^{h_m} u(x)\psi(x)\,\mathrm{d}x &= -\int_\Omega u(x)\Delta_k^{-h_m}\psi(x)\,\mathrm{d}x\\ &\to\; -\int_\Omega u(x)D_k\psi(x)\,\mathrm{d}x, \qquad \text{with } h_m\to 0.\end{aligned}$$

Hence

$$\int_\Omega \vartheta_k(x)\psi(x)\,\mathrm{d}x = -\int_\Omega u(x)D_k\psi(x)\,\mathrm{d}x, \qquad \forall\,\psi\in C_0^\infty(\Omega).$$

Therefore $D_k u = \vartheta_k$ in the weak sense. □

Remark 3.8. If the assumptions of Lemma 3.6 are satisfied and $\omega \le v$, then we have

$$\|\Delta_k^h u\|_{L^2(\Omega',\omega)} \le \|\Delta_k^h u\|_{L^2(\Omega',v)} \le C\|D_k u\|_{L^2(\Omega',\omega)}.$$

We are able now to prove the main result of this paper.

Theorem 3.9. *Let $u\in W^{1,2}(\Omega,\omega,v)$ be a weak solution of the equation $Lu = g$ in Ω, and assume that*

(a) *$g/v \in L^2(\Omega,v)$;*

(b) *The pair of weights (v,ω) is uniformly S_2 in each coordinate;*

(c) *$|\Delta_k^h a_{ij}(x)| \le C_1 v(x)$, $x\in\Omega'\subset\subset\Omega$ a.e., $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$, with constant C_1 is independent of Ω' and h.*
Then for any subdomain $\Omega'\subset\subset\Omega$, we have $u\in W^{2,2}(\Omega',\omega,v)$ and

$$\|u\|_{W^{2,2}(\Omega',\omega,v)} \le \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega,v)} + \|g/v\|_{L^2(\Omega,v)}\right) \tag{3.5}$$

for $\mathbf{C} = \mathbf{C}(n, C_M, C_1, d')$, and $d' = \operatorname{dist}(\Omega',\partial\Omega)$.

Proof. Since $u\in W^{1,2}(\Omega,\omega,v)$ is a weak solution of the equation $Lu = g$, then by Definition 2.11 we have,

$$\int_\Omega a_{ij}(x)D_i u(x)D_j\varphi(x)\,\mathrm{d}x = \int_\Omega g(x)\varphi(x)\,\mathrm{d}x, \tag{3.6}$$

for all $\varphi\in W_0^{1,2}(\Omega,\omega,v)$ (in particular for all $\varphi\in C_0^\infty(\Omega)$).

In (3.6) let us replace φ by $\Delta_k^{-h}\varphi$ $(1 \leq k \leq n)$, with $\varphi \in C_0^\infty(\Omega)$, $\operatorname{supp}(\varphi) \subset\subset \Omega$ and let $|2h| < \operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega)$. Then, by Lemma 3.2, we obtain

$$\begin{aligned}
-\int_\Omega g(x)\Delta_k^{-h}\varphi(x)\,\mathrm{d}x &= -\int_\Omega a_{ij}(x)D_i u(x) D_j(\Delta_k^{-h}\varphi(x))\,\mathrm{d}x \\
&= -\int_\Omega a_{ij}(x)D_i u(x)\Delta_k^{-h}(D_j\varphi)(x)\,\mathrm{d}x \\
&= \int_\Omega \Delta_k^h(a_{ij}D_i u)(x) D_j\varphi(x)\,\mathrm{d}x \\
&= \int_\Omega \big(a(x+he_k)\Delta_k^h D_i u(x) + D_i u(x)\Delta_k^h a_{ij}(x)\big)\, D_j\varphi(x)\,\mathrm{d}x \\
&= \int_\Omega \big([h\Delta_k^h a_{ij}(x) + a_{ij}(x)]\Delta_k^h D_i u(x) + D_i u(x)\Delta_k^h a_{ij}(x)\big) \\
&\qquad \cdot D_j\varphi(x)\,\mathrm{d}x.
\end{aligned} \tag{3.7}$$

By Lemma 3.6, if $u \in W^{1,2}(\Omega, \omega, v)$ we have

$$\|\Delta_k^h u\|_{L^2(\Omega',v)} \leq C\|D_k u\|_{L^2(\Omega,\omega)} = \tilde{C}, \qquad \forall\, \Omega' \subset\subset \Omega. \tag{3.8}$$

Since $u \in L^2(\Omega, v)$ and $v \in A_2$ (see Remark 2.3 and Remark 2.7), by Lemma 3.7 we have that

$$\|D_k u\|_{L^2(\Omega',v)} \leq \|D_k u\|_{L^2(\Omega,v)} \leq \tilde{C} = C\|D_k u\|_{L^2(\Omega,\omega)}, \quad \forall\, \Omega' \subset\subset \Omega. \tag{3.9}$$

Consequently, in (3.7), we obtain

$$\begin{aligned}
&\int_\Omega a_{ij}(x) D_i(\Delta_k^h u(x)) D_j\varphi(x)\,\mathrm{d}x \\
&\quad = -\int_\Omega g(x)\Delta_k^{-h}\varphi(x)\,\mathrm{d}x - \int_\Omega \Delta_k^h a_{ij}(x) D_i u(x) D_j\varphi(x)\,\mathrm{d}x \\
&\qquad - \int_\Omega h\Delta_k^h a_{ij}(x)\Delta_k^h D_i u(x) D_j\varphi(x)\,\mathrm{d}x \\
&\quad \leq \int_\Omega |g(x)|\left|\Delta_k^{-h}\varphi(x)\right|\,\mathrm{d}x + \int_\Omega \left|\Delta_k^h a_{ij}(x)\right| |D_i u(x)|\, |D_j\varphi(x)|\,\mathrm{d}x \\
&\qquad + |h|\int_\Omega \left|\Delta_k^h a_{ij}(x)\right| \left|\Delta_k^h D_i u(x)\right| |D_j\varphi(x)|\,\mathrm{d}x
\end{aligned} \tag{3.10}$$

We have, by Lemma 3.6,

$$\begin{aligned}
\int_\Omega |g(x)|\left|\Delta_k^{-h}\varphi(x)\right|\,\mathrm{d}x &= \int_\Omega \left(\frac{|g(x)|}{v(x)}\right) v^{1/2}(x)\left|\Delta_k^{-h}\varphi(x)\right| v^{1/2}(x)\,\mathrm{d}x \\
&\leq \left\|\frac{g}{v}\right\|_{L^2(\Omega,v)} \|\Delta_k^{-h}\varphi\|_{L^2(\operatorname{supp}\varphi,v)} \\
&\leq C\left\|\frac{g}{v}\right\|_{L^2(\Omega,v)} \|D_k\varphi\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

And, using (3.9), we obtain

$$\begin{aligned}\int_\Omega \left|\Delta_k^h a_{ij}(x)\right| |D_i u(x)|\, |D_j\varphi(x)|\,\mathrm{d}x &\\ &\le C_1 \int_{\operatorname{supp}\varphi} v(x)\,|D_i u(x)|\,|D_j\varphi(x)|\,\mathrm{d}x\\ &\le C_1 \int_{\operatorname{supp}\varphi} |D_i u(x)| v^{1/2}(x) |D_j\varphi(x)| v^{1/2}(x)\,\mathrm{d}x\\ &\le C_1 C^2 \left(\int_\Omega |D_i u(x)|^2\omega(x)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |D_i\varphi(x)|^2\omega(x)\,\mathrm{d}x\right)^{1/2}\end{aligned}$$

Hence, in (3.10), we obtain

$$\begin{aligned}\int_\Omega a_{ij}(x) D_i(\Delta_k^h u(x)) D_j\varphi(x)\,\mathrm{d}x &\\ &\le C\left(\|u\|_{W^{1,2}(\Omega,\omega,v)} + \left\|\frac{g}{v}\right\|_{L^2(\Omega,\omega)}\right)\|D\varphi\|_{L^2(\Omega,\omega)} \qquad (3.11)\\ &\quad + C_1|h|\int_\Omega v(x)|\Delta_k^h D_i u(x)||D_j\varphi(x)|\,\mathrm{d}x\end{aligned}$$

Let $\Omega' \subset\subset \Omega$. To proceed further let us take a function $\eta \in C_0^\infty(\Omega)$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ in Ω' and with $\|D\eta\|_\infty \le 2/d'$, where $d' = \operatorname{dist}(\Omega', \partial\Omega)$ and set $\varphi = \eta^2 \Delta_k^h u$ (with $|2h| < \operatorname{dist}(\operatorname{supp}(\eta), \partial\Omega)$). We have

$$D_j\varphi = 2\eta D_j\eta \Delta_k^h u + \eta^2 D_j(\Delta_k^h u).$$

We denote by $a = \|u\|_{W^{1,2}(\Omega,\omega,v)} + \left\|\frac{g}{v}\right\|_{L^2(\Omega,v)}$. In (3.11) we obtain

$$\begin{aligned}&\int_\Omega \langle A\eta D(\Delta_k^h u), \eta D(\Delta_k^h u)\rangle\,\mathrm{d}x\\ &= \int_\Omega a_{ij}(x)[\eta(x) D_i(\Delta_k^h u(x))][\eta(x) D_j(\Delta_k^h u(x))]\,\mathrm{d}x\\ &\le C\left(\|u\|_{W^{1,2}(\Omega,\omega,v)} + \left\|\frac{g}{v}\right\|_{L^2(\Omega,v)}\right)\|2\eta D_j\eta\Delta_k^h u\\ &\quad + \eta^2\Delta_k^h(D_j u)\|_{L^2(\Omega,\omega)} + 2\int_\Omega |a_{ij}(x)||\eta D_i\Delta_k^h u||D_j\eta\Delta_k^h u|\,\mathrm{d}x\\ &\quad + C_1|h|\int_\Omega v|\Delta_k^h D_i u||2\eta D_j\eta\Delta_k^h u + \eta^2 D_j\Delta_k^h u|\,\mathrm{d}x\\ &\le Ca\left(2\|\eta D_j\eta\Delta_k^h u\|_{L^2(\operatorname{supp}\eta,\omega)} + \|\eta D_j(\Delta_k^h u)\|_{L^2(\operatorname{supp}\eta,\omega)}\right)\\ &\quad + 2I_2 + C_1|h|I_3\\ &\le Ca\left(\|D_j\eta\|_\infty\|\Delta_k^h u\|_{L^2(\operatorname{supp}\eta,\omega)} + \|\eta D_j\Delta_k^h u\|_{L^2(\operatorname{supp}\eta,\omega)}\right)\\ &\quad + 2I_2 + C_1|h|I_3\\ &\le Ca\|D_j\eta\|_\infty\|D_k u\|_{L^2(\Omega,\omega)} + Ca\|\eta D_j\Delta_k^h u\|_{L^2(\Omega,\omega)} + 2I_2 + C_1|h|I_3,\end{aligned}$$

i. e.

$$\int_\Omega \langle A\eta D(\Delta_k^h u),\eta D(\Delta_k^h u)\rangle\,\mathrm{d}x \le C\ \ a^2\|D_j\eta\|_\infty + Ca\|\eta D_j\Delta_k^h u\|_{L^2(\Omega,\omega)} + 2I_2 + C_1|h|I_3. \tag{3.12}$$

Let us estimate the integrals I_2 and I_3. By (1.2), we have that $|a_{ij}(x)| \le Cv(x)$ a.e. in Ω. Using (3.8), (3.9) and Remark 3.8, we obtain

$$\begin{aligned}
I_2 = &\int_\Omega |a_{ij}||\eta D_i\Delta_k^h u||D_j\eta\Delta_k^h u|\,\mathrm{d}x\\
\le\ & C\int_{\mathrm{supp}\eta} v|\eta D_i\Delta_k^h u||D_j\eta\Delta_k^h u|\,\mathrm{d}x\\
\le\ & C\|\eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\|D_j\eta\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\\
\le\ & C\|D_i(\eta\Delta_k^h u) - D_i\eta\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\|D_j\eta\|_\infty\|\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\\
\le\ & C\left(C_M\|D_i\eta\Delta_k^h u + \eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,\omega)} + C\|u\|_{W^{1,2}(\Omega,\omega,v)}\right)\\
& \cdot\|u\|_{W^{1,2}(\Omega,\omega,v)}\\
\le\ & C\left(\|D_k u\|_{L^2(\Omega,\omega)} + \|\eta D_i\Delta_k^h u\|_{L^2(\Omega,\omega)} + \|u\|_{W^{1,2}(\Omega,\omega,v)}\right)\\
& \cdot\|u\|_{W^{1,2}(\Omega,\omega,v)}\\
\le\ & C\left(\|u\|_{W^{1,2}(\Omega,\omega,v)} + \|\eta D_i\Delta_k^h u\|_{L^2(\Omega,\omega)}\right)\|u\|_{W^{1,2}(\Omega,\omega,v)}\\
\le\ & C\|u\|^2_{W^{1,2}(\Omega,\omega,v)} + C\|\eta D_i\Delta_k^h u\|_{L^2(\Omega,\omega)}\|u\|_{W^{1,2}(\Omega,\omega,v)}\\
\le\ & Ca^2 + Ca\|\eta D_j\Delta_k^h u\|_{L^2(\Omega,\omega)}.
\end{aligned}\tag{3.13}$$

We also have,

$$\begin{aligned}
I_3 = &\int_\Omega v|\Delta_k^h D_i u||2\eta D_j\eta\Delta_k^h u + \eta^2 D_j\Delta_k^h u|\,\mathrm{d}x\\
\le\ & 2\int_{\mathrm{supp}\eta} v|\Delta_k^h D_i u||\eta D_j\eta\Delta_k^h u|\,\mathrm{d}x + \int_{\mathrm{supp}\eta} v|\Delta_k^h D_i u||\eta^2 D_j\Delta_k^h u|\,\mathrm{d}x\\
=\ & 2\int_{\mathrm{supp}\eta} v|\eta\Delta_k^h D_i u||D_j\eta\Delta_k^h u|\,\mathrm{d}x + \int_{\mathrm{supp}\eta} v|\eta D_i\Delta_k^h u||\eta D_j\Delta_k^h u|\mathrm{d}x\\
\le\ & 2\|\eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\|D_j\eta\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\\
& + \|\eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\|\eta D_j\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}.
\end{aligned}\tag{3.14}$$

Using (3.8) and (3.9), we obtain

$$\begin{aligned}
\|D_j\eta\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)} &\le \|D_j\eta\|_\infty\|\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\\
&\le C\|D_j\eta\|_\infty\|D_k u\|_{L^2(\Omega,\omega)}\\
&\le C\|D_j\eta\|_\infty\|u\|_{W^{1,2}(\Omega,\omega,v)}\\
&\le Ca\|D_j\eta\|_\infty.
\end{aligned}\tag{3.15}$$

And we also have,

$$\begin{aligned}
\|\eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)} &= \|D_i(\eta\Delta_k^h u)-\Delta_k^h u D_i\eta\|_{L^2(\mathrm{supp}\eta,v)}\\
&\le \|D_i(\eta\Delta_k^h u)\|_{L^2(\mathrm{supp}\eta,v)}+\|\Delta_k^h u D_i\eta\|_{L^2(\mathrm{supp}\eta,v)}\\
&\le C_M\|D_i(\eta\Delta_k^h u)\|_{L^2(\mathrm{supp}\eta,\omega)}+\|D_i\eta\|_\infty\|\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,v)}\\
&\le C_M\|D_i\eta\Delta_k^h u+\eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,\omega)}\\
&\quad + C_M\|D_i\eta\|_\infty\|D_k u\|_{L^2(\Omega,\omega)}\\
&\le Ca\|D_j\eta\|_\infty + C_M\|\eta D_i\Delta_k^h u\|_{L^2(\mathrm{supp}\eta,\omega)}. \qquad (3.16)
\end{aligned}$$

By condition (1.2) we have,

$$\begin{aligned}
\int_\Omega a_{ij}(x)[\eta(x)D_j(\Delta_k^h u(x))][\eta(x)D_i(\Delta_k^h u(x))]\,\mathrm{d}x &\ge \int_\Omega |\eta(x)D(\Delta_k^h u(x))|^2\omega(x)\,\mathrm{d}x\\
&= \int_\Omega |\eta(x)\Delta_k^h(Du(x))|^2\omega(x)\,\mathrm{d}x. \qquad (3.17)
\end{aligned}$$

We denote by $b=\|\eta D(\Delta_k^h u)\|_{L^2(\Omega,\omega)}$. By (3.12), (3.13), (3.14), (3.15), (3.16) and (3.17), and using Young's inequality, we obtain

$$\begin{aligned}
b^2 &\le Ca^2+Cab+C|h|a^2+Cab|h|+C|h|b^2\\
&\le Ca^2+C\frac{\varepsilon^{-2}}{2}a^2+C|h|a^2+C\frac{\varepsilon^2}{2}b^2+C\frac{\varepsilon^{-2}}{2}a^2+C\frac{\varepsilon^2}{2}b^2|h|^2+C|h|b^2\\
&= \left(C+C\varepsilon^{-2}+C|h|\right)a^2+\left(C\frac{\varepsilon^2}{2}+C\frac{\varepsilon^2}{2}|h|^2+C|h|\right)b^2.
\end{aligned}$$

Choose $\varepsilon>0$ and h such that

$$\frac{C\varepsilon^2}{2}+\frac{C\varepsilon^2}{2}|h|^2+C|h|\le\frac{1}{2},$$

we obtain,

$$\int_\Omega |\eta\Delta_k^h Du|^2\omega\,\mathrm{d}x \le \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega,v)}+\left\|\frac{g}{v}\right\|_{L^2(\Omega,v)}\right)^2.$$

Using $\eta\equiv 1$ in Ω', we have

$$\int_{\Omega'} |\Delta_k^h Du|^2\omega\,\mathrm{d}x \le \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega,v)}+\left\|\frac{g}{v}\right\|_{L^2(\Omega,v)}\right)^2.$$

Then we conclude that

$$\begin{aligned}
\|D_j(\Delta_k^h u)\|^2_{L^2(\Omega',\omega)} = \|\Delta_k^h D_j u\|^2_{L^2(\Omega',\omega)} &\le \|\Delta_k^h Du\|^2_{L^2(\Omega',\omega)}\\
&\le \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega,v)}+\left\|\frac{g}{v}\right\|_{L^2(\Omega,v)}\right)^2.
\end{aligned}$$

By Lemma 3.7, we have there exists $D_{jk}u$ and

$$\|D_{jk}u\|_{L^2(\Omega',\omega)} \le \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega,v)}+\left\|\frac{g}{v}\right\|_{L^2(\Omega,v)}\right).$$

Therefore $u\in W^{2,2}(\Omega',\omega,v)$, $\forall\,\Omega'\subset\subset\Omega$. $\square$

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