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ISOLATED SUBSEMIGROUPS IN THE VARIANTS OF $\mathcal{T}_n$

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Abstract. We classify all isolated, completely isolated and convex subsemigroups in the semigroup $\mathcal{T}_n$ of all transformations of an n-element set, considered as the semigroup with respect to a sandwich operation.

1. Introduction and description of the results

For a semigroup, $(S, \cdot)$, and an element, $a \in S$, one can consider the *variant* $(S, *_a)$ of S, for which the *deformed* or *sandwich* multiplication $*_a : S \times S \to S$ with the sandwich element a is defined as follows: $x *_a y = xay$ for all $x, y \in S$. This construction was proposed in [**10**] and studied later on for various classes of semigroups by several authors, see for example [**1, 2, 6, 7, 8, 11, 14, 15, 16, 17, 18, 9**] and others. One of the main motivations for this study is the fact that usually the variants of the semigroups have more interesting and richer structure than the original semigroups.

The basic question, which one faces in the study of variants of the semigroup S is: *For which $a, b \in S$ are the variants $(S, *_a)$ and $(S, *_b)$ isomorphic?* For such classical transformation semigroups on a given set, N, as the full symmetric inverse semigroup $\mathcal{IS}(N)$, the full transformation semigroup $\mathcal{T}(N)$, and the semigroup $\mathcal{PT}(N)$ of all partial transformations, the isomorphism criteria for their variants were obtained in [**15, 16, 18**]. In particular, in all the above cases it was shown that (up to isomorphism) one can always assume that a is an idempotent of S.

A natural class of subsemigroups of the given semigroup S is formed by the so-called *isolated* subsemigroups. A subsemigroup, T, of S is called *isolated* provided that the following condition is satisfied: $x^n \in T$ for some $n \in \mathbb{N}$ implies $x \in T$ for all $x \in S$, see for example [**13**]. This class contains various subclasses, for example the class of completely isolated subsemigroups and some others, see Section 2 or [**13**] for details. Isolated subsemigroups of some classical semigroups are known. For example, a complete classification of isolated and completely isolated subsemigroups of $\mathcal{IS}(N)$ in the case of finite N was obtained in [**4**]. For the factor power $\mathcal{FP}(\mathcal{S}_n)$ of the symmetric group $\mathcal{S}_n$ the same problem was solved in [**5**], and for the Brauer semigroup $\mathfrak{B}_n$ in [**12**].

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In the present paper we present a complete description of the isolated subsemigroups for all (up to isomorphism) variants of the full transformation semigroup $\mathcal{T}(N)$ in the case of a finite set N. We also classify the classes of completely isolated, right convex, left convex and convex subsemigroups, see Section 2 for the corresponding definitions.

Let us now briefly describe our approach, results and the structure of the paper. The results of [**4, 5, 12**] suggest to split the study of the isolated subsemigroups into three different cases. In [**4, 5, 12**] these are: the case of invertible elements, the case of maximal subgroups corresponding to maximal non-invertible idempotents, and the rest. For the variants of semigroups the above cases should first be adjusted as all non-trivial variants never containing any unit and, as a consequence, any invertible elements. However, in the case of transformation semigroups there is a natural "substitution" for this. For transformation semigroups there is a natural notion of the rank (the cardinality of the image) and stable rank (the rank of the unique idempotent in the cyclic subsemigroup, generated by the given element). Using these notions we split our study into the following three cases: the case of the maximal possible stable rank l (which equals the rank of the sandwich element), the case of the stable rank $l-1$, and the rest. Even in the first case we obtain quite a striking difference from the classical case. In the classical case all invertible elements obviously form a minimal isolated subsemigroup. In contrast to this, for every variant of $\mathcal{T}_n$ we show in Subsection 4.1 that the set $\mathcal{T}_n^{(l)}$ of all elements of stable rank l is a subsemigroup, and that there is a natural congruence on $\mathcal{T}_n^{(l)}$, the quotient modulo which is a rectangular band. Moreover, the natural projection onto the quotient induces a bijection between the set of all (isolated) subsemigroups of this rectangular band (which is relatively easy to describe) and the set of all isolated subsemigroups of $\mathcal{T}_n^{(l)}$.

Not completely unexpected, but it turned out that the most interesting and the most complicated case of our study is the case of the elements of stable rank $l-1$ (in the classical case certain subsemigroups, consisting of elements of rank $l-1$, make the difference between the classes of isolated and completely isolated subsemigroups). Such subsemigroups are studied in Subsection 4.2, which is the heart of our paper. We show that the isolated subsemigroups of this form split into three classes. In every class the semigroups are indexed by several discrete parameters which are either elements or subsets of some (not very complicated) sets. The structure of the simplest of these three classes is quite similar to the structure of isolated subsemigroups in the case of $\mathcal{T}_n^{(l)}$. Two other classes are, in some sense, "composed" of several copies of the first class, which leads to quite nice but non-trivial description of isolated subsemigroups in the case of stable rank $l-1$.

The last case, that is the case of stable rank at most $l-2$, is studied in Subsection 4.3. This case is relatively "poor" with respect to the structure of the isolated subsemigroups. In fact we show that if an isolated subsemigroup contains an element, whose stable rank does not exceed $l-2$, then it contains all elements, whose stable ranks do not exceed $l-1$ (and the last $l-1$ is not a typo, it is indeed $l-1$ and not $l-2$). In Subsection 4.4 we present our main theorem about the classification of all isolated subsemigroups in all variants of $\mathcal{T}_n$. In Section 5 we use this theorem to classify all completely isolated and convex subsemigroups. We finish the paper with deriving the corollaries from our results for the original semigroup $\mathcal{T}_n$ in Section 6.

The basic generalities on isolated subsemigroups are collected in Section 2. In Section 3 we fix the notation and collect some generalities on the semigroup $\mathcal{T}_n$ and its variants. We would like to emphasize once more that we always assume the sandwich element to be an idempotent, which we can do up to isomorphism of the sandwich semigroup $\mathcal{T}_n$. Among other basic facts, in Subsection 3.3 we describe all idempotents and the corresponding maximal subgroups in all variants of $\mathcal{T}_n$.

2. Generalities on isolated subsemigroups

In this section we collect some generalities on isolated and completely isolated subsemigroups, which we failed to find in the literature and which we think might be of independent interest. Some of these basic properties will be often used in the paper without reference.

Let S be a semigroup. A subsemigroup, $T \subset S$, is called

- *isolated* provided that for all $x \in S$ the condition $x^n \in T$ for some $n > 0$ implies $x \in T$;
- *completely isolated* provided that $xy \in T$ implies $x \in T$ or $y \in T$ for all $x, y \in S$;
- *right convex* provided that $xy \in T$ implies $y \in T$ for all $x, y \in S$;
- *left convex* provided that $xy \in T$ implies $x \in T$ for all $x, y \in S$;
- *convex* provided that $xy \in T$ implies $x \in T$ and $y \in T$ for all $x, y \in S$.

We refer the reader to [**3, 13**] for details on these definitions. Denote by $\mathfrak{I}(S)$, $\mathfrak{CI}(S)$, $\mathfrak{RC}(S)$, $\mathfrak{LC}(S)$ and $\mathfrak{C}(S)$, the sets of all isolated, completely isolated, right convex, left convex or convex subsemigroups of S respectively. For $X \subset S$ we denote by $\overline{X}$ the complement of X in S.

From the definition one immediately obtains the following properties (here everywhere T is assumed to be a subsemigroup of S):

(I) $\mathfrak{I}(S)$ is closed with respect to (non-empty) intersections, in particular, for every $x \in S$ there exists the unique isolated subsemigroup $\mathtt{I}(x)$ of S such that $x \in \mathtt{I}(x)$ and which is minimal with respect to this property.

(II) $T \in \mathfrak{I}(S)$ if and only if $\overline{T}$ is either empty or a (not necessarily disjoint) union of subsemigroups of S.

(III) $\mathfrak{CI}(S)$ is closed with respect to taking the (non-empty) complement in S.

(IV) $T \in \mathfrak{CI}(S)$ if and only if $\overline{T}$ is either empty or a subsemigroup of S.

(V) $T \in \mathfrak{RC}(S)$ if and only if $\overline{T}$ is either empty or a left ideal.

(VI) $T \in \mathfrak{LC}(S)$ if and only if $\overline{T}$ is either empty or a right ideal.

(VII) $T \in \mathfrak{C}(S)$ if and only if $\overline{T}$ is either empty or a (two-sided) ideal.

(VIII) If S_1 and S_2 are two semigroups and $f : S_1 \to S_2$ is a homomorphism then taking the pre-image (provided that it is non-empty) preserves the properties of being isolated, completely isolated, left convex, right convex and convex, that is, it induces a map from $\mathfrak{I}(S_2)$ to $\mathfrak{I}(S_1)$ and so on.

(IX)

$$\begin{tikzcd}
& \mathfrak{RC}(S) \arrow[dr, hook] & & \\
\mathfrak{C}(S) \arrow[ur, hook] \arrow[dr, hook] & & \mathfrak{CI}(S) \arrow[r, hook] & \mathfrak{I}(S) \\
& \mathfrak{LC}(S) \arrow[ur, hook] & &
\end{tikzcd}$$

We remark that for $T_1, T_2 \in \mathfrak{CI}(S)$ the subsemigroup $T_1 \cap T_2$ does not belong to $\mathfrak{CI}(S)$ in general. An example can be found in Section 5. From Section 5 it also follows that all the five classes above can be different. Now we would like to characterize some of the above subsemigroups in terms of certain homomorphisms and congruences on S.

Proposition 1. (a) *$T \in \mathfrak{C}(S)$ if and only if there exists $f : S \to (\mathbb{Z}_2, \cdot)$ such that $T = f^{-1}(1)$.*

(b) *Let $\mathfrak{C}(S) = \{S, T_1, \dots, T_k\}$. Then $\{X_1 \cap \dots \cap X_k \neq \emptyset : X_i = T_i$ or $X_i = \overline{T}_i\}$ defines a congruence, ρ, on S, the quotient modulo which embeds into $(\mathbb{Z}_2, \cdot)^k$.*

(c) *For $x \in S$ the congruence class ρ_x of ρ, containing x, is an isolated subsemigroup of S, in particular, the congruence ρ gives rise to a decomposition of S into a disjoint union of isolated subsemigroups.*

Proof. Let T be a proper convex subsemigroup of S. Consider the equivalence relation on S with two equivalence classes: T and $\overline{T}$. This relation is a congruence on S and the corresponding quotient is isomorphic to $(\mathbb{Z}_2, \cdot)$ (the isomorphism is given by $T \mapsto 1$, $\overline{T} \mapsto 0$). Conversely, note that $\{1\} \in \mathfrak{C}((\mathbb{Z}_2, \cdot))$. Hence if $f : S \to (\mathbb{Z}_2, \cdot)$ then (VIII) implies that $f^{-1}(1) \in \mathfrak{C}(S)$, which proves (a).

For $i = 1, \dots, k$ let $f_i : S \to \mathbb{Z}_2$ denote the homomorphism such that $f_i(T_i) = 1$ and $f_i(\overline{T}_i) = 0$. Then by (a) the kernel of the homomorphism $(f_1, \dots, f_k)$ is just ρ. This proves (b).

(c) follows from (I), (IX) and the fact that $T_i, \overline{T}_i \in \mathfrak{CI}(S)$ for all i. □

Proposition 2. *Let $\mathfrak{K}$ be one of the classes $\mathfrak{I}(S)$, $\mathfrak{CI}(S)$, $\mathfrak{RC}(S)$, $\mathfrak{LC}(S)$ or $\mathfrak{C}(S)$. If T is a subsemigroup of S and is a (not necessarily disjoint) union of subsemigroups from $\mathfrak{K}$, then $T \in \mathfrak{K}$.*

Proof. The result follows from the fact that the conditions on $\overline{T}$, which appear in (II), (IV), (V), (VI), and (VII), are closed with respect to intersections. □

We would like to remark that the union of isolated subsemigroups (or any other type of subsemigroups defined above) does not need to be a semigroup in general. One easily constructs examples using our results, presented in Section 4 and Section 5.

Denote by $E(S)$ the set of all idempotents of S and for $e \in E(S)$ define

$$\sqrt{e} = \{x \in S : x^m = e \text{ for some } m > 0\}.$$

Proposition 3. *If $\sqrt{e}$ is a subsemigroup of S then $\sqrt{e}$ is a minimal (with respect to inclusions) isolated subsemigroup of S, in particular, $\sqrt{e} = \mathtt{I}(x)$ for every $x \in \sqrt{e}$.*

Proof. Assume that $\sqrt{e}$ is a subsemigroup of S. If $y^m \in \sqrt{e}$ for some $m > 0$ then $(y^m)^k = e$ for some $k > 0$ and hence $y \in \sqrt{e}$ implying that $\sqrt{e}$ is isolated.

Let now $x \in \sqrt{e}$. Then $\mathtt{I}(x)$ contains all powers of x, in particular, it contains e. Since $\mathtt{I}(x)$ is isolated, it follows that $\mathtt{I}(x)$ must contain $\sqrt{e}$. However, as we have already shown, $\sqrt{e}$ is isolated itself. Hence $\sqrt{e} = \mathtt{I}(x)$ by the minimality of $\mathtt{I}(x)$. This completes the proof. □

3. $\mathcal{T}_n$ AND ITS VARIANTS

3.1. Notation. Throughout the paper we fix a positive integer n and set $N = \{1, 2, \dots, n\}$. We denote by $\mathcal{T}_n$ the semigroup of all maps $\beta : N \to N$ with respect to the composition $\cdot$ of maps from the left to the right (in contrast with the standard composition $\circ$ which works from the right to the left). We adopt the standard notation $\beta(x)$ for the value of β on the element $x \in N$. In particular, we always have $\beta\gamma(x) = \gamma(\beta(x))$ for all $\beta, \gamma \in \mathcal{T}_n$ and $x \in N$.

For $\beta \in \mathcal{T}_n$ we denote by $\mathrm{im}(\beta)$ the *image* of β, that is the set $\{\beta(x) : x \in N\}$, and define the *rank* of β via $\mathrm{rank}(\beta) = |\,\mathrm{im}(\beta)|$. Define also the equivalence relation Λ_β on N as follows: $x\Lambda_\beta y$, $x, y \in N$, if and only if $\beta(x) = \beta(y)$. Sometimes it will be convenient to consider Λ_β as an unordered (disjoint) union of equivalence classes and use the notation

$$\Lambda_\beta = B_1^\beta \cup B_2^\beta \cup \cdots \cup B_k^\beta$$

if $\mathrm{rank}(\beta) = k$. If the element β is clear from the context, we sometimes may skip it as the upper index. The relation Λ_β is called the *kernel* of β.

If $\beta \in \mathcal{T}_n$ and $N = N_1 \cup N_2 \cup \cdots \cup N_k$ is a decomposition of N into a disjoint union of subsets such that for all $i = 1, \dots, k$ we have $\beta(x) = \beta(y) = b_i$ for all $x, y \in N_i$ and for some $b_i \in N$, we will use the following notation for β:

$$\beta = \begin{pmatrix} N_1 & N_2 & \dots & N_k \\ b_1 & b_2 & \dots & b_k \end{pmatrix}.$$

Note that we do not require $\mathrm{rank}(\beta) = k$, which means that some of b_i's may coincide in general.

For $i \in N$ we denote by θ_i the element of $\mathcal{T}_n$ defined via $\theta_i(x) = i$ for all $x \in N$. Note that θ_i is a right zero for $\mathcal{T}_n$. For a set X we denote by $\mathcal{S}(X)$ the symmetric group on X.

3.2. Variants of $\mathcal{T}_n$ and their isomorphism. For $\alpha \in \mathcal{T}_n$ let $(\mathcal{T}_n, *_\alpha)$ be the *variant* of $\mathcal{T}_n$, that is the semigroup $\mathcal{T}_n$ with the *sandwich operation* $\beta *_\alpha \gamma = \beta\alpha\gamma$, $\beta, \gamma \in \mathcal{T}_n$, with respect to the *sandwich element* α. The following statement is proved in [**15, 16**]:

Theorem 4. *Let $\alpha_1, \alpha_2 \in \mathcal{T}_n$. Then the following statements are equivalent:*

1. $(\mathcal{T}_n, *_{\alpha_1}) \cong (\mathcal{T}_n, *_{\alpha_2})$.
2. $\alpha_1\sigma = \tau\alpha_2$ *for some* $\sigma, \tau \in \mathcal{S}(\mathcal{N})$.
3. *For every* $i = 1, \dots, n$ *the decompositions* Λ_{α_1} *and* Λ_{α_2} *contain the same number of blocks of cardinality* i.

From Theorem 4 it follows that every variant of $\mathcal{T}_n$ is isomorphic to a variant, the sandwich element of which is an idempotent. Hence from now on we fix $\alpha \in E(\mathcal{T}_n)$ of rank l. Set $* = *_\alpha$ and let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_l \\ a_1 & a_2 & \dots & a_l \end{pmatrix}.$$

Then $\Lambda_\alpha = A_1 \cup A_2 \cup \dots \cup A_l$, $\mathrm{im}(\alpha) = \{a_1, \dots, a_l\}$, and $a_i \in A_i$ for all i. We also set $A = \{a_1, \dots, a_l\}$.

Let $\beta \in \mathcal{T}_n$ and $\varepsilon \in E(\mathcal{T}_n, *)$ be the unique idempotent in the cyclic subsemigroup generated by β. We define the *stable rank* of β as $\mathrm{strk}(\beta) = \mathrm{rank}(\varepsilon)$, and set $\Lambda^{\mathrm{st}}_\beta = \Lambda_\varepsilon$, $\mathrm{im}^{\mathrm{st}}(\beta) = \mathrm{im}(\varepsilon)$. Note that $\Lambda^{\mathrm{st}}_\beta = \Lambda^{\mathrm{st}}_{\beta\alpha}$ and $\mathrm{im}^{\mathrm{st}}(\beta) = \mathrm{im}^{\mathrm{st}}(\alpha\beta)$ for all $\beta \in (\mathcal{T}_n, *)$. For $\beta \in \mathcal{T}_n$ and $m \in \mathbb{N}$ we set

$$\beta^m = \underbrace{\beta\beta\dots\beta}_{m \text{ times}} \qquad \text{and} \qquad \beta^{*m} = \underbrace{\beta * \beta * \dots * \beta}_{m \text{ times}}.$$

For $i = 1, \dots, l$ we denote by $\mathcal{T}_n^{(i)}$ the set of all elements in $(\mathcal{T}_n, *)$ of stable rank i.

For some other properties of $(\mathcal{T}_n, *)$, for example for the description of Green's relations or the automorphism group, we refer the reader to [**15, 17**].

3.3. Description of idempotents and maximal subgroups. To proceed with the study of isolated subsemigroups of $(\mathcal{T}_n, *)$ we need to describe the idempotents in this semigroup.

Theorem 5. (a) *Let*

$$\varepsilon = \begin{pmatrix} E_1 & E_2 & \dots & E_k \\ e_1 & e_2 & \dots & e_k \end{pmatrix} \in \mathcal{T}_n \tag{1}$$

be an element of rank k. Then $\varepsilon \in E(\mathcal{T}_n, *)$ if and only if there exists an injection, $f : \{1, \dots, k\} \to \{1, \dots, l\}$, such that the following two conditions are satisfied:

(i) $e_i \in A_{f(i)}$ *for all* $i = 1, \dots, k$,

(ii) $a_{f(i)} \in E_i$ *for all* $i = 1, \dots, k$.

(b)

$$|E(\mathcal{T}_n, *)| = \sum_{\emptyset \neq X \subset \{1,\dots,l\}} \left(\prod_{i \in X} |A_i| \right) \cdot |X|^{n-|X|}.$$

Proof. If $\varepsilon \in E(\mathcal{T}_n, *)$ then $\varepsilon = \varepsilon * \varepsilon = \varepsilon\alpha\varepsilon$. Therefore $\mathrm{rank}(\varepsilon) = \mathrm{rank}(\varepsilon\alpha)$ and hence all e_i's belong to different blocks of Λ_α. Define $f : \{1, \dots, k\} \to \{1, \dots, l\}$

via the requirement $e_i \in A_{f(i)}$ for all $i = 1, \dots, k$ and (ai) follows. On the other hand, let $\varepsilon(a_{f(i)}) = e_j$. Then

$$\varepsilon(a_{f(i)}) = \varepsilon\alpha\varepsilon(a_{f(i)}) = \alpha\varepsilon(e_j) = \varepsilon(a_{f(j)}),$$

which implies $f(i) = f(j)$ and thus $a_{f(i)} \in E_i$ for all $i = 1, \dots, k$ giving (aii). Conversely, if both (ai) and (aii) are satisfied, a direct calculation shows that $\varepsilon \in E(\mathcal{T}_n, *)$. This proves (a).

To prove (b) we count the number of idempotents, for which $\mathrm{im}(f) = X \subset \{1, \dots, l\}$, where f is taken from (a). If X is fixed, we should independently choose elements $e_i \in A_i$ for all $i \in X$, and then every element from the set $N \setminus \{a_{f(i)} : i \in X\}$ should be associated to some block of Λ_ε. The formula of (b) now follows by an application of the product rule and the sum rules. $\square$

The above description of idempotents allows us to describe the maximal subgroups in $(\mathcal{T}_n, *)$:

Corollary 6. *Let ε be an idempotent in $(\mathcal{T}_n, *)$ of rank k, defined via (1). Then the corresponding maximal subgroup $G(\varepsilon)$ of $(\mathcal{T}_n, *)$ is isomorphic to the symmetric group $\mathcal{S}(\{e_1, \dots, e_k\})$ and consists of all $\beta \in \mathcal{T}_n$ which satisfy the following two conditions:*

(i) $\Lambda_\beta = \Lambda_\varepsilon$,
(ii) $\mathrm{im}(\beta) = \mathrm{im}(\varepsilon)$.

Proof. $\beta \in G(\varepsilon)$ if and only if $\beta \in \sqrt{\varepsilon}$ and $\varepsilon * \beta = \beta * \varepsilon = \beta$. In particular, it follows that $\mathrm{rank}(\beta) = \mathrm{rank}(\varepsilon)$. For such β the condition (i) is equivalent to $\varepsilon * \beta = \beta$ and (ii) is equivalent to $\beta * \varepsilon = \beta$. This completes the proof. $\square$

3.4. Some homomorphisms from $(\mathcal{T}_n, *)$ to $\mathcal{T}_n$ and $\mathcal{T}(A)$. In this subsection we construct two homomorphisms from $(\mathcal{T}_n, *)$ to the usual $\mathcal{T}_n$ and an epimorphism to $\mathcal{T}_m$ for some $m \le n$. For every $\beta \in \mathcal{T}_n$ set

$$\varphi_l(\beta) = \alpha\beta, \qquad \varphi_r(\beta) = \beta\alpha.$$

Lemma 7. (a) *Both φ_l and φ_r are homomorphisms from $(\mathcal{T}_n, *)$ to $\mathcal{T}_n$.*
(b) *For every $\beta \in \mathcal{T}_n$ we have $\mathrm{im}^{\mathrm{st}}(\varphi_l(\beta)) = \mathrm{im}^{\mathrm{st}}(\beta)$ and $\Lambda^{\mathrm{st}}_{\varphi_r(\beta)} = \Lambda^{\mathrm{st}}_\beta$, in particular,*

$$\mathrm{strk}(\beta) = \mathrm{strk}(\varphi_l(\beta)) = \mathrm{strk}(\varphi_r(\beta)).$$

Proof. We prove both statements for, say, φ_r. In other cases the arguments are similar. For $\beta, \gamma \in \mathcal{T}_n$ we have:

$$\varphi_r(\beta * \gamma) = (\beta * \gamma)\alpha = \beta\alpha\gamma\alpha = (\beta\alpha)(\gamma\alpha) = \varphi_r(\beta)\varphi_r(\gamma),$$

which proves (a).

We prove (b) for φ_r using the following argument. First we observe that $\beta * \beta = \beta\alpha * \beta$ as $\alpha\alpha = \alpha$. Therefore for every $m \ge 1$ we have $\beta^{*(m+1)} = (\beta\alpha)^{*m}\beta = (\beta\alpha)^m\beta$, which implies that every block of $\Lambda^{\mathrm{st}}_\beta$ is a union of some blocks of $\Lambda^{\mathrm{st}}_{\varphi_r(\beta)}$. On the other hand $(\beta\alpha)^{*(m)} = (\beta\alpha)^{(m)} = (\beta)^{*m}\alpha$, which implies that every block of $\Lambda^{\mathrm{st}}_{\varphi_r(\beta)}$ is a union of some blocks of $\Lambda^{\mathrm{st}}_\beta$. Hence $\Lambda^{\mathrm{st}}_\beta = \Lambda^{\mathrm{st}}_{\varphi_r(\beta)}$, which yields $\mathrm{strk}(\beta) = \mathrm{strk}(\varphi_r(\beta))$ and completes the proof. $\square$

Let $\mathcal{T}(A)$ denote the full transformation semigroup on A and observe that $\alpha\beta(A) \subset A$ for all $\beta \in \mathcal{T}_n$. Hence from Lemma 7(a) we immediately obtain:

Corollary 8. *The map* $\overline{\varphi} : (\mathcal{T}_n, *) \to \mathcal{T}(A)$, *defined via*

$$\overline{\varphi}(\beta) = \varphi_r(\beta)|_A,$$

is an epimorphism.

Note that the congruence, which corresponds to $\overline{\varphi}$, was also used in [**15**].

4. Classification of the isolated subsemigroups

From the definition of $*$ it follows that the stable rank of an element from $(\mathcal{T}_n, *)$ does not exceed l. The results of [**4, 5, 12**] suggest that for the description of the minimal isolated subsemigroups in $(\mathcal{T}_n, *)$ it would be convenient to consider three different cases, namely the elements of stable rank l, $l-1$, and at most $l-2$, separately.

4.1. The case of stable rank l.

Lemma 9. (a) *Let* $\beta \in (\mathcal{T}_n, *)$. *Then* β *has stable rank* l *if and only if* $\overline{\varphi}(\beta)$ *is a bijection.*

(b) *Let* $\beta \in (\mathcal{T}_n, *)$ *be an element of stable rank* l.

$$\Lambda^{\mathrm{st}}_{\beta} = \Lambda_{\beta\alpha}, \qquad \text{and} \qquad \mathrm{im}^{\mathrm{st}}(\beta) = \mathrm{im}(\alpha\beta) = \beta(\{a_1, \dots, a_l\}).$$

(c) *Let* $\beta, \gamma \in (\mathcal{T}_n, *)$ *be two elements of stable rank* l. *Then*

$$\Lambda^{\mathrm{st}}_{\beta*\gamma} = \Lambda^{\mathrm{st}}_{\beta}, \qquad \text{and} \qquad \mathrm{im}^{\mathrm{st}}(\beta * \gamma) = \mathrm{im}^{\mathrm{st}}(\gamma).$$

Proof. If $\overline{\varphi}(\beta)$ is a bijection, then $\overline{\varphi}(\beta)^m$ is a bijection for any m. Hence the stable rank of β is at least l, which implies that it is exactly l. On the other hand, the stable rank of β does not exceed the rank of $\overline{\varphi}(\beta)$, which means that the stable rank of β is strictly smaller than l in the case when $\overline{\varphi}(\beta)$ is not a bijection. This proves (a).

Since $\mathrm{rank}(\alpha) = l$ we have both that $|\,\mathrm{im}(\alpha\beta)| \leq l$ and that the number of blocks in $\Lambda_{\beta\alpha}$ does not exceed l. However, since $\mathrm{strk}(\beta) = l$ by our assumption, it follows that $|\,\mathrm{im}(\alpha\beta)| = l$ and that the number of blocks in $\Lambda_{\beta\alpha}$ equals l. Since, obviously, $\Lambda_{\beta\alpha} \subset \Lambda^{\mathrm{st}}_{\beta}$ and $\mathrm{im}^{\mathrm{st}}(\beta) \subset \mathrm{im}(\alpha\beta)$, we derive $\Lambda^{\mathrm{st}}_{\beta} = \Lambda_{\beta\alpha}$ and $\mathrm{im}^{\mathrm{st}}(\beta) = \mathrm{im}(\alpha\beta)$. The statement (b) follows.

Since, obviously, $\Lambda_{\beta\alpha} \subset \Lambda^{\mathrm{st}}_{\beta*\gamma}$ and $\mathrm{im}^{\mathrm{st}}(\beta * \gamma) \subset \mathrm{im}(\alpha\gamma)$, the statement (b) implies $\Lambda^{\mathrm{st}}_{\beta} \subset \Lambda^{\mathrm{st}}_{\beta*\gamma}$ and $\mathrm{im}^{\mathrm{st}}(\beta * \gamma) \subset \mathrm{im}^{\mathrm{st}}(\gamma)$. On the other hand, as $\mathrm{strk}(\beta) = \mathrm{strk}(\gamma) = l$, the statement (a) implies that $\overline{\varphi}(\beta * \gamma)$ is a bijection and hence $\mathrm{strk}(\beta * \gamma) = l$ again by (a). The last equality, however, says that different blocks of $\Lambda^{\mathrm{st}}_{\beta}$ should remain different in $\Lambda^{\mathrm{st}}_{\beta*\gamma}$ since the total number of such blocks equals l. Thus $\Lambda^{\mathrm{st}}_{\beta} = \Lambda^{\mathrm{st}}_{\beta*\gamma}$. We also have $\mathrm{im}^{\mathrm{st}}(\beta * \gamma) = \mathrm{im}^{\mathrm{st}}(\gamma)$ as both sets have the same cardinality. This completes the proof. □

Let $\mathcal{X}$ be the set of all unordered partitions of N into l disjoint blocks satisfying the condition that all a_i's belong to different blocks. We interpret the elements of

$\mathcal{X}$ as equivalence relations on N. Let further $\mathcal{Y}$ be the set of all sets of the form $\{b_1, \dots, b_l\}$ where $b_i \in A_i$ for all $i = 1, \dots, l$. Now we are ready to describe the minimal isolated subsemigroups in $\mathcal{T}_n^{(l)}$.

Proposition 10. (a) *For every $\Lambda \in \mathcal{X}$ and $I \in \mathcal{Y}$ there exists a unique idempotent, $\varepsilon(\Lambda, I) \in (\mathcal{T}_n, *)$, of stable rank l such that $\Lambda_{\varepsilon(\Lambda,I)} = \Lambda$ and $\mathrm{im}(\varepsilon(\Lambda, I)) = I$.*

(b) *For every $\Lambda \in \mathcal{X}$ and $I \in \mathcal{Y}$ the set $\sqrt{\varepsilon(\Lambda, I)}$ is an isolated subsemigroup of $(\mathcal{T}_n, *)$ and $\sqrt{\varepsilon(\Lambda, I)} = \mathtt{I}(\beta)$ for every $\beta \in \sqrt{\varepsilon(\Lambda, I)}$.*

Proof. The statement (a) follows from the description of idempotents in $(\mathcal{T}_n, *)$, obtained in Theorem 5.

To prove (b) we first note that, by definition, the set $\sqrt{\varepsilon(\Lambda, I)}$ contains exactly elements with stable kernel Λ and stable image I. Let $\beta, \gamma \in \sqrt{\varepsilon(\Lambda, I)}$. Then Lemma 9(c) implies that $\beta * \gamma \in \sqrt{\varepsilon(\Lambda, I)}$ as well, in particular, $\sqrt{\varepsilon(\Lambda, I)}$ is a subsemigroup of $(\mathcal{T}_n, *)$. Hence (b) follows from Proposition 3. This completes the proof. □

Proposition 11. (a) *$\mathcal{T}_n^{(l)}$ is a subsemigroup of $(\mathcal{T}_n, *)$.*

(b) *The decomposition*

$$\mathcal{T}_n^{(l)} = \bigcup_{(\Lambda,I)\in\mathcal{X}\times\mathcal{Y}} \sqrt{\varepsilon(\Lambda, I)} \tag{2}$$

into a disjoint union of subsets defines a congruence on $\mathcal{T}_n^{(l)}$, the quotient modulo which is isomorphic to the rectangular band $\mathcal{X} \times \mathcal{Y}$.

(c) *Let $\psi : \mathcal{T}_n^{(l)} \to \mathcal{X} \times \mathcal{Y}$ be the natural projection, given by* (b). *Then ψ^{-1} induces a bijection between the set of all subsemigroups of $\mathcal{X} \times \mathcal{Y}$ and $\mathfrak{I}(\mathcal{T}_n^{(l)})$.*

(d) $|\mathfrak{I}(\mathcal{T}_n^{(l)})| = 2^{|\mathcal{X}|+|\mathcal{Y}|} - 2^{|\mathcal{X}|} - 2^{|\mathcal{Y}|} + 1.$

We remark that $|\mathcal{X}| = l^{n-l}$ and $|\mathcal{Y}| = \prod_{i=1}^{l} |A_i|$. Note also that $T \subset \mathcal{X} \times \mathcal{Y}$ is a subsemigroup if and only if the following condition is satisfied: for all $(\Lambda, I), (\Lambda', I') \in \mathcal{X} \times \mathcal{Y}$ we have that $(\Lambda, I), (\Lambda', I') \in T$ implies both $(\Lambda, I') \in T$ and $(\Lambda', I) \in T$.

Proof. The statements (a) and (b) follow from Lemma 9(c). The rectangular band $\mathcal{X}\times\mathcal{Y}$ consists of idempotents, and hence every subsemigroup of it is isolated. Now (VIII) implies that ψ^{-1} (i.e. taking the pre-image with respect to ψ) induces a map from $\mathfrak{I}(\mathcal{X} \times \mathcal{Y})$ to $\mathfrak{I}(\mathcal{T}_n^{(l)})$, which is injective since ψ is surjective. Let now $T \in \mathfrak{I}(\mathcal{T}_n^{(l)})$. Since the congruence classes are minimal isolated subsemigroups of $\mathcal{T}_n^{(l)}$ by Proposition 10, it follows that T must be the union of those congruence classes, which have a non-trivial intersection with T. Hence $\psi(T)$ is a subsemigroup of $\mathcal{X} \times \mathcal{Y}$ and $\psi^{-1}(\psi(T)) = T$, implying that the map, induced by ψ^{-1}, is even bijective. This proves (c).

Because of (c) we just compute the cardinality of the set $\mathfrak{T}(\mathcal{X} \times \mathcal{Y})$ of all subsemigroups in $\mathcal{X} \times \mathcal{Y}$. It is easy to see that every subsemigroup in $\mathcal{X} \times \mathcal{Y}$ is uniquely determined by the sets of the left and the right coordinates of its elements.

So, there are exactly $(2^{|\mathcal{X}|}-1)(2^{|\mathcal{Y}|}-1)$ subsemigroups in $\mathcal{X}\times\mathcal{Y}$. The statement (d) follows and the proof of our proposition is complete. □

For $\emptyset\neq X\subset\mathcal{X}$ and $\emptyset\neq Y\subset\mathcal{Y}$ let $F(X,Y)$ denote the union of all $\sqrt{\varepsilon}$ for which $\Lambda_\varepsilon\in X$ and $\mathrm{im}(\varepsilon)\in Y$. From Proposition 11 we obtain:

Corollary 12. *Every isolated subsemigroup of $(\mathcal{T}_n,*)$, which is contained in $\mathcal{T}_n^{(l)}$, is equal to $F(X,Y)$ for appropriate X and Y.*

4.2. The case of stable rank $l-1$. Let $\varepsilon\in E(\mathcal{T}_n,*)$ be some idempotent of rank $l-1$, given by (1). From Theorem 5 it follows that there exists a unique element, $i\in\{1,\dots,l-1\}$, and a unique pair, $\{m,k\}\subset\{1,\dots,l\}$, $m\neq k$, such that $\{a_m,a_k\}\subset E_i$ and $e_i\in A_m$. We call the pair $\{a_m,a_k\}$ *distinguished*, the element a_k *trifle* and the element a_m *burdened* (for the idempotent ε) and denote them by $\mathfrak{d}(\varepsilon)$, $\mathfrak{t}(\varepsilon)$ and $\mathfrak{b}(\varepsilon)$ respectively. By $\mathfrak{nt}(\varepsilon)$ and $\mathfrak{nb}(\varepsilon)$ we denote the elements of $\{1,\dots,l\}$ such that $\mathfrak{t}(\varepsilon)=a_{\mathfrak{nt}(\varepsilon)}$ and $\mathfrak{b}(\varepsilon)=a_{\mathfrak{nb}(\varepsilon)}$ respectively. It will be convenient now to adjust (1) in the following way: if necessary, we can rearrange the indexes of e_i's to obtain

$$\varepsilon=\begin{pmatrix} E_1 & \dots & E_{\mathfrak{nt}(\varepsilon)-1} & E_{\mathfrak{nt}(\varepsilon)+1} & \dots & E_l\\ e_1 & \dots & e_{\mathfrak{nt}(\varepsilon)-1} & e_{\mathfrak{nt}(\varepsilon)+1} & \dots & e_l\end{pmatrix}, \tag{3}$$

where $a_i\in E_i$ for all $i=1,\dots,\mathfrak{nt}(\varepsilon)-1,\mathfrak{nt}(\varepsilon)+1,\dots,l$. Then, in particular, $\mathfrak{t}(\varepsilon)\in E_{\mathfrak{nb}(\varepsilon)}$.

For $\beta\in\sqrt{\varepsilon}$ we set $\mathfrak{d}(\beta)=\mathfrak{d}(\varepsilon)$, $\mathfrak{t}(\beta)=\mathfrak{t}(\varepsilon)$, $\mathfrak{b}(\beta)=\mathfrak{b}(\varepsilon)$, $\mathfrak{nt}(\beta)=\mathfrak{nt}(\varepsilon)$, and $\mathfrak{nb}(\beta)=\mathfrak{nb}(\varepsilon)$. For $k=1,\dots,l$ let $\mathcal{T}_n^{(l-1,k)}$ denote the set of all elements $\beta\in\mathcal{T}_n^{(l-1)}$ such that $\mathfrak{t}(\beta)=a_k$. For $k,m=1,\dots,l$, $k\neq m$, let $\mathcal{T}_n^{(l-1,k,m)}$ denote the set of all elements $\beta\in\mathcal{T}_n^{(l-1,k)}$ such that $\mathfrak{b}(\beta)=a_m$.

Lemma 13. *Let ε be an idempotent in $(\mathcal{T}_n,*)$ of rank $l-1$ given by (3) and $\beta\in\sqrt{\varepsilon}$. Then*

(a) $\mathrm{im}^{\mathrm{st}}(\beta)=\beta(A\setminus\{\mathfrak{t}(\beta)\})$.
(b) *If* $\mathrm{rank}(\beta\alpha)=l-1$ *then* $\Lambda_\varepsilon=\Lambda_\beta^{\mathrm{st}}=\Lambda_{\beta\alpha}$.
(c) *If* $\mathrm{rank}(\beta\alpha)=l$ *then every block of $\Lambda_{\beta\alpha}$ contains at most two elements from A, moreover, there is the unique block, namely $(\beta\alpha)^{-1}(\mathfrak{t}(\beta))$, which does not contain any element from A, and there is the unique block which contains exactly two elements from A, namely the elements from $\mathfrak{d}(\beta)$.*
(d) *$\beta\alpha$ induces a permutation on $A\setminus\{\mathfrak{t}(\beta)\}$*

Proof. We obviously have $\Lambda_{\beta\alpha}\subset\Lambda_\beta^{\mathrm{st}}$ and in the case $\mathrm{rank}(\beta\alpha)=l-1$ the equality follows since the number of blocks in $\Lambda_{\beta\alpha}$ and $\Lambda_\beta^{\mathrm{st}}$ coincide. This proves (b). Moreover, in this case $\mathrm{im}^{\mathrm{st}}(\beta)=\beta(A\setminus\{\mathfrak{t}(\beta)\})$ follows immediately from the definition of $\mathfrak{t}(\beta)$, which proves the corresponding part of (a).

Since $\mathrm{strk}(\beta)\neq l$, in the case $\mathrm{rank}(\beta\alpha)=l$ it is not possible that every block of $\Lambda_{\beta\alpha}$ contains an element from A. On the other hand, every block of $\Lambda_\beta^{\mathrm{st}}$ contains at least one element from A and hence, since $\mathrm{strk}(\beta)=l-1$, it follows that all blocks of $\Lambda_\beta^{\mathrm{st}}$ except one contain exactly one element from A, and the exceptional block contains two elements from A. This implies the uniqueness of the block with two elements from A and of the block without any elements from A for $\Lambda_{\beta\alpha}$. From the definition of $\mathfrak{t}(\beta)$ it follows that $(\beta\alpha)^{-1}(\mathfrak{t}(\beta))$ does not contain any element of A, and from the definition of $\mathfrak{d}(\beta)$ it follows that $\mathfrak{d}(\beta)$ is contained in some block of $\Lambda_{\beta\alpha}$. This proves (c). Since the pre-image of $\mathfrak{t}(\beta)$ does not intersect A, $\mathfrak{t}(\beta)$ can not belong to $\mathrm{im}^{\mathrm{st}}(\beta\alpha)$, which completes the proof of (a) as well. Finally, (d) follows from (a) and the lemma is proved. □

Corollary 14. *Let $\beta, \gamma \in \mathcal{T}_n^{(l-1)}$. Then $\mathrm{rank}(\beta\alpha\gamma\alpha) \geq \mathrm{rank}(\beta\alpha) - 1$ and $\Lambda_{\beta\alpha\gamma\alpha}$ is obtained from $\Lambda_{\beta\alpha}$ by uniting $(\beta\alpha)^{-1}(\mathfrak{b}(\gamma))$ and $(\beta\alpha)^{-1}(\mathfrak{t}(\gamma))$.*

Proof. $\beta\alpha$ maps the blocks of $\Lambda_{\beta\alpha}$ to $\mathrm{im}(\beta\alpha) \subset A$. Computing $\Lambda_{\beta\alpha\gamma\alpha}$ we should unite those blocks of $\Lambda_{\beta\alpha}$, whose images belong to the same block of $\Lambda_{\gamma\alpha}$. However, Lemma 13(c) says that there is a unique block in $\Lambda_{\gamma\alpha}$, which contains more than one element from A, and this block contains $\mathfrak{d}(\gamma)$. The statement follows. □

Corollary 15. *Let $\beta, \gamma \in \mathcal{T}_n^{(l-1)}$ be such that $\beta * \gamma \in \mathcal{T}_n^{(l-1)}$. Then*
(a) *$\mathfrak{d}(\beta * \gamma) = \mathfrak{d}(\beta)$.*
(b) *$\mathfrak{t}(\beta * \gamma) = \mathfrak{t}(\gamma)$.*

Proof. (a) follows from Lemma 13(c). To prove (b) consider $\beta\alpha\gamma\alpha$. From Lemma 13(c) we have $(\gamma\alpha)^{-1}(\mathfrak{t}(\gamma)) \cap A = \emptyset$, which obviously implies that $(\beta\alpha\gamma\alpha)^{-1}(\mathfrak{t}(\gamma)) = \emptyset$. The statement now follows from the definition of the trifle element. □

Now we can describe all minimal isolated subsemigroups in $\mathcal{T}_n^{(l-1)}$.

Corollary 16. *Let $\varepsilon \in \mathcal{T}_n^{(l-1)}$ be an idempotent given by* (3)*. Then $\sqrt{\varepsilon}$ is a minimal isolated subsemigroup of $(\mathcal{T}_n, *)$.*

Proof. Because of Proposition 3 we need only to prove that $\sqrt{\varepsilon}$ is a semigroup. Let $\beta, \gamma \in \sqrt{\varepsilon}$. We obviously have $\Lambda_{\beta\alpha} \subset \Lambda_{\beta\alpha\gamma\alpha}$. By Corollary 14, $\Lambda_{\beta\alpha\gamma\alpha}$ is obtained from $\Lambda_{\beta\alpha}$ uniting the pre-image of $\mathfrak{t}(\gamma)$ $(= \mathfrak{t}(\beta))$ with that of $\mathfrak{b}(\gamma)$ $(= \mathfrak{b}(\beta))$. By Lemma 13(d), the element $\beta\alpha\gamma\alpha$ induces a permutation on the set $A \setminus \{\mathfrak{t}(\beta)\}$, in particular, $\mathrm{rank}(\beta\alpha\gamma\alpha) \geq l-1$. On the other hand, it follows that $\mathrm{rank}(\beta\alpha\gamma\alpha) \leq l-1$. Hence $\mathrm{rank}(\beta\alpha\gamma\alpha) = l-1$. Note that $(\beta\alpha\gamma\alpha)^{*m} = (\beta\alpha\gamma\alpha)^m$ since $\alpha\alpha = \alpha$. Applying the previous arguments inductively we get $\Lambda_{(\beta\alpha\gamma\alpha)^m} = \Lambda_{\beta\alpha\gamma\alpha}$ and $\mathrm{im}((\beta\alpha\gamma\alpha)^m) = \mathrm{im}(\beta\alpha\gamma\alpha)$ for all m, in particular, it follows that the stable rank of $\beta\alpha\gamma\alpha$ equals $l-1$. As both $\beta\alpha\gamma\alpha$ and γ have stable rank $l-1$, we also deduce that $\mathrm{im}^{\mathrm{st}}(\beta * \gamma) = \mathrm{im}^{\mathrm{st}}(\gamma)$. Using $\mathfrak{d}(\beta) = \mathfrak{d}(\gamma)$, we further get $\Lambda^{\mathrm{st}}_{\beta*\gamma} = \Lambda^{\mathrm{st}}_{\beta} = \Lambda_{\beta\alpha\gamma\alpha}$. In particular, if m is such that $(\beta\alpha\gamma\alpha)^{*m} = \nu$ is an idempotent, then $\Lambda_\nu = \Lambda_\varepsilon$ and $\mathrm{im}^{\mathrm{st}}(\nu) = \mathrm{im}^{\mathrm{st}}(\varepsilon)$, which implies that $\nu = \varepsilon$ since every idempotent in $(\mathcal{T}_n, *)$ is uniquely defined by the corresponding Λ and the image, see Theorem 5. Therefore $\beta * \gamma \in \sqrt{\varepsilon}$ and hence $\sqrt{\varepsilon}$ is indeed a semigroup. The statement follows. □

Proposition 17. (a) *For every $k, m \in \{1, \dots, l\}$, $k \neq m$, the set $\mathcal{T}_n^{(l-1,k,m)}$ is an isolated subsemigroup of $(\mathcal{T}_n, *)$.*

(b) *For every* $k \in \{1,\dots,l\}$ *the set* $\mathcal{T}_n^{(l-1,k)}$ *is an isolated subsemigroup of* $(\mathcal{T}_n,*)$.

(c) *For every* $k,m \in \{1,\dots,l\}$, $k \neq m$, *the set* $\mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$ *is an isolated subsemigroup of* $(\mathcal{T}_n,*)$.

(d) *Let* $\varepsilon_1 \neq \varepsilon_2$ *be two idempotents in* $\mathcal{T}_n^{(l-1)}$. *Assume that* $\mathfrak{d}(\varepsilon_1) \neq \mathfrak{d}(\varepsilon_2)$ *and* $\mathfrak{t}(\varepsilon_1) \neq \mathfrak{t}(\varepsilon_2)$. *Then* $\sqrt{\varepsilon_1} * \sqrt{\varepsilon_2}$ *is not contained in* $\mathcal{T}_n^{(l-1)}$.

Proof. The statement (a) follows from (b) and (c) using (I).

To prove (b) let us first consider the case when $\alpha = \mathrm{id}$. In this case the only idempotents $\mathcal{T}_n^{(l-1,k)}$ contains are $\varepsilon_{m,k}$, $m \in \{1,\dots,k-1,k+1,\dots,n\}$, defined as follows:

$$\varepsilon_{m,k} = \begin{pmatrix} 1 & 2 & \dots & k-1 & k & k+1 & \dots & n \\ 1 & 2 & \dots & k-1 & m & k+1 & \dots & n \end{pmatrix}. \tag{4}$$

If $\beta \in \mathcal{T}_n$ is such that $\beta^j = \varepsilon_{m,k}$ for some m and $j > 0$ then $\mathrm{rank}(\beta) = n-1$ and β induces a permutation on $N \setminus \{k\}$. On the other hand, if β is not invertible and induces a permutation on $N \setminus \{k\}$, it must have rank $n-1$. It follows that the set $\cup_{m \neq k} \sqrt{\varepsilon_{m,k}}$ consists of all noninvertible elements in $\mathcal{T}_n$, which induce a permutation on $N \setminus \{k\}$. In particular, this set is closed with respect to composition and hence is a subsemigroup. Since it is a union of isolated subsemigroups by Corollary 16, it is an isolated subsemigroup itself by Proposition 2. Hence (b) is true in the case of invertible α. In the general case consider the homomorphism $\overline{\varphi} : (\mathcal{T}_n,*) \to \mathcal{T}(A)$. Let $\mathcal{T}(A)^{(l-1,k)}$ denote the set of all elements in $\mathcal{T}(A)$ of stable rank $l-1$ and with the trifle element a_k. Let $\beta \in \mathcal{T}_n^{(l-1,k)}$. Then from Lemma 13 it follows that $\alpha\beta\alpha$ has rank $l-1$ and induces a permutation on $A \setminus \{a_k\}$. In particular, $\overline{\varphi}(\beta) \in \mathcal{T}(A)^{(l-1,k)}$. On the other hand, if $\overline{\varphi}(\beta) \in \mathcal{T}(A)^{(l-1,k)}$ then $\mathrm{strk}(\beta) = l-1$ and $\mathfrak{t}(\beta) = a_k$, that is $\beta \in \mathcal{T}_n^{(l-1,k)}$. This implies that $\mathcal{T}_n^{(l-1,k)} = \overline{\varphi}^{-1}(\mathcal{T}(A)^{(l-1,k)})$. Since $\mathcal{T}(A)^{(l-1,k)}$ is an isolated subsemigroup of $\mathcal{T}(A)$, we get, using (VIII), that $\mathcal{T}_n^{(l-1,k)}$ is an isolated subsemigroup of $(\mathcal{T}_n,*)$. This proves (b). The statement (c) is proved by analogous arguments (that is one first checks it in the case $\alpha = \mathrm{id}$ and then uses $\overline{\varphi}^{-1}$).

Finally, to prove (d) we show that $\mathrm{strk}(\varepsilon_1 * \varepsilon_2) < l-1$ if $\mathfrak{d}(\varepsilon_1) \neq \mathfrak{d}(\varepsilon_2)$ and $\mathfrak{t}(\varepsilon_1) \neq \mathfrak{t}(\varepsilon_2)$. Obviously $\mathrm{strk}(\varepsilon_1 * \varepsilon_2) \leq \mathrm{rank}(\overline{\varphi}(\varepsilon_1)\overline{\varphi}(\varepsilon_2))$ and $\overline{\varphi}(\varepsilon_i) = \varepsilon_{\mathfrak{b}(\varepsilon_i),\mathfrak{t}(\varepsilon_i)}$ for $i = 1,2$ (here we use the notation similar to (4)). Under our assumption a direct calculation shows that $\mathrm{rank}((\varepsilon_{\mathfrak{b}(\varepsilon_1),\mathfrak{t}(\varepsilon_1)}\varepsilon_{\mathfrak{b}(\varepsilon_2),\mathfrak{t}(\varepsilon_2)})^2) = l-2$. The statement (d) follows and hence the proof is complete. $\square$

Now we would like to present various constructions of isolated subsemigroups in $\mathcal{T}_n^{(l-1)}$. We start with the one which corresponds to the case described in Proposition 17(a).

Fix $k,m \in \{1,\dots,l\}$, $k \neq m$. Let $\mathcal{U}^{(k,m)}$ denote the set of all unordered partitions of N into $l-1$ disjoint blocks satisfying the condition that every block contains at least one element from A and that a_k and a_m are contained in the same block. Note that $\mathcal{U}^{(k,m)} = \mathcal{U}^{(m,k)}$. Let further $\mathcal{V}^{(k)}$ denote the set of all sets of the form $\{b_1,\dots,b_{k-1},b_{k+1},\dots,b_l\}$, where $b_i \in A_i$ for all $i = 1,\dots,k-1,k+1,\dots,l$.

For $k, m \in \{1, \dots, l\}$, $k \neq m$, $\emptyset \neq X \subset \mathcal{U}^{(k,m)}$, and $\emptyset \neq Y \subset \mathcal{V}^{(k)}$, let $H(k, m, X, Y)$ denote the union of all $\sqrt{\varepsilon}$, where $\varepsilon \in \mathcal{T}_n^{(l-1,k,m)}$ is an idempotent such that $\Lambda_\varepsilon \in X$ and $\mathrm{im}(\varepsilon) \in Y$. For $H(k, m, X, Y)$ we can prove an analogue of Proposition 11 and Corollary 12.

Proposition 18. *Let $k, m \in \{1, \dots, l\}$, $k \neq m$.*

(a) *For every $\Lambda \in \mathcal{U}^{(k,m)}$ and $I \in \mathcal{V}^{(k)}$ there exists a unique idempotent, $\varepsilon(\Lambda, I)$, such that $\Lambda_{\varepsilon(\Lambda,I)} = \Lambda$ and $\mathrm{im}(\varepsilon(\Lambda, I)) = I$.*

(b) *The decomposition*

$$\mathcal{T}_n^{(l-1,k,m)} = \bigcup_{(\Lambda,I)\in\mathcal{U}^{(k,m)}\times\mathcal{V}^{(k)}} \sqrt{\varepsilon(\Lambda, I)}$$

into a disjoint union of subsets defines a congruence on $\mathcal{T}_n^{(l-1,k,m)}$, the quotient modulo which is isomorphic to the rectangular band $\mathcal{U}^{(k,m)} \times \mathcal{V}^{(k)}$.

(c) *For every $\emptyset \neq X \subset \mathcal{U}^{(k,m)}$ and $\emptyset \neq Y \subset \mathcal{V}^{(k)}$ the set $H(k, m, X, Y)$ is an isolated subsemigroup of $(\mathcal{T}_n, *)$.*

(d) *Every isolated subsemigroup of $\mathcal{T}_n^{(l-1,k,m)}$ is equal to $H(k, m, X, Y)$ for appropriate X and Y.*

(e) *$\mathcal{T}_n^{(l-1,k,m)}$ contains $2^{|\mathcal{U}^{(k,m)}|+|\mathcal{V}^{(k)}|} - 2^{|\mathcal{U}^{(k,m)}|} - 2^{|\mathcal{V}^{(k)}|} + 1$ isolated subsemigroups.*

Proof. The statement (a) follows from Theorem 5. Let $(\Lambda, I), (\Lambda', I') \in \mathcal{U}^{(k,m)} \times \mathcal{V}^{(k)}$, $\beta \in \sqrt{\varepsilon(\Lambda, I)}$ and $\gamma \in \sqrt{\varepsilon(\Lambda', I')}$. From Proposition 17(a) we have $\beta * \gamma \in \mathcal{T}_n^{(l-1,k,m)}$, in particular, $\mathrm{strk}(\beta*\gamma) = l-1$. Hence, using $\mathfrak{b}(\beta) = \mathfrak{b}(\gamma)$, $\mathfrak{t}(\beta) = \mathfrak{t}(\gamma)$, and Corollary 14, we get $\Lambda^{\mathrm{st}}_{\beta*\gamma} = \Lambda^{\mathrm{st}}_{\beta}$. Moreover, Corollary 15(b) and Lemma 13(a) imply $\mathrm{im}^{\mathrm{st}}(\beta * \gamma) = \mathrm{im}^{\mathrm{st}}(\gamma)$. Thus $\beta * \gamma \in \sqrt{\varepsilon(\Lambda, I')}$, which implies (b). The rest is proved by the same arguments as in Proposition 11 and Corollary 12. □

Our next construction corresponds to the case described in Proposition 17(c). For $k, m \in \{1, \dots, l\}$, $k \neq m$, and $\emptyset \neq X \subset \mathcal{U}^{(k,m)} = \mathcal{U}^{(m,k)}$, let $K(\{k, m\}, X)$ denote the union of all $\sqrt{\varepsilon}$, where $\varepsilon \in \mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$ is an idempotent such that $\Lambda_\varepsilon \in X$.

Proposition 19. *Let $k, m \in \{1, \dots, l\}$, $k \neq m$.*

(a) *For every X as above the set $K(\{k, m\}, X)$ is an isolated subsemigroup of $\mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$.*

(b) *Every isolated subsemigroup of $\mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$, which intersects both $\mathcal{T}_n^{(l-1,k,m)}$ and $\mathcal{T}_n^{(l-1,m,k)}$ in a non-trivial way, is equal to the semigroup $K(\{k, m\}, X)$ for an appropriate X.*

(c) *The number of isolated subsemigroups of $\mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$, which intersect both subsemigroups $\mathcal{T}_n^{(l-1,k,m)}$ and $\mathcal{T}_n^{(l-1,m,k)}$ in a non-trivial way, equals $2^{(l-1)^{n-l}} - 1$.*

Proof. That both $K(\{k, m\}, X) \cap \mathcal{T}_n^{(l-1,k,m)}$ and $K(\{k, m\}, X) \cap \mathcal{T}_n^{(l-1,m,k)}$ are subsemigroups follows from Proposition 18. Let $\Lambda, \Lambda' \in X$, $I \in \mathcal{V}^{(k)}$, $I' \in \mathcal{V}^{(m)}$, $\beta \in \sqrt{\varepsilon(\Lambda, I)}$ and $\gamma \in \sqrt{\varepsilon(\Lambda', I')}$. From Proposition 17(c) we have $\mathrm{strk}(\beta * \gamma) =$

$l-1$. Hence, using $\mathfrak{d}(\beta)=\mathfrak{d}(\gamma)$ and Corollary 14, we get $\Lambda^{\mathrm{st}}_{\beta*\gamma}=\Lambda^{\mathrm{st}}_{\beta}=\Lambda\in X$. Further, $\mathfrak{d}(\beta*\gamma)=\mathfrak{d}(\beta)$ by Corollary 15(a). Thus $\beta*\gamma\in K(\{k,m\},X)$ and the claim (a) follows.

Let now T be an isolated subsemigroup of $\mathcal{T}_n^{(l-1,k,m)}\cup\mathcal{T}_n^{(l-1,m,k)}$, which intersects both $\mathcal{T}_n^{(l-1,k,m)}$ and $\mathcal{T}_n^{(l-1,m,k)}$ in a non-trivial way. Set

$$X=\{\Lambda^{\mathrm{st}}_{\beta}:\beta\in T\}.$$

Then $T\subset K(\{k,m\},X)$.

Let now $\Lambda\in X$ and $\beta\in T\cap\mathcal{T}_n^{(l-1,m,k)}$ be an idempotent such that $\Lambda_\beta=\Lambda$. Then for any $\gamma\in\mathcal{T}_n^{(l-1,m,k)}\cap T$ from Corollary 15 it follows that $\beta*\gamma\in\mathcal{T}_n^{(l-1,m,k)}$, and as both β and $\beta*\gamma$ have stable rank $l-1$, it also follows that $\Lambda^{\mathrm{st}}_{\beta}=\Lambda^{\mathrm{st}}_{\beta*\gamma}$. This means that

$$X=\{\Lambda^{\mathrm{st}}_{\beta}:\beta\in T\cap\mathcal{T}_n^{(l-1,m,k)}\}=\{\Lambda^{\mathrm{st}}_{\beta}:\beta\in T\cap\mathcal{T}_n^{(l-1,k,m)}\}. \tag{5}$$

Now let ε be an idempotent of $\mathcal{T}_n^{(l-1,k,m)}\cap T$ given by (3). Let $i\in\{1,\dots,\mathfrak{nt}(\varepsilon)-1,\mathfrak{nt}(\varepsilon)+1,\dots,l\}$ and $e'_i\in A_i$ be some element different from e_i. Consider the idempotent ε' defined as follows:

$$\varepsilon'(x)=\begin{cases}\varepsilon(x), & x\notin E_i;\\ e'_i, & \text{otherwise.}\end{cases}$$

Obviously $\Lambda_{\varepsilon'}=\Lambda_\varepsilon$.

Lemma 20. $\varepsilon'\in T$.

Proof. Because of (5) we can fix an idempotent $\beta\in\mathcal{T}_n^{(l-1,m,k)}\cap T$ such that $\Lambda_\beta=\Lambda_\varepsilon$. Consider now the element $\gamma\in\mathcal{T}_n$ defined as follows:

$$\gamma(x)=\begin{cases}e'_i, & x=\mathfrak{t}(\varepsilon);\\ e_i, & x\in E_{\mathfrak{nb}(\varepsilon)},\ x\neq\mathfrak{t}(\varepsilon);\\ e_{\mathfrak{nb}(\varepsilon)}, & x\in E_i, \quad x\neq\mathfrak{t}(\varepsilon);\\ \varepsilon(x), & \text{otherwise.}\end{cases}$$

From the definitions we have $\gamma*\gamma=\varepsilon$ and hence $\gamma\in T$. In particular, $\beta*\gamma\in T$. As both $\beta*\gamma$ and β have rank $l-1$, we have $\Lambda_{\beta*\gamma}=\Lambda_\beta=\Lambda_\varepsilon=\Lambda_{\varepsilon'}$.

We further have $\mathrm{strk}(\beta*\gamma)=l-1$, $\mathrm{im}(\gamma)=\mathrm{im}(\varepsilon)\cup\{e'_i\}$, and, finally, $e_i\notin\mathrm{im}(\beta*\gamma)$ because $(\alpha\gamma)^{-1}(e_i)=A_{\mathfrak{b}(\varepsilon)}=A_{\mathfrak{t}(\beta)}$. Together these imply that the stable image of $\beta*\gamma$ is

$$\mathrm{im}^{\mathrm{st}}(\beta*\gamma)=(\mathrm{im}(\varepsilon)\cup\{e'_i\})\setminus\{e_i\}=\mathrm{im}^{\mathrm{st}}(\varepsilon').$$

Thus the element $\beta*\gamma\in T$ satisfies $\Lambda_{\beta*\gamma}=\Lambda_{\varepsilon'}$ and $\mathrm{im}^{\mathrm{st}}(\beta*\gamma)=\mathrm{im}^{\mathrm{st}}(\varepsilon')$, implying $\beta*\gamma\in\sqrt{\varepsilon'}$, which means $\varepsilon'\in T$. □

Lemma 20 basically says that if T contains an idempotent with a certain kernel and a certain image, then T also contains an idempotent with the same kernel, but in which the image is different by one element, arbitrarily chosen in arbitrary block

A_i. From this fact, formula (5), and the definition of the semigroup $K(\{k,m\},X)$ it follows easily that we have the inclusion $\mathcal{T}_n^{(l-1,k,m)} \cap K(\{k,m\},X) \subset T$. Analogously we obtain $\mathcal{T}_n^{(l-1,m,k)} \cap K(\{k,m\},X) \subset T$ and hence $K(\{k,m\},X) = T$, which proves the statement (b).

To prove (c) we only have to observe that the (non-empty) set X can be chosen in $2^{(l-1)^{n-l}} - 1$ different ways. $\square$

Our last construction corresponds to the case given by Proposition 17(b). For $k \in \{1,\dots,l\}$, $M \subset \{a_1,\dots,a_{k-1},a_{k+1},\dots,a_l\}$, $|M| > 1$, and $\emptyset \neq Y \subset \mathcal{V}^{(k)}$, let $L(k,M,Y)$ denote the union of all $\sqrt{\varepsilon}$, where $\varepsilon \in \mathcal{T}_n^{(l-1,k)}$ is an idempotent such that $\mathfrak{b}(\varepsilon) \in M$, $\mathrm{im}(\varepsilon) \in Y$.

Proposition 21. *Let $k \in \{1,\dots,l\}$.*

(a) *For every M and Y as above the set $L(k,M,Y)$ is an isolated subsemigroup of $\mathcal{T}_n^{(l-1,k)}$.*
(b) *Every isolated subsemigroup of $\mathcal{T}_n^{(l-1,k)}$, which is not contained in any $\mathcal{T}_n^{(l-1,k,m)}$, is equal to the semigroup $L(k,M,Y)$ for appropriate M and Y.*
(c) *The number of isolated subsemigroups of $\mathcal{T}_n^{(l-1,k)}$, which are not contained in any $\mathcal{T}_n^{(l-1,k,m)}$, equals $(2^{l-1}-l)(2^p-1)$, where $p = \prod_{i\neq k} |A_i|$.*

Proof. One proves (a) using Lemma 13, Corollary 15 and Proposition 17(b). The statement (c) follows from (b) by a standard combinatorial calculation. We prove (b).

Let T be an isolated subsemigroup of $\mathcal{T}_n^{(l-1,k)}$, which is not contained in any $\mathcal{T}_n^{(l-1,k,m)}$. Set

$$M = \{\mathfrak{b}(\beta) : \beta \in T\}, \qquad \text{and} \qquad Y = \{\mathrm{im}^{\mathrm{st}}(\beta) : \beta \in T\}.$$

Then $|M| > 1$ by our assumption and thus $T \subset L(k,M,Y)$. The non-trivial part is to prove that $L(k,M,Y) \subset T$. Lemma 13, Corollary 15 and Proposition 17(b) tell us that for $\beta,\gamma \in T$ we have $\mathfrak{b}(\beta * \gamma) = \mathfrak{b}(\beta)$ and $\mathrm{im}^{\mathrm{st}}(\beta * \gamma) = \mathrm{im}^{\mathrm{st}}(\gamma)$. In particular, this implies that for fixed $a_m \in M$ and $I \in Y$ there always exists some idempotent $\varepsilon \in T$ given by (3), such that $\mathfrak{b}(\varepsilon) = a_m$ and $\mathrm{im}(\varepsilon) = I$.

As $|M| > 1$, for the same reason we can also fix another idempotent $\tau \in T$ such that $\mathfrak{b}(\tau) \neq a_m$ and $\mathrm{im}(\tau) = I$. Set

$$\Lambda = \Lambda_\varepsilon = E_1 \cup \dots \cup E_{k-1} \cup E_{k+1} \cup \dots \cup E_l$$

and recall that $a_i \in E_i$ for all $i = 1,\dots,k-1,k+1,\dots,l$, $a_k \in E_m$. We will need the following statement:

Lemma 22. *Let $x \in N \setminus A$ and assume that $x \in E_s$. Let $t \in \{1,\dots,k-1,k+1,\dots,l\}$, $t \neq s$. Define*

$$\Lambda' = E'_1 \cup \dots \cup E'_{k-1} \cup E'_{k+1} \cup \dots \cup E'_l$$

in the following way: for $i = 1, \dots, k-1, k+1, \dots, l$ set

$$E'_i = \begin{cases} E_i, & i \neq s, t; \\ E_s \setminus \{x\}, & i = s; \\ E_t \cup \{x\}, & i = t. \end{cases}$$

Then T contains some γ such that $\mathfrak{b}(\gamma) = a_m$, $\mathrm{im}^{\mathrm{st}}(\gamma) = I$ and $\Lambda^{\mathrm{st}}_\gamma = \Lambda'$.

Proof. Let σ be any permutation of $1, \dots, k-1, k+1, \dots, l$ such that $\sigma(t) = \mathfrak{nb}(\tau)$ and $\sigma(s) = \mathfrak{nb}(\varepsilon)$ (such σ exists since $\mathfrak{b}(\varepsilon) \neq \mathfrak{b}(\tau)$). Consider the element $\beta \in \mathcal{T}_n$ defined as follows:

$$\begin{aligned} \beta(x) &= a_k, \\ \beta(E_s \setminus \{x\}) &= e_{\sigma(s)}, \\ \beta(E_i) &= e_{\sigma(i)} \quad \text{for all } i \in \{1, \dots, l\} \setminus \{k, s\}. \end{aligned}$$

One checks that $\beta \in \sqrt{\varepsilon} \subset T$, thus $\gamma = \beta * \tau \in T$. However, Corollary 14 implies that $\Lambda_\gamma = \Lambda'$. That $\mathfrak{b}(\gamma) = a_m$, $\mathrm{im}(\gamma) = I$ follows from Corollary 15. This completes the proof. □

Using Lemma 22 inductively (starting from ε), one shows that for every $\tilde{\Lambda} \in \mathcal{U}^{(k,m)}$ the semigroup T contains an idempotent $\tilde{\varepsilon}$ such that $\Lambda_{\tilde{\varepsilon}} = \tilde{\Lambda}$, $\mathfrak{b}(\tilde{\varepsilon}) = a_m$, and $\mathrm{im}(\tilde{\varepsilon}) = I$. The statement of the proposition follows. □

Now we can gather the harvest of the hard work above.

Corollary 23. *Let T be an isolated subsemigroup of $(\mathcal{T}_n^{(l-1)}, *)$. Then either $T = H(k, m, X, Y)$ for some appropriate k, m, X, and Y, or $T = K(\{k, m\}, X)$ for some appropriate $\{k, m\}$ and X, or $T = L(k, M, Y)$ for some appropriate k, M, Y.*

Proof. Let T be an isolated subsemigroup of $(\mathcal{T}_n^{(l-1)}, *)$. Proposition 17(d) implies that either $T \subset \mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$ for some appropriate m and k or $T \subset \mathcal{T}_n^{(l-1,k)}$ for some appropriate k.

If $T \subset \mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$ and $T \subset \mathcal{T}_n^{(l-1,k)}$ at the same time, then $T \subset \mathcal{T}_n^{(l-1,k,m)}$ and hence $T = H(k, m, X, Y)$ for some appropriate X and Y by Proposition 18.

If $T \subset \mathcal{T}_n^{(l-1,k,m)} \cup \mathcal{T}_n^{(l-1,m,k)}$ but $T \not\subset \mathcal{T}_n^{(l-1,k)}$ and $T \not\subset \mathcal{T}_n^{(l-1,m)}$, then $T = K(\{k, m\}, X)$ for some appropriate X by Proposition 19.

Finally, if $T \subset \mathcal{T}_n^{(l-1,k)}$ but $T \not\subset \mathcal{T}_n^{(l-1,k,m)}$ for any m, then $T = L(k, M, Y)$ for some appropriate M and Y by Proposition 21. This completes the proof. □

4.3. The case of stable rank at most $l-2$. During this subsection we assume that $l > 2$.

Proposition 24. *Let T be an isolated subsemigroup of $(\mathcal{T}_n, *)$ such that $T \not\subset \mathcal{T}_n^{(l)} \cup \mathcal{T}_n^{(l-1)}$. Then $T \supset \mathcal{T}_n \setminus \mathcal{T}_n^{(l)}$.*

To prove this we will need the following lemmas, in which without loss of generality we assume that $e_i \in A_i$ for all $i = 1, \dots, k$:

Lemma 25. *Let T be an isolated subsemigroup of $(\mathcal{T}_n, *)$. Assume that T contains an idempotent ε of rank k, given by* (1)*, where $1 < k < l-1$. Then T contains an idempotent of rank $k-1$ or less.*

Proof. Assume first that at least one of E_i's contains at least three elements from A. Without loss of generality we may assume that $E_1 \supset \{a_1, a_l, a_{l-1}\}$. Consider the following elements:

$$\beta = \begin{pmatrix} E_1 \setminus \{a_{l-1}\} & E_2 & \dots & E_k & a_{l-1} \\ e_1 & e_2 & \dots & e_k & a_l \end{pmatrix},$$
$$\gamma = \begin{pmatrix} E_1 \setminus \{a_l\} & E_2 & \dots & E_k & a_l \\ e_1 & e_2 & \dots & e_k & a_{l-1} \end{pmatrix},$$
$$\nu = \begin{pmatrix} E_1 \setminus \{a_l\} & E_2 & E_3 & \dots & E_k & a_l \\ e_1 & a_l & e_3 & \dots & e_k & e_2 \end{pmatrix}.$$

We have $\beta * \beta = \gamma * \gamma = \varepsilon$ and hence $\beta, \gamma \in T$. Thus T contains the element $\gamma * \beta$. But $\nu * \nu = \gamma * \beta$, implying $\nu \in T$. At the same time $\mathrm{rank}(\nu * \beta * \nu * \beta) = k-1$ and the statement follows.

Let us now assume that none of the idempotents in T of rank at most $l-2$ contains a block in the kernel having three or more elements from A. Then, without loss of generality, we can assume that $\{a_1, a_l\} \subset E_1$ and $\{a_2, a_{l-1}\} \subset E_2$. Consider the following elements:

$$\beta = \begin{pmatrix} E_1 \setminus \{a_l\} & E_2 & E_3 & \dots & E_k & a_l \\ e_2 & e_1 & e_3 & \dots & e_k & a_{l-1} \end{pmatrix},$$
$$\gamma = \begin{pmatrix} E_1 & E_2 \setminus \{a_{l-1}\} & E_3 & \dots & E_k & a_{l-1} \\ e_2 & e_1 & e_3 & \dots & e_k & a_l \end{pmatrix},$$
$$\nu = \begin{pmatrix} E_1 & E_2 \setminus \{a_{l-1}\} & E_3 & \dots & E_k & a_{l-1} \\ e_2 & e_1 & e_3 & \dots & e_k & a_{l-1} \end{pmatrix}.$$

We have $\beta * \beta = \gamma * \gamma = \varepsilon$ and hence $\beta, \gamma \in T$, in particular, $\gamma * \beta \in T$. At the same time $\nu * \nu = \gamma * \beta$, forcing $\nu \in T$. But then a_1, a_l and a_{l-1} belong to the same block of $\Lambda_{\nu * \varepsilon}$. Thus the element $\nu * \varepsilon * \nu * \varepsilon$, which is an idempotent in T of rank at most $l-2$, has a block with at least three elements from A. The obtained contradiction completes the proof. □

Lemma 26. *Let T be an isolated subsemigroup of $(\mathcal{T}_n, *)$. Assume that T contains an idempotent of rank 1 and $l > 2$. Then T contains all idempotents of $(\mathcal{T}_n, *)$, whose rank does not exceed $l-1$.*

Proof. First let us show that T contains all idempotents of rank 1. Without loss of generality we can assume that T originally contains θ_1 (the element with image $\{1\}$) and $1 \in A_1$. Let $y \in A_3$. Consider the following elements:

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_l \\ 1 & a_1 & \dots & a_{l-1} \end{pmatrix},$$
$$\gamma = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_{l-1} & A_l \\ 1 & y & a_4 & \dots & a_l & 1 \end{pmatrix},$$
$$\nu = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & \dots & A_l \\ y & y & 1 & a_4 & \dots & a_l \end{pmatrix}.$$

We have $\beta^{*n} = \gamma^{*n} = \theta_1$ and hence $\beta, \gamma \in T$. In particular, $\beta * \gamma \in T$. But $\nu * \nu = \beta * \gamma$, hence $\nu \in T$. At the same time $\theta_1 * \nu = \theta_y$ and thus $\theta_y \in T$. This actually shows that $\theta_y \in T$ for all $y \notin A_1$. But then, taking any idempotent of rank one with the image outside A_1, from the above it also follows that $\theta_y \in T$ for all $y \in A_1$. Observe that the construction above does not work in the cases $l = 1, 2$.

Now let ε be an idempotent of $(\mathcal{T}_n, *)$ of rank $k \leq l-1$ given by (1). Without loss of generality we may assume $a_l \in E_1$. Consider the following elements:

$$\beta = \begin{pmatrix} E_1 & E_2 & E_3 & \dots & E_{k-1} & E_k \\ e_1 & a_3 & a_4 & \dots & a_k & a_l \end{pmatrix},$$
$$\gamma = \begin{pmatrix} N \setminus \{a_3, \dots, a_k, a_l\} & a_3 & a_4 & \dots & a_k & a_l \\ e_1 & e_2 & e_3 & \dots & e_{k-1} & e_k \end{pmatrix}.$$

From the first part of the proof we have $\theta_{e_1} \in T$. We further have $\beta^{*n} = \gamma^{*n} = \theta_{e_1}$ and hence $\beta, \gamma \in T$. At the same time $\beta * \gamma = \varepsilon$ also must belong to T. This completes the proof. □

Now we are ready to prove Proposition 24:

Proof of Proposition 24. If $T \not\subset \mathcal{T}_n^{(l)} \cup \mathcal{T}_n^{(l-1)}$, it must contain an idempotent of rank $l-2$ or less. Applying Lemma 25 inductively we obtain that T must contain an idempotent of rank 1. From Lemma 26 it now follows that T contains all idempotents of rank $\leq l-1$. Since T is isolated, we derive that $T \supset \mathcal{T}_n \setminus \mathcal{T}_n^{(l)}$ and complete the proof. □

4.4. General classification of isolated subsemigroups. After the hard work in Subsections 4.1–4.3 we can now complete the classification of all isolated subsemigroups of $(\mathcal{T}_n, *)$ relatively easily. We write $\mathcal{T}_n$ as the disjoint union of the sets $\mathcal{T}_n^{(l)}$, $\mathcal{T}_n^{(l-1)}$, and $Z = \mathcal{T}_n \setminus (\mathcal{T}_n^{(l)} \cup \mathcal{T}_n^{(l-1)})$, and for every collection of these sets we will classify all isolated subsemigroups, which have non-trivial intersections exactly with the elements of this collection.

Theorem 27. *Let T be an isolated subsemigroup of $(\mathcal{T}_n, *)$. Then T is one of the following:*

(i) *$F(X, Y)$ for appropriate $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ (see Subsection 4.1);*
(ii) *$H(k, m, X, Y)$ for appropriate $k, m \in \{1, \dots, l\}$, $k \neq m$, $X \subset \mathcal{U}^{(k,m)}$, and $Y \subset \mathcal{V}^{(k)}$, if $l > 1$ (see Subsection 4.2);*

(iii) *$K(\{k,m\},X)$ for appropriate $\{k,m\}\subset\{1,\dots,l\}$, $k\neq m$, and $X\subset\mathcal{U}^{(k,m)}$, if $l>1$ (see Subsection 4.2);*

(iv) *$L(k,M,Y)$ for appropriate $k\in\{1,\dots,l\}$, $Y\subset\mathcal{V}^{(k)}$, and*

$$M\subset\{a_1,\dots,a_{k-1},a_{k+1},\dots,a_l\},\qquad |M|>1,$$

if $l>1$ (see Subsection 4.2);

(v) *$\mathcal{T}_n\setminus\mathcal{T}_n^{(l)}$, if $l>2$;*

(vi) *$F(X,Y)\cup(\mathcal{T}_n\setminus\mathcal{T}_n^{(l)})$, for appropriate $X\subset\mathcal{X}$ and $Y\subset\mathcal{Y}$, if $l>1$.*

All subsemigroups in the above list are different, in particular, the list above gives a complete classification of isolated subsemigroups of $(\mathcal{T}_n,*)$.

Proof. We have to consider the following cases:

Case 1: T intersects only with $\mathcal{T}_n^{(l)}$. In this case Corollary 12 gives (i).

Case 2: T intersects only with $\mathcal{T}_n^{(l-1)}$. This makes sense only if $l>1$. In this case Corollary 23 gives (ii), (iii) and (iv).

Case 3: T intersects only with Z. This makes sense only if $l>2$, when it is not possible by Proposition 24.

Case 4: T intersects only with Z and $\mathcal{T}_n^{(l)}$. This makes sense only if $l>2$, when it is not possible by Proposition 24.

Case 5: T intersects only with Z and $\mathcal{T}_n^{(l-1)}$. This makes sense only for $l>2$. Since $\mathcal{T}_n\setminus\mathcal{T}_n^{(l)}$ is an ideal and the complement of an isolated subsemigroup, it is an isolated subsemigroup itself. Hence in this case Proposition 24 gives (v).

Case 6: T intersects only with $\mathcal{T}_n^{(l)}$ and $\mathcal{T}_n^{(l-1)}$. This makes sense only for $l>1$. Let $\varepsilon\in T$ be an idempotent of rank l and $\nu\in T$ be an idempotent of rank $l-1$. Corollary 6 implies that for any permutation on A there exists an element from $G(\varepsilon)$ which induces this fixed permutation on A. In particular, if $l>2$ then, multiplying ν from the left and from the right with the elements from $G(\varepsilon)$, we can change $\mathfrak{b}(\nu)$ and $\mathfrak{t}(\nu)$ in an arbitrary way. Thus we can find some $\nu'\in T\cap\mathcal{T}_n^{(l-1)}$ such that neither $\mathfrak{d}(\nu)=\mathfrak{d}(\nu')$ nor $\mathfrak{t}(\nu)=\mathfrak{t}(\nu')$ holds. From Proposition 17(d) it follows that such T must intersect with Z, a contradiction. Hence for $l>2$ this case is not possible.

Let us now assume that $l=2$. Then $T\cap\mathcal{T}_n^{(2)}=F(X,Y)$ for appropriate X and Y because of Corollary 12. Further, Proposition 19(b) implies that $T\cap\mathcal{T}_n^{(1)}=\mathcal{T}_n^{(1)}=\mathcal{T}_n\setminus\mathcal{T}_n^{(2)}$, which gives the case (vi) for $l=2$.

Case 7: T intersects with $\mathcal{T}_n^{(l)}$, $\mathcal{T}_n^{(l-1)}$ and Z. This makes sense only if $l>2$. In this case Proposition 24 implies that T contains the ideal $\mathcal{T}_n\setminus\mathcal{T}_n^{(l)}$. The intersection with $\mathcal{T}_n^{(l)}$ should be one of $F(X,Y)$ and since $\mathcal{T}_n\setminus\mathcal{T}_n^{(l)}$ is an ideal, it follows that the union $F(X,Y)\cup(\mathcal{T}_n\setminus\mathcal{T}_n^{(l)})$ is a semigroup for all possible X and Y. It is isolated by Proposition 2. This gives (vi).

Since all possible cases are considered, the statement follows. □

Corollary 28. *Let $\varepsilon\in E(\mathcal{T}_n,*)$. Then $\sqrt{\varepsilon}$ is a subsemigroup if and only if* $\mathrm{rank}(\varepsilon)=l$ *or* $\mathrm{rank}(\varepsilon)=l-1$.

Corollary 29. *Set* $p = \prod_{i=1}^{l} |A_i|$ *and for* $k = 1, \dots, l$ *set* $p_k = \prod_{i\neq k} |A_i|$. *Then*
(a) *for* $l = 1$ *the semigroup* $(\mathcal{T}_n, *)$ *contains* $2^n - 1$ *isolated subsemigroups;*
(b) *for* $l = 2$ *the semigroup* $(\mathcal{T}_n, *)$ *contains*

$$2\left(2^{2^{n-2}+p} - 2^{2^{n-2}} - 2^p\right) + 2^{p_1} + 2^{p_2} + 1$$

isolated subsemigroups;
(c) *for* $l > 2$ *the semigroup* $(\mathcal{T}_n, *)$ *contains*

$$\binom{l}{2}\left(2^{(l-1)^{(n-l)}} - 1\right) + \sum_{k=1}^{l}(l-1)\cdot(2^{(l-1)^{n-l}+p_k} - 2^{(l-1)^{n-l}} - 2^{p_k} + 1)$$
$$+ \sum_{k=1}^{l}(2^{l-1} - l)(2^{p_k} - 1) + 2\cdot(2^{l^{n-l}+p} - 2^{l^{n-l}} - 2^p + 1) + 1$$

isolated subsemigroups.

Proof. All statements follow from Theorem 27, Proposition 11(d), Proposition 18(e), Proposition 19(c), and Proposition 21(c) using standard combinatorial computations. □

5. Classification of other types of isolated subsemigroups

5.1. The case when the sandwich element has rank $l > 1$.

Theorem 30. *Let* $l > 1$. *Then*
1. $\mathfrak{CI}(\mathcal{T}_n, *) = \{\mathcal{T}_n, \mathcal{T}_n^{(l)}, \mathcal{T}_n \setminus \mathcal{T}_n^{(l)}\}$
$\cup\{F(X, \mathcal{Y}), F(X, \mathcal{Y}) \cup (\mathcal{T}_n \setminus \mathcal{T}_n^{(l)}) : X \subset \mathcal{X}, X \neq \emptyset, \mathcal{X}\}$
$\cup\{F(\mathcal{X}, Y), F(\mathcal{X}, Y) \cup (\mathcal{T}_n \setminus \mathcal{T}_n^{(l)}) : Y \subset \mathcal{Y}, Y \neq \emptyset, \mathcal{Y}\}$.
2. $\mathfrak{RC}(\mathcal{T}_n, *) = \{\mathcal{T}_n\} \cup \{F(\mathcal{X}, Y) : Y \subset \mathcal{Y}, Y \neq \emptyset\}$.
3. $\mathfrak{LC}(\mathcal{T}_n, *) = \{\mathcal{T}_n\} \cup \{F(X, \mathcal{Y}) : X \subset \mathcal{X}, X \neq \emptyset\}$.
4. $\mathfrak{C}(\mathcal{T}_n, *) = \{\mathcal{T}_n, \mathcal{T}_n^{(l)}\}$.

Proof. We prove 1. and leave the proof of the other statements to the reader. Because of (IX), to prove 1. it is enough to check which of the semigroups listed in Theorem 27 are completely isolated, that is they have a complement, which is a semigroup itself.

$\overline{F(X,Y)}$ is a subsemigroup if and only if $\mathcal{T}_n^{(l)} \setminus F(X, Y)$ is a subsemigroup. Using Proposition 11(c) we easily obtain that the latter is equivalent to $X = \mathcal{X}$ or $Y = \mathcal{Y}$, that is $F(X, Y) \in \mathfrak{CI}(\mathcal{T}_n, *)$ if and only if $X = \mathcal{X}$ or $Y = \mathcal{Y}$.

If $H(k, m, X, Y)$ is completely isolated, then $\overline{H(k, m, X, Y)}$ is completely isolated as well, in particular, it is isolated. However, it contains $\mathcal{T}_n^{(l)}$ and intersects with $\mathcal{T}_n^{(l-1)}$, which, by Theorem 27, implies that $\overline{H(k, m, X, Y)} = \mathcal{T}_n$, a contradiction. Hence $H(k, m, X, Y)$ is not completely isolated. The same arguments also imply that neither the subsemigroup $K(\{k, m\}, X)$ nor $L(k, M, Y)$ is completely isolated.

$\mathcal{T}_n \setminus \mathcal{T}_n^{(l)}$ is completely isolated as the complement to the completely isolated subsemigroup $\mathcal{T}_n^{(l)} = F(\mathcal{X}, \mathcal{Y})$.

Finally, the complement to $F(X,Y) \cup (\mathcal{T}_n \setminus \mathcal{T}_n^{(l)})$ belongs to $\mathcal{T}_n^{(l)}$ and is completely isolated if and only if $F(X,Y) \cup (\mathcal{T}_n \setminus \mathcal{T}_n^{(l)})$ is. The above classification of completely isolated subsemigroups of $\mathcal{T}_n^{(l)}$ implies that $F(X,Y) \cup (\mathcal{T}_n \setminus \mathcal{T}_n^{(l)})$ is completely isolated if and only if $X = \mathcal{X}$ or $Y = \mathcal{Y}$. This completes the proof. □

5.2. The case when the sandwich element has rank 1.

Proposition 31. *Assume that $\alpha = \theta_1$. Then*

(a) $E(\mathcal{T}_n, *) = \{\theta_i : i = 1, \dots, n\}$.
(b) $\mathfrak{I}(\mathcal{T}_n, *) = \mathfrak{CI}(\mathcal{T}_n, *) = \mathfrak{RC}(\mathcal{T}_n, *) = \left\{\bigcup_{i \in X} \sqrt{\theta_i} : \emptyset \neq X \subset N\right\}$.
(c) $\mathfrak{LC}(\mathcal{T}_n, *) = \mathfrak{C}(\mathcal{T}_n, *) = \{\mathcal{T}_n\}$.

Proof. The statement (a) follows from Theorem 5(a). The statement about $\mathfrak{I}(\mathcal{T}_n, *)$ follows from Theorem 27. Lemma 9(c) implies that in the case $\alpha = \theta_1$ every isolated subsemigroup of $(\mathcal{T}_n, *)$ is right convex, in particular, is completely isolated. This proves (b). Again Lemma 9(c) implies that in the case $\alpha = \theta_1$ every proper subsemigroup of $(\mathcal{T}_n, *)$ can neither be left convex nor convex. This proves (c) and completes the proof. □

6. The case of the classical $\mathcal{T}_n$

In this section we separately formulate the main statement for the extreme case $l = n$, that is for the case of the classical $\mathcal{T}_n$, since this is clearly a case of independent interest. For $m, k \in N$, $m \neq k$, let $\varepsilon_{m,k}$ be defined as in (4). In this subsection we assume that $\alpha = \mathrm{id}$, that is $(\mathcal{T}_n, *) = \mathcal{T}_n$. Recall (see Corollary 6) that the maximal subgroup $G(\varepsilon_{m,k})$ consists of all $\beta \in \mathcal{T}_n$ such that $\beta(m) = \beta(k)$ and which induce a permutation on $N \setminus \{k\}$.

Proposition 32. (a) $\mathfrak{I}(\mathcal{T}_1) = \mathfrak{CI}(\mathcal{T}_1) = \mathfrak{LC}(\mathcal{T}_1) = \mathfrak{RC}(\mathcal{T}_1) = \mathfrak{C}(\mathcal{T}_1) = \{\mathcal{T}_1\}$.

(b) (i) $\mathfrak{I}(\mathcal{T}_2) = \{\mathcal{T}_2, \mathcal{S}(\{1,2\}), \mathcal{T}_2 \setminus \mathcal{S}(\{1,2\}), G(\varepsilon_{1,2}), G(\varepsilon_{2,1})\}$.
(ii) $\mathfrak{CI}(\mathcal{T}_2) = \{\mathcal{T}_2, \mathcal{S}(\{1,2\}), \mathcal{T}_2 \setminus \mathcal{S}(\{1,2\})\}$.
(iii) $\mathfrak{LC}(\mathcal{T}_2) = \mathfrak{RC}(\mathcal{T}_2) = \mathfrak{C}(\mathcal{T}_2) = \{\mathcal{S}(\{1,2\}), \mathcal{T}_2\}$.

(c) *Let $n > 2$. Then*
(i)
$$\begin{aligned}\mathfrak{I}(\mathcal{T}_n) = &\{\mathcal{T}_n, \mathcal{S}(N), \mathcal{T}_n \setminus \mathcal{S}(N)\}\\ &\cup \left\{\bigcup_{m \in M} G(\varepsilon_{m,k}) : k \in N, \emptyset \neq M \subset N \setminus \{k\}\right\}\\ &\cup \{G(\varepsilon_{m,k}) \cup G(\varepsilon_{k,m}) : m, k \in N, m \neq k\}.\end{aligned}$$
(ii) $\mathfrak{CI}(\mathcal{T}_n) = \{\mathcal{T}_n, \mathcal{S}(N), \mathcal{T}_n \setminus \mathcal{S}(N)\}$.
(iii) $\mathfrak{LC}(\mathcal{T}_n) = \mathfrak{RC}(\mathcal{T}_n) = \mathfrak{C}(\mathcal{T}_n) = \{\mathcal{S}(N), \mathcal{T}_n\}$.

Proof. The statement (c) follows from Theorem 27 and Theorem 30. The statements (a) and (b) are obvious. □

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