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GENERALIZED DIEUDONNÉ AND HONDA CRITERIA FOR PRIMARY ABELIAN GROUPS

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Dedicated to the occasion of the 70th anniversary of Professor Tibor Katriňák

Abstract. We prove two theorems of the type of Dieudonné for the classes of p-torsion n-Σ-groups and weakly n-summable groups, respectively. These claims extend the classical Dieudonné criterion (Portugal. Math., 1952) and Honda criterion (Comment. Math. Univ. St. Pauli, 1964) as well as generalize recent results of ours in (Acta Math. Univ. Ostrav., 2006; 2007) and (Alg. Colloq., 2008).

Introduction and Principal Known Facts

Throughout the rest of this paper, let it be agreed that all groups taken into consideration are assumed additive p-primary abelian groups for an arbitrary but fixed prime p. For such a group A and any ordinal α the p^α-th power $p^\alpha A$ of A is defined inductively as follows: $pA = \{pa|a \in A\}$, $p^\alpha A = p(p^{\alpha-1}A)$ if $\alpha - 1$ exists and $p^\alpha A = \cap_{\beta<\alpha} p^\beta A$ otherwise. Following [**9**], the subgroup N of A is called nice in A if $p^\gamma(A/N) = (p^\gamma A + N)/N$ for each ordinal number γ. In terms of intersections, this equality is obviously tantamount to the equality $\cap_{\delta<\tau}(N + p^\delta A) = N + p^\tau A$ for every limit ordinal τ.

The explicitly unexplained terminology and the unstated notation are standard and are the same as those either in [**9**] or in [**2**].

As for the known facts on the subject explored, the first important result is the criterion of Dieudonné [**8**] which refines the classical Kulikov's criterion for direct sums of cyclic groups [**12**] in a very convenient form for applications.

Theorem 1. (Dieudonné, 1952) *Let A be a group with a subgroup G such that A/G is a direct sum of cyclic groups. Then A is a direct sum of cyclic groups $\iff$ $G = \cup_{n<\omega} G_n, G_n \subseteq G_{n+1} \leq G$, $\forall\ n \geq 1 : G_n \cap p^n A = 0$.*

Note that such a subgroup G is of necessity nice in A.

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We have further strengthened in [**2**] the previous statement to the class of σ-summable groups which properly contains the class of direct sums of cyclic groups.

Theorem 2. (Danchev, 2005) *Let A be a group with a nice subgroup G such that A/G is σ-summable. Then A is σ-summable $\iff$ $G = \cup_{n<\omega} G_n$, $G_n \subseteq G_{n+1} \leq G$, $\forall n \geq 1$, $\exists \alpha_n < \text{length}(A) : G_n \cap p^{\alpha_n} A = 0$.*

In particular, if G is a σ-summable balanced subgroup of A so that A/G is σ-summable, then A is σ-summable.

For that extension but in terms of valuated groups we refer the reader to [**3**].

Furthermore, we improved in [**7**] (see [**6**] too) the Dieudonné assertion for the classes of summable groups and Σ-groups, respectively; the variant for valuated groups was considered in [**5**] and [**4**], respectively.

Theorem 3. (Danchev, 2008) *Let A be a group of countable length with a nice subgroup G equipped with a valuation produced by the restricted height valuation on A such that A/G is summable. Then A is summable $\iff$ $G[p] = \cup_{n<\omega} G_n$, $G_n \subseteq G_{n+1} \leq G$, $\forall n \geq 1$, G_n are height-finite in A.*

In particular, if G is a balanced subgroup of A so that A/G is summable of countable length, then A is summable of countable length if and only if G is summable of countable length.

Theorem 4. (Danchev, 2008) *Let A be a group with a nice subgroup G such that $G \cap p^{\omega+1} A = p^{\omega+1} G$ and A/G is a Σ-group. Then A is a Σ-group $\iff$ $G[p] = \cup_{n<\omega} G_n$, $G_n \subseteq G_{n+1} \leq G$, $\forall n \geq 1$, G_n are with a finite number of finite heights as computed in A, that is, $G_n \cap p^n A \subseteq p^\omega A$.*

In particular, if G is a balanced subgroup of A so that A/G is a Σ-group, then A is a Σ-group if and only if G is a Σ-group.

The goal of the present paper is to enlarge the last two affirmations, namely Theorems 3 and 4, to new classes of abelian groups called weakly n-summable groups and n-Σ-groups, respectively. However, the proofs are based on developments of our ideas given in the aforementioned articles [**5**] and [**4**].

The Main Results

1. Generalized Dieudonné and Honda criteria for weakly n-summable groups

A group C of countable length is said to be *a weakly n-summable group* provided that $C[p^n] = \cup_{i<\omega} C_i, C_i \subseteq C_{i+1} \leq C[p^n]$ and all C_i are with finite height spectrum in C. When $n = 1$ this is precisely the classical criterion for summability of groups with countable lengths due to Honda [**11**] (see [**9**], v. II, p. 123, Theorem 84.1, too). Thus weakly 1-summable groups of countable length correspond with summable groups in the sense of Honda. Besides, each weakly n-summable group is a summable group while the converse implication fails for $n \geq 2$. There exist many interesting properties of this class of groups some of which are similar to the summable ones and some of which are specific. However, this is the theme of another research article.

We are now ready to prove our first main result.

Theorem 1.1. *Suppose the group A possesses a nice subgroup G of countable length equipped with a valuation produced by the restricted height valuation on A such that A/G is a weakly n-summable group of countable length. Then A is a weakly n-summable group of countable length if and only if $G[p^n] = \cup_{i<\omega} G_i, G_i \subseteq G_{i+1} \leq G[p^n]$ and all G_i are height-finite in A.*

Proof. "$\Rightarrow$". Write $A[p^n] = \cup_{i<\omega} A_i$, $A_i \subseteq A_{i+1} \leq A[p^n]$ and all A_i are height-finite in A. Consequently, $G[p^n] = \cup_{i<\omega} G_i$ by setting $G_i = A_i \cap G$ and clearly $G_i \subseteq G_{i+1} \leq G[p^n]$ with all G_i height-finite in A.

"$\Leftarrow$". Since A/G is of countable length, there is an ordinal α with the property that $p^\alpha(A/G) = (p^\alpha A + G)/G = 0$, whence $p^\alpha A \subseteq G$. Moreover, since G is of countable length, there exists an ordinal β so that $p^\beta G = 0$. Hence $p^{\alpha+\beta} A = 0$ and, since $\alpha + \beta$ is also countable, we have reached the conclusion that A is with countable length. Next, write $(A/G)[p^n] = \cup_{i<\omega}(B_i/G)$, where $B_i \subseteq B_{i+1} \leq A$ with $p^n B_i \subseteq G$ and, for all indices i, B_i/G are height-finite in A/G. That is why

$$(B_i/G) \setminus \{G\} \subseteq [p^{\gamma_1}(A/G) \setminus p^{\gamma_1+1}(A/G)] \cup \cdots \cup [p^{\gamma_i}(A/G) \setminus p^{\gamma_i+1}(A/G)]$$

for some ordinals $\gamma_1, \cdots, \gamma_i$. On the other hand, we write down

$$G_i \setminus \{0\} \subseteq [p^{\delta_1} A \setminus p^{\delta_1+1} A] \cup \cdots \cup [p^{\delta_i} A \setminus p^{\delta_i+1} A]$$

for some ordinals $\delta_1, \cdots, \delta_i$. Since $(A[p^n] + G)/G \subseteq (A/G)[p^n]$, it is easily verified that

$$A[p^n] = \bigcup_{i<\omega} B_i[p^n] = \bigcup_{i<\omega} A_i$$

by substituting $A_i = B_i[p^n]$.

Now we choose an ascending family $(F_i)_{i<\omega}$ of subgroups of $A[p^n]$ such that $F_i \cap G = 0$ and such that $(F_i \oplus G)/G = (B_i/G) \cap [(A[p^n] + G)/G]$. With the classical modular law from ([**9**], v. I) at hand the last is equivalent to the equality $F_i \oplus G = B_i[p^n] + G$ where $F_i \leq B_i[p^n]$. We claim that $A[p^n] = \cup_{i<\omega}(F_i \oplus G_i)$. In order to check that, letting $a \in A[p^n]$ hence $a + G \in (A[p^n] + G)/G \subseteq (A/G)[p^n]$. Thus $a + G \in (B_l/G) \cap [(A[p^n] + G)/G]$ for some index l, whence by our choice we have $a + G \subseteq F_l \oplus G$. Thus, $a \in F_l \oplus G[p^n]$ and hence $a \in F_s \oplus G_s$ for some index s. So, the claim has sustained.

Furthermore, because of the niceness of G in A, we obtain

$$\begin{aligned}[(F_i \oplus G)/G] \setminus \{G\} \subseteq\ & (B_i/G) \setminus \{G\} \\ \subseteq\ & [((p^{\gamma_1} A + G)/G) \setminus ((p^{\gamma_1+1} A + G)/G)] \cup \cdots \\ & \cup [((p^{\gamma_i} A + G)/G) \setminus ((p^{\gamma_i+1} A + G)/G)]\end{aligned}$$

which is tantamount to

$$\begin{aligned}(F_i \oplus G) \setminus G &\subseteq [(p^{\gamma_1}A + G) \setminus (p^{\gamma_1+1}A + G)] \cup \cdots \cup [(p^{\gamma_i}A + G) \setminus (p^{\gamma_i+1}A + G)]\\ &\subseteq [(p^{\gamma_1}A \setminus p^{\gamma_1+1}A) + G] \cup \cdots \cup [(p^{\gamma_i}A \setminus p^{\gamma_i+1}A) + G]\\ &= [(p^{\gamma_1}A \setminus p^{\gamma_1+1}A) \cup \cdots \cup (p^{\gamma_i}A \setminus p^{\gamma_i+1}A)] + G.\end{aligned}$$

Therefore,

$$F_i \setminus G = F_i \setminus \{0\} \subseteq [(p^{\gamma_1}A \setminus p^{\gamma_1+1}A) \cup \cdots \cup (p^{\gamma_i}A \setminus p^{\gamma_i+1}A)] + G$$

and since $(p^\varepsilon A + G)[p^n] = (p^\varepsilon A)[p^n] + G[p^n]$ for each ordinal ε we have

$$F_i \setminus G \subseteq [(p^{\gamma_1}A \setminus p^{\gamma_1+1}A) \cup \cdots \cup (p^{\gamma_i}A \setminus p^{\gamma_i+1}A)] + G[p^n].$$

Next, we select an ascending sequence $(L_i)_{i<\omega}$ of subgroups of $A[p^n]$ so that $L_i \subseteq F_i$ for each index i with

$$\bigcup_{i<\omega} L_i = \bigcup_{i<\omega} F_i$$

and so that

$$L_i \setminus \{0\} \subseteq [(p^{\gamma_1}A \setminus p^{\gamma_1+1}A) \cup \cdots \cup (p^{\gamma_i}A \setminus p^{\gamma_i+1}A)] + G_i$$

with $(L_i \oplus G_i + p^{\tau+1}A) \cap G[p^n] \subseteq G_i$ for every ordinal $\tau \notin \{\gamma_1, \cdots, \gamma_i; \delta_1, \cdots, \delta_i\}$. It is plainly verified that $A[p^n] = \cup_{i<\omega}(L_i \oplus G_i)$ with $L_i \oplus G_i \subseteq L_{i+1} \oplus G_{i+1}$. What remains to prove is that every member $L_i \oplus G_i$ of the union is height-finite in A. Doing that, given $x \in L_i \oplus G_i$, hence $x = a_i + g_i$ where $a_i \in L_i$ and $g_i \in G_i$. Thus

$$a_i \in L_i \setminus \{0\} \subseteq F_i \setminus \{0\} \subseteq (p^{\gamma_1}A \setminus p^{\gamma_1+1}A) \cup \cdots \cup (p^{\gamma_i}A \setminus p^{\gamma_i+1}A)$$

and

$$g_i \in G_i \setminus \{0\} \subseteq (p^{\delta_1}A \setminus p^{\delta_1+1}A) \cup \cdots \cup (p^{\delta_i}A \setminus p^{\delta_i+1}A).$$

Note that when either $a_i = 0$ or $g_i = 0$, everything is done.

Besides, we see that if $\mathrm{height}_A(a_i) \neq \mathrm{height}_A(g_i)$, then

$$\mathrm{height}_A(x) = \min\{\mathrm{height}_A(a_i), \mathrm{height}_A(g_i)\} \in \{\gamma_1, \cdots, \gamma_i; \delta_1, \cdots, \delta_i\}.$$

Otherwise, if $\mathrm{height}_A(a_i) = \mathrm{height}_A(g_i)$, we assume in a way of contradiction that $(L_i \oplus G_i) \cap (p^\tau A \setminus p^{\tau+1}A) \neq \emptyset$ for some $\tau \notin \{\gamma_1, \cdots, \gamma_i; \delta_1, \cdots, \delta_i\}$, i.e. that $\mathrm{height}_A(x) = \tau$ for some x as above. But since

$$x + G \in [(B_i/G) \setminus \{G\}] \cap [(p^\tau A + G)/G] = [(B_i/G) \setminus \{G\}] \cap [p^\tau(A/G)],$$

it must be that

$$x + G \in p^{\tau+1}(A/G) = (p^{\tau+1}A + G)/G.$$

Hence

$$x \in (p^{\tau+1}A + G)[p^n] = (p^{\tau+1}A)[p^n] + G[p^n]$$

and so $x = a_\tau + g$ for some $a_\tau \in (p^{\tau+1}A)[p^n]$ and $g \in G[p^n] \setminus \{0\}$. Thus

$$x - a_\tau = g \in (L_i \oplus G_i + p^{\tau+1}A) \cap G[p^n] \subseteq G_i$$

and $0 \neq g \in G_i$ with $\mathrm{height}_A(g) = \tau$ since

$$\mathrm{height}_A(x) = \tau \qquad \text{and} \qquad \mathrm{height}_A(a_\tau) \geq \tau + 1 > \tau.$$

Consequently,
$$(G_i \setminus \{0\}) \cap (p^\tau A \setminus p^{\tau+1} A) \neq \emptyset$$
for τ as above, which is false.

We then finally conclude that
$$(L_i \oplus G_i) \setminus \{0\} \subseteq (p^{\gamma_1} A \setminus p^{\gamma_1+1} A) \cup \cdots \cup (p^{\gamma_i} A \setminus p^{\gamma_i+1} A) \\ \cup (p^{\delta_1} A \setminus p^{\delta_1+1} A) \cup \cdots \cup (p^{\delta_i} A \setminus p^{\delta_i+1} A),$$
whence they are height-finite in A, as wanted. □

The following consequence is immediate.

Corollary 1.2. *Suppose that G is a balanced subgroup of the group A such that A/G is a weakly n-summable group. Then A is a weakly n-summable group if and only if G is.*

2. Generalized Dieudonné and Honda criteria for n-Σ-groups

A group C is called *an n-Σ-group* provided that $C[p^n] = \cup_{i<\omega} C_i$, $C_i \subseteq C_{i+1} \leq C[p^n]$ and all C_i are with a finite number of finite heights as calculated in C, that is $C_i \cap p^i C \subseteq p^\omega C$. When $n = 1$ this is precisely the criterion for Σ-groups due to the author [**1**]. Thereby 1-Σ-groups correspond with Σ-groups in the classical sense. Moreover, each n-Σ-group is a Σ-group whereas the converse implication fails for $n \geq 2$. However, it is well-known by [**13**] that Σ-groups with countable first Ulm subgroups are of necessity with first Ulm factor which is a direct sum of cyclic groups, hence they are n-Σ-groups for any $n \in \mathbb{N}$.

Clearly, each weakly n-summable group (of countable length), defined as in Section 1, is an n-Σ-group. It can be showed via a concrete counterexample that the converse claim is wrong for groups with lengths $\geq \omega \cdot 2$. Nevertheless, we shall illustrate below that for lengths $< \omega \cdot 2$ these two classes of groups do coincide (for $n = 1$, the interested reader can see [**1**]).

Proposition 2.1. *Let $n \geq 1$. Then every n-Σ-group of length $< \omega \cdot 2$ is weakly n-summable and vice versa.*

Proof. For an n-Σ-group C we write $C[p^n] = \cup_{i<\omega} C_i, C_i \subseteq C_{i+1} \leq C[p^n]$ and, $\forall\ i < \omega$, $C_i \cap p^i C \subseteq p^\omega C$. Since length$(p^\omega C) < \omega$, i.e., $p^\omega C$ is bounded, there is some $k \geq 1$ such that $p^{\omega+k} C = 0$. Consequently, it is readily observed that C_i has only finitely many height values in C or, in other words, C_i are height-finite in C. That is why, by definition, C is weakly n-summable, as asserted. □

Now, we shall proceed to prove our second main result.

Theorem 2.2. *Suppose the group A possesses a nice subgroup G such that $G \cap p^{\omega+n} A = p^{\omega+n} G$ and A/G is an n-Σ-group. Then A is an n-Σ-group if and only if $G[p^n] = \cup_{i<\omega} G_i, G_i \subseteq G_{i+1} \leq G[p^n]$ and, for all $i < \omega$, $G_i \cap p^i A \subseteq p^\omega A$.*

Proof. "$\Rightarrow$". Write $A[p^n] = \cup_{i<\omega} A_i$, $A_i \subseteq A_{i+1} \leq A[p^n]$ and, $\forall\ i < \omega$, $A_i \cap p^i A \subseteq p^\omega A$. Hence $G[p^n] = \cup_{i<\omega} G_i$, where $G_i = G \cap A_i$. Since $G_i \subseteq G_{i+1} \leq G[p^n]$ and, $\forall\ i < \omega$, $G_i \cap p^i A \subseteq A_i \cap p^i A \subseteq p^\omega A$, the proof is finished.

"$\Leftarrow$". Write

$$(A/G)[p^n] = \bigcup_{i<\omega} (B_i/G)$$

where $B_i \subseteq B_{i+1} \leq A$ with $p^n B_i \subseteq G$ and, $\forall\ i < \omega$,

$$(B_i/G) \bigcap p^i(A/G) \subseteq p^\omega(A/G).$$

The last inclusion, in view of the classical modular law from ([**9**, v. I]), is tantamount to the relation $B_i \cap p^i A \subseteq p^\omega A + G$. Furthermore, because it is obvious that $(A[p^n] + G)/G \subseteq (A/G)[p^n]$, we obtain that

$$A[p^n] = \bigcup_{i<\omega} B_i[p^n] = \bigcup_{i<\omega} A_i,$$

by putting $A_i = B_i[p^n]$. Next, we select an ascending family $(K_i)_{i<\omega}$ of subgroups of $A[p^n]$ such that $K_i \cap G = 0$ and such that

$$(K_i \oplus G)/G = (B_i/G) \cap [(A[p^n] + G)/G].$$

Applying the classical modular law (see cf. [**9**]), the last equality is equivalent to

$$K_i \oplus G = B_i[p^n] + G$$

where $K_i \subseteq B_i[p^n]$.

We claim that

$$A[p^n] = \bigcup_{i<\omega} (K_i \oplus G_i).$$

In order to check this, letting $a \in A[p^n]$ hence

$$a + G \in (A[p^n] + G)/G \subseteq (A/G)[p^n]$$

and even more

$$a + G \in (B_t/G) \cap [(A[p^n] + G)/G]$$

for some index t. Therefore, via our selection, $a + G \subseteq K_t \oplus G$ and $a \in K_t \oplus G$. Finally, $a \in K_t \oplus G[p^n]$, whence $a \in K_m \oplus G_m$ for some index m. This substantiates our claim.

Furthermore, we choose an ascending tower $(P_i)_{i<\omega}$ of subgroups of $A[p^n]$ so that for all indices i the following dependencies hold: $P_i \subseteq K_i$ with $\cup_{i<\omega} P_i = \cup_{i<\omega} K_i$ and so that $(P_i \oplus G_i) \cap p^i A \subseteq p^\omega A + G_i$. The choice is possible because of the inclusions

$$K_i \oplus G_i \subseteq K_i \oplus G \subseteq B_i$$

with

$$(K_i \oplus G_i) \cap p^i A \subseteq B_i \cap p^i A \subseteq (p^\omega A + G)[p^n] = (p^\omega A)[p^n] + G[p^n] \subseteq p^\omega A + G[p^n].$$

It is straightforward that

$$A[p^n] = \bigcup_{i<\omega} (P_i \oplus G_i)$$

where $P_i \oplus G_i \subseteq P_{i+1} \oplus G_{i+1}$. What suffices to compute is that $(P_i \oplus G_i) \cap p^i A \subseteq p^\omega A$. In fact, with the aid of the classical modular law (e.g., [**9**], v. I), we have

$$(P_i \oplus G_i) \cap p^i A \subseteq (p^\omega A + G_i) \cap p^i A = p^\omega A + (G_i \cap p^i A) = p^\omega A$$

since by hypothesis

$$G_i \cap p^i A \subseteq p^\omega A.$$

□

The following consequence is direct.

Corollary 2.3. *Suppose that G is a balanced subgroup of the group A such that A/G is an n-Σ-group. Then A is an n-Σ-group if and only if G is an n-Σ-group.*

Remarks. In [**3**], [**4**] and [**5**] the subgroup is furnished with the restricted height valuation on the whole group. So, the height condition $G \cap p^{\omega+n}A = p^{\omega+n}G$ is automatically satisfied. Certainly, in [**4**] the Theorem can be equivalently formulated as follows: "Suppose G is a group with a nice valuated subgroup A endowed with the valuation produced by the restricted height valuation of G. If G/A is a Σ-group, then G is a Σ-group if and only if A is a Σ-group", which formulation is identical to those from [**3**] and [**5**]. Moreover, in [**7**] it was given a different proof when $n = 1$ but again with the usage of that height restriction on G and A, namely $G \cap p^{\omega+1}A = p^{\omega+1}G$.

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