Acta Math. Univ. Comenianae
Vol. LXXVII, 1(2008), pp. 93–122

LIFTING MAPPINGS OVER INVARIANTS OF FINITE GROUPS

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Abstract. We characterize those regular, holomorphic or formal maps into the orbit space V/G of a complex representation of a finite group G which admit a regular, holomorphic or formal lift to the representation space V. In particular, the case of complex reflection groups is investigated.

1. Introduction

The problem of lifting of morphisms to the orbit space for real or complex representations of compact Lie groups was studied in several papers.

The existence of lifts of smooth diffeomorphisms of orbit spaces for a real representation of a compact Lie group G in a real vector space V was investigated in [**2**], [**12**], and [**6**]. The condition for the existence of lifts of smooth and analytic curves in orbit spaces for real representations of compact Lie groups was found in [**1**]. Note that in the real case the orbit space V/G has a natural structure of a real semialgebraic subset of some space $\mathbb{R}^m$ and its stratification coincides with the isotropy type stratification of V/G. This stratification plays the main role in the results mentioned above.

In the complex case the lifting problem is more complicated and only the case of a representation of a finite group G in a complex vector space V was studied. In this case the orbit space $Z = V/G$ coincides with the categorical quotient $V/\!\!/G$ which is a normal affine variety. Therefore the orbit space Z has the natural structure of a complex analytic set and there are several types of morphisms into $V/\!\!/G$, like regular, rational or holomorphic. To formulate the condition of lifting one needs to use the isotropy type (Luna) stratification of the orbit space V/G which is finer than its stratification as affine variety. The conditions of lifting for holomorphic automorphisms of orbit spaces were found for the Weyl groups in [**9**] and for any finite groups in [**4**]. In [**7**] it was proved that each holomorphic lift of a regular automorphism of the orbit space is regular.

Received November 13, 2006.
2000 *Mathematics Subject Classification.* Primary 14L24, 14L30.
Key words and phrases. invariants, lifts.
M. L., P. W. M., and A. R. were supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 14195 MAT, P 17108-N04'.

In this paper we consider the conditions for lifts of germs of holomorphic morphisms at 0 from $\mathbb{C}^p$ to Z, for lifts of regular maps from $\mathbb{C}^p$ to Z, and for lifts of formal morphisms from $\mathbb{C}^p$ to Z, i.e., the morphisms of the $\mathbb{C}$-algebra $\mathbb{C}[Z]$ to the ring of formal power series $\mathbb{C}[[X_1,\dots,X_p]]$ in variables $X_1,\dots,X_p$. These conditions are formulated with the use of the spaces $J_0^q(\mathbb{C}^p,Z)$ and $J_0^\infty(\mathbb{C}^p,Z)$ of jets at $0\in\mathbb{C}^p$ of orders q and ∞, respectively. In particular, we consider these conditions in the case when G is a finite group generated by complex reflections. Finally, we calculate the above conditions for some simple cases.

Algebraically, the above problems of lifting could be reformulated as partial cases of the general problem of extension of morphisms in the category of $\mathbb{C}$-algebras from a subalgebra to the whole algebra.

Note that, by Lefschetz's principle (see, for example, [**14**]), all conditions of lifting which are formulated in algebraic terms are valid for any field of characteristic zero.

In Section 2 we formulate the types of lifting problems which are solved in the paper and introduce the tools we need for this: Luna's stratification and the jet spaces for affine varieties, in particular, for a G-module V and for the orbit space $Z=V/G$.

In Section 3 we define the functions $\widetilde{T}(A_1,\dots,A_d)$ and $\widetilde{\mathfrak{T}}(A_1,\dots,A_d)$, which are used in the conditions of lifting for holomorphic, regular, and formal lifts and study their properties.

In Section 4 we obtain sufficient conditions for the existence of local and global holomorphic lifts and regular lifts.

In Section 5 we obtain a sufficient condition for the existence of formal lifts.

In Section 6 we consider the above sufficient conditions for lifting for complex reflection groups and calculate some of these conditions for a reflection group in $\mathbb{C}$ and for the dihedral groups.

2. Preliminaries

2.1. Luna stratification of orbit spaces. Let V be an n-dimensional complex vector space, G a finite subgroup of $GL(V)$, and $\mathbb{C}[V]^G$ the algebra of G-invariant polynomials on V.

The following facts are well known (see, for example [**11**]). Denote by Z the categorical quotient $V/\!\!/G$, i.e., the normal affine algebraic variety with the coordinate ring $\mathbb{C}[V]^G$. Since the group G is finite, the categorical quotient $V/\!\!/G$ is the geometric one, i.e., $V/\!\!/G$ is the orbit space V/G. Let $\pi=\pi_V:V\to Z$ be the quotient projection. The affine algebraic variety Z has the natural structure of a complex analytic space: Let $\sigma_1,\dots,\sigma_m$ be the minimal system of homogeneous generators of the algebra $\mathbb{C}[V]^G$ and let $\sigma:V\to\sigma(V)\subseteq\mathbb{C}^m$ be the corresponding morphism. Then $\sigma(V)$ is an irreducible Zariski-closed subset of $\mathbb{C}^m$ which is isomorphic to the affine variety Z. For this presentation of Z the morphism $\sigma:V\to\sigma(V)$ coincides with the projection π.

In the sequel we assume that the minimal system of homogeneous generators $\sigma_1,\dots,\sigma_m$ is fixed and they are called the basic generators.

Let K be a subgroup of G. We denote by $V_{(K)}$ the set of points of V whose isotropy groups are conjugate to K. By definition, $V_{(K)} \subseteq \cup_{g \in G} V^{gKg^{-1}}$, where V^K is the subspace of V of fixed points of the action of K on V. Put $Z_{(K)} := \pi(V_{(K)})$. It is known that $\{Z_{(K)} : K < G\}$ is a finite stratification of Z into locally closed irreducible smooth algebraic subvarieties. This is the simplest case of a Luna stratification, see [**8**]. Put $V_0 := V_{(K)}$ for $K = \{\mathrm{id}\}$ and $Z_0 := \pi(V_0)$.

Denote by $Z_{>i}$ the union of the strata of codimension greater than i and put $Z_{\le i} := Z \setminus Z_{>i}$. Then $Z_{>i}$ is a Zariski-closed subset of Z and $Z_0 = Z_{\le 0} = Z \setminus Z_{>0}$ is a stratum of Z called the principal stratum. Points in Z_0 and in V_0 are called regular points. The following proposition is evident.

Proposition. *Z_0 is a Zariski-open smooth subvariety of Z and the restriction of π to V_0 is an étale morphism onto Z_0.*

2.2. Polarizations. Let α be a homogeneous G-invariant polynomial of degree d on V and let $\alpha^s : V^d \to \mathbb{C}$ be the corresponding symmetric d-linear form on V. For $v_1, \dots, v_k \in V$ we have

$$\begin{aligned}\alpha(t_1 v_1 + \dots + t_k v_k) &= \alpha^s(t_1 v_1 + \dots + t_k v_k, \dots, t_1 v_1 + \dots + t_k v_k) \\ &= \sum_{i_1, \dots, i_d} t_{i_1} \dots t_{i_d}\, \alpha^s(v_{i_1}, \dots, v_{i_d}) \\ &= \sum_{r_1 + \dots + r_k = d} t_1^{r_1} \dots t_k^{r_k}\, \alpha_{r_1, \dots, r_k}(v_1, \dots, v_k),\end{aligned}$$

where

$$\alpha_{r_1, \dots, r_k}(v_1, \dots, v_k) := \frac{d!}{r_1! \dots r_k!}\, \alpha^s(\underbrace{v_1, \dots, v_1}_{r_1 \text{ times}}, \dots, \underbrace{v_k, \dots, v_k}_{r_k \text{ times}})$$

and $t_1, \dots, t_k$ are variables.

The polynomials $\alpha_{r_1, \dots, r_k}$ are the usual polarizations of α on V^k.

2.3. Invariant coordinates on V. For each regular point $z_0 \in Z_0$ there is a system of regular functions $z_1, \dots, z_n$ on Z such that each $y_i := z_i \circ \pi$ equals one of the generators σ_j and the functions $z_i - z_i(z_0)$ are local parameters at z_0. Then the y_i are local coordinates on V in a neighborhood of each point $v \in \pi^{-1}(z_0)$. By definition, the functions y_i are G-invariant. These coordinates y_i are called *invariant coordinates* on V.

Since we fixed the basic generators $\sigma_1, \dots, \sigma_m$, there are only finitely many choices of such invariant coordinates on V.

Let e_i $(i = 1, \dots, n)$ be a basis of V and u_i the corresponding coordinates on V. Denote by $J \in \mathbb{C}[V]$ the Jacobian $\det\left(\frac{\partial y_i}{\partial u_j}\right)$. It is clear that J is a homogeneous polynomial.

Proposition. *For each integer $k > 0$ there is a G-invariant polynomial $\Delta_k \in \mathbb{C}[V]^G$ of minimal degree such that J^k divides Δ_k and the sets of zeros of J and Δ_k coincide. The polynomial Δ_k is unique up to a nonzero factor $c \in \mathbb{C}$.*

Proof. Let $J = f_1^{n_1} \dots f_s^{n_s}$ be a decomposition of J into the product of linearly independent irreducible polynomials $f_l \in \mathbb{C}[V]$. Consider the principal effective divisor $(J) = n_1(f_1) + \cdots + n_s(f_s)$ of the polynomial J on V. Since for each $g \in G$ we have $J \circ g = \det(g_j^i).J$, where (g_j^i) is the matrix of g in the basis e_i, the divisor (J) is G-invariant. Then each $g \in G$ permutes the prime divisors (f_p) $(p = 1, \dots, s)$. This implies that, if $g(f_p) = (f_q)$, the coefficients n_p and n_q of the divisor (J) are equal. Let $\{m_1, \dots, m_l\}$ be the set of distinct coefficients of the divisor (J) and let, for each m_α, Φ_α be the product of distinct factors f_p of J having the same power m_α in the above decomposition of J. Then we have $J = \prod_{\alpha=1}^l \Phi_\alpha^{m_\alpha}$. By the above arguments, for each $\alpha = 1, \dots, l$ the divisor (Φ_α) of the polynomial Φ_α is G-invariant.

Since the group G is finite, for each $\alpha = 1, \dots, l$ there is a minimal integer $p_\alpha > 0$ such that the polynomial $\Phi_\alpha^{p_\alpha}$ is G-invariant. For $\alpha = 1, \dots, l$ let $km_\alpha = s_\alpha p_\alpha + r_\alpha$, where s_α and r_α are unique nonnegative integers such that $0 \le r_\alpha < p_\alpha$. Then we have $J^k = \prod_{\alpha=1}^l \Phi_\alpha^{s_\alpha p_\alpha + r_\alpha}$. Let μ_α be the least common multiple of r_α and p_α. Then $\Delta_k = \prod_{\alpha=1}^l \Phi_\alpha^{s_\alpha p_\alpha + \mu_\alpha}$ is a G-invariant polynomial of minimal degree such that the sets of zeros of J and Δ_k coincide and J^k divides Δ_k.

By the above formula for J^k, for each G-invariant polynomial P such that the sets of zeros of J and P coincide and J^k divides P, Δ_k divides P. $\square$

We denote by $\widetilde{\Delta}_k$ the regular function on Z such that $\widetilde{\Delta}_k \circ \pi = \Delta_k$. By definition, we have $\widetilde{\Delta}_k(z_0) \neq 0$ for each k. Conversely, let y_i be invariant coordinates on V, let z_i be the corresponding regular functions on Z, and for some positive integer k let $\widetilde{\Delta}_k$ be the corresponding regular function on Z. If, for a point $z \in Z$, we have $\widetilde{\Delta}_k(z) \neq 0$, then $z \in Z_0$.

Later, for the sake of simplicity, we put $\Delta := \Delta_1$ and $\widetilde{\Delta} := \widetilde{\Delta}_1$.

Denote by $V(\widetilde{\Delta})$ the set of zeros of $\widetilde{\Delta}$. Thus $Z_{>0}$ is the intersection of the Zariski-closed subsets $V(\widetilde{\Delta})$ obtained from all choices of invariant coordinates constructed from the basic generators of $\mathbb{C}[V]^G$. The similar statement is true for $V \setminus V_0$ if we replace $\widetilde{\Delta}$ by Δ.

2.4. Jet spaces. Now, for an affine variety X, we will define the space $J_0^q(\mathbb{C}^p, X)$ of q-jets at 0 of morphisms from $\mathbb{C}^p$ to X. It is similar to the corresponding notion of jet spaces for affine schemes of finite type over $\mathbb{C}$ in the case $p = 1$ (see, for example, [**10**] and [**5**]).

We will consider now the category of $\mathbb{C}$-algebras.

Let $\mathbb{C}[X_1, \dots, X_p]$ be the $\mathbb{C}$-algebra of polynomials in variables $X_1, \dots, X_p$ with complex coefficients and let $\mathfrak{m}_p$ be the ideal of $\mathbb{C}[X_1, \dots, X_p]$ generated by the $X_1, \dots, X_p$. Put $\mathfrak{m}_p^q := (\mathfrak{m}_p)^q$. Then

$$\mathfrak{J}_p^q := \mathbb{C}[X_1, \dots, X_p]/\mathfrak{m}_p^{q+1}$$

is the truncated ring of polynomials, the model jet algebra. In particular, $\mathfrak{J}_p^0 = \mathbb{C}[X_1, \dots, X_p]/\mathfrak{m}_p = \mathbb{C}$.

Let $A = (a_1, \dots, a_s)$ for $a_1, \dots, a_s \in \{1, \dots, p\}$ be a (unordered) multi-index of order $|A| := s$. In particular, for $s = 0$ we put $A := \emptyset$. Denote by $\mathfrak{A}_{p,q}$ the set of multi-indices A of orders $\leq q$. By definition, each $P \in \mathfrak{J}_p^q$ can be written as $P = \sum_{A \in \mathfrak{A}_{p,q}} p_A \, X_A$, where $p_A \in \mathbb{C}$ and $X_A := X_{a_1} \dots X_{a_s}$. The natural bijection $P \mapsto (p_A)_{A \in \mathfrak{A}_{p,q}}$ between $\mathfrak{J}_p^q$ and $\mathbb{C}^{\mathfrak{A}_{p,q}}$ is an isomorphism of vector spaces and defines a structure of affine space on $\mathfrak{J}_p^q$. For $q \leq r$, consider the natural morphism $\rho_{r,q} : \mathfrak{J}_p^r \to \mathfrak{J}_p^q$.

For an affine variety X over $\mathbb{C}$, the set of $\mathfrak{J}_p^q$-valued points of X, i.e., morphisms from the coordinate ring $\mathbb{C}[X]$ of X to the ring $\mathfrak{J}_p^q$, is called the *space of q-jets* of morphisms from $\mathbb{C}^p$ to X at $0 \in \mathbb{C}^p$ and is denoted by $J_0^q(\mathbb{C}^p, X)$. In particular, we have $J_0^0(\mathbb{C}^p, X) = X$ and $J_0^1(\mathbb{C}^p, X) = TX$, the total tangent bundle of X.

It is evident that each polynomial function on $\mathfrak{J}_p^q$ defines a function on $J_0^q(\mathbb{C}^p, X)$ and these functions generate a ring of $\mathbb{C}$-valued functions on $J_0^q(\mathbb{C}^p, X)$. It is clear that this ring is a finitely generated $\mathbb{C}$-algebra. Then $J_0^q(\mathbb{C}^p, X)$ supplied with this ring has a structure of an affine variety (not necessarily irreducible). For two affine varieties X_1 and X_2 and for a morphism $\varphi : X_1 \to X_2$, we have the natural morphism $J_0^q(\mathbb{C}^p, \varphi) : \bar{J}_0^q(\mathbb{C}^p, X_1) \to \bar{J}_0^q(\mathbb{C}^p, X_2)$ of affine varieties. Thus one can consider $J_0^q(\mathbb{C}^p, \quad)$ as a covariant functor from the category of affine varieties to itself.

For each $h \in J_0^q(\mathbb{C}^p, X)$ there is a unique point $x \in X$ such that the corresponding maximal ideal $\mathfrak{m}_x$ coincides with the kernel of the composition $\rho_{q,0} \circ h$. Then the morphism h can be extended uniquely to a morphism from $\mathcal{O}_x$ to $\mathfrak{J}_p^q$ vanishing on $\mathfrak{m}_x^{q+1}$ and hence induces a morphism $h_{x,q} : \mathcal{O}_x/\mathfrak{m}_x^{q+1} \to \mathfrak{J}_p^q$ which, in turn, determines the initial morphism h uniquely. Therefore one can view $J_0^q(\mathbb{C}^p, X)$ as the set of morphisms from the local rings $\mathcal{O}_x/\mathfrak{m}_x^{q+1}$ to $\mathfrak{J}_p^q$ for all $x \in X$.

$$\begin{tikzcd}
& \mathbb{C}[X] \arrow[r, "h"] \arrow[d] & \mathfrak{J}_p^q \\
\mathfrak{m}_x^{q+1} \arrow[r, hook] & \mathcal{O}_x \arrow[r] \arrow[ur, dashed, "h"] & \mathcal{O}_x/\mathfrak{m}_x^{q+1} \arrow[u, dashed, "h_{x,q}"']
\end{tikzcd}$$

Assume that X is presented as a Zariski-closed subset of $\mathbb{C}^m$ defined by an ideal $(\Phi_1, \dots, \Phi_r)$ of the ring $\mathbb{C}[W_1, \dots, W_m]$ of polynomials with complex coefficients in variables $W_1, \dots, W_m$.

A morphism $h : \mathbb{C}[X] \to \mathfrak{J}_p^q$ is defined by a morphism $h' : \mathbb{C}[W_1, \dots, W_m] \to \mathfrak{J}_p^q$ with $h'(\Phi_l) = 0$ for $l = 1, \dots, r$. It is determined by $h'(W_i) = \sum_{A \in \mathfrak{A}_{p,q}} W_{i,A} \, X_A$, where $i = 1, \dots, m$ and $W_{i,A} \in \mathbb{C}$. The condition $h'(\Phi_l) = 0$ is equivalent to the vanishing of all the coefficients of the variables X_A in $h(\Phi_l)$. Thus the map $J_0^q(\mathbb{C}^p, X) \to (\mathbb{C}^m)^{\mathfrak{A}_{p,q}}$ given by $h \mapsto (W_{i,A})_{i=1,\dots,m,\, A \in \mathfrak{A}_{p,q}}$ induces a bijective correspondence between $J_0^q(\mathbb{C}^p, X)$ and the Zariski-closed subset of $(\mathbb{C}^m)^{\mathfrak{A}_{p,q}}$ defined by $r|\mathfrak{A}_{p,q}|$ many polynomial equations, where $|\mathfrak{A}_{p,q}|$ denotes the cardinality of the set $\mathfrak{A}_{p,q}$. By definition, this correspondence is an isomorphism of affine varieties.

The homomorphism $\rho_{r,q}$ induces the morphism

$$p_{X,r,q} : J_0^r(\mathbb{C}^p, X) \to J_0^q(\mathbb{C}^p, X).$$

In particular, we have the morphism $p_{X,q,0} : J_0^q(\mathbb{C}^p, X) \to X$.

The projective limit $J_0^\infty(\mathbb{C}^p, X) = \varprojlim J_0^q(\mathbb{C}^p, X)$ is called the *space of ∞-jets* at $0 \in \mathbb{C}^p$ of morphisms from $\mathbb{C}^p$ to X or the *space of formal morphisms* from $\mathbb{C}^p$ to X. By the definition of a projective limit we have natural projections $p_{X,\infty,q} : J_0^\infty(\mathbb{C}^p, X) \to J_0^q(\mathbb{C}^p, X)$. By definition, one can consider a point of $J_0^\infty(\mathbb{C}^p, X)$ either as a morphism $\mathbb{C}[X] \to \mathbb{C}[[X_1, \dots, X_p]]$ or as a morphism from the completion $\widetilde{\mathcal{O}_x}$ of the local ring $\mathcal{O}_x$ for some $x \in X$ to $\mathbb{C}[[X_1, \dots, X_p]]$.

In particular, for the above presentation of X as a Zariski-closed subset of $\mathbb{C}^m$, each $h \in J_0^\infty(\mathbb{C}^p, X)$ is uniquely defined by a morphism $h' : \mathbb{C}[W_1, \dots, W_m] \to \mathbb{C}[[X_1, \dots, X_p]]$ with $h'(\Phi_l) = 0$ for $l = 1, \dots, r$. It is defined by $h'(W_i) = h_i \in \mathbb{C}[[X_1, \dots, X_p]]$, where $\Phi_l(h_i) = 0$ for each $l = 1, \dots, r$.

Consider X as a complex analytic set and let $h : \mathbb{C}^p, 0 \to X$ be a germ of a holomorphic map at $0 \in \mathbb{C}^p$. Denote by $\mathfrak{F}_{\mathbb{C}^p,0}$ and $\mathfrak{F}_{X,x}$ the rings of germs of holomorphic functions on $\mathbb{C}^p$ at 0 and on X at x respectively. We may identify the ring $\mathfrak{F}_{\mathbb{C}^p,0}$ with a subring of the ring $\mathbb{C}[[X_1, \dots, X_p]]$. Consider the morphism $h^* : \mathfrak{F}_{X,x} \to \mathfrak{F}_{\mathbb{C}^p,0} \subseteq \mathbb{C}[[X_1, \dots, X_p]]$ induced by h. The restriction of h^* to $\mathcal{O}_{X,x}$, which is denoted by $j_0^\infty h$, belongs to $J_0^\infty(\mathbb{C}^p, X)$ and is called the ∞-jet of h at 0. Put $j_0^q h := p_{X,\infty,q}(j_0^\infty h)$ and call $j_0^q h$ the q-jet of h at 0.

Denote by $x_1, \dots, x_p$ the standard coordinates in $\mathbb{C}^p$. Let $A = (a_1, \dots, a_s)$ be a multi-index, W a finite dimensional complex vector space, and $F : \mathbb{C}^p, 0 \to W$ a germ of a holomorphic map, i.e., $F \in W \otimes \mathfrak{F}_{\mathbb{C}^p,0}$. We denote by ∂_A a linear operator on $W \otimes \mathfrak{F}_{\mathbb{C}^p,0}$ which is equal to the tensor product of the identical operator on W and the operator $\frac{\partial^{|A|}}{\partial x_{a_1} \dots \partial x_{a_s}}$ on $\mathfrak{F}_{\mathbb{C}^p,0}$. In particular, we have $\partial_\emptyset F = F$ and we write $\partial_a F$ instead of $\partial_{(a)} F$.

For the above presentation of X as a Zariski-closed subset in $\mathbb{C}^m$ the holomorphic germ $h : \mathbb{C}^p, 0 \to X$ can be given by a holomorphic map F from a neighborhood of $0 \in \mathbb{C}^p$ to $\mathbb{C}^m$ such that $\Phi_l \circ F = 0$ for each $l = 1, \dots, r$. Denote by $\mathfrak{A}_p$ the set of all multi-indices $A = (a_1, \dots, a_s)$. By definition, the ∞-jet $j_0^\infty h$ is uniquely determined by the indexed set $(\partial_A F(0))_{A \in \mathfrak{A}_p}$ of complex numbers. The complex numbers $\partial_A F(0)$ satisfy the equations $\partial_A(\Phi_l \circ F)(0) = 0$ for $A \in \mathfrak{A}_p$ and do not depend on the choice of F. Similarly, the q-jet $j_0^q F$ is determined by the indexed set $(\partial_A F(0))_{A \in \mathfrak{A}_{p,q}}$ of complex numbers satisfying the equations $\partial_A(\Phi_l \circ F)(0) = 0$ for all $A \in \mathfrak{A}_{p,q}$. The above considerations show that for a smooth point $x \in X$ our notion of jets coincide with the usual one.

Note that the jet spaces of holomorphic functions and of regular functions on affine varieties coincide.

Later we denote by ∂_A also the linear operator on $W \otimes \mathbb{C}[[X_1, \dots, X_p]]$ which is equal to the tensor product of the identical operator on W and the operator $\frac{\partial^{|A|}}{\partial x_{a_1} \dots \partial x_{a_s}}$ on $\mathbb{C}[[X_1, \dots, X_p]]$.

2.5. Consider a G-module V, the spaces of q-jets $J_0^q(\mathbb{C}^p, V)$, and $J_0^q(\mathbb{C}^p, Z)$, and the sets of formal morphisms $J_0^\infty(\mathbb{C}^p, V)$ and $J_0^\infty(\mathbb{C}^p, Z)$. The projection $\pi : V \to Z$ induces the morphism $J_0^q(\mathbb{C}^p, \pi) : J_0^q(\mathbb{C}^p, V) \to J_0^q(\mathbb{C}^p, Z)$ and the map $J_0^\infty(\mathbb{C}^p, \pi) : J_0^\infty(\mathbb{C}^p, V) \to J_0^\infty(\mathbb{C}^p, Z)$.

The standard action of the group G on $\mathbb{C}[V]$ induces an action of G on $J_0^q(\mathbb{C}^p, V)$ by automorphisms of $J_0^q(\mathbb{C}^p, V)$ as an affine variety and on $J_0^\infty(\mathbb{C}^p, V)$. The inclusion $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$, the morphism $J_0^q(\mathbb{C}^p, \pi)$, and the map $J_0^\infty(\mathbb{C}^p, \pi)$ induce the morphism

$$\pi^q : J_0^q(\mathbb{C}^p, V)/G \to J_0^q(\mathbb{C}^p, Z)$$

and the map

$$\pi^\infty : J_0^\infty(\mathbb{C}^p, V)/G \to J_0^\infty(\mathbb{C}^p, Z).$$

Denote by $\bar{J}_0^q(\mathbb{C}^p, Z)$ the Zariski closure of $\pi^q(J_0^q(\mathbb{C}^p, V)/G)$ in $J_0^q(\mathbb{C}^p, Z)$.

Proposition. *The morphism $\pi^q : J_0^q(\mathbb{C}^p, V)/G \to J_0^q(\mathbb{C}^p, Z)$ induces a birational morphism of $J_0^q(\mathbb{C}^p, V)/G$ to $\bar{J}_0^q(\mathbb{C}^p, Z)$.*

Proof. The group G acts freely on the open subset $p_{V,q,0}^{-1}(V_0) \subseteq J_0^q(\mathbb{C}^p, V)$. Since all points of V_0 and Z_0 are smooth, for each $v \in V_0$ the morphism $J_0^q(\mathbb{C}^p, \pi)$ induces a bijective map of $p_{V,q,0}^{-1}(v)$ onto $p_{Z,q,0}^{-1}(\pi(v))$. Thus the morphism π^q maps the Zariski-open subset $p_{V,q,0}^{-1}(V_0)/G$ of $J_0^q(\mathbb{C}^p, V)/G$ onto the Zariski-open subset $p_{Z,q,0}^{-1}(Z_0)$ of $J_0^q(\mathbb{C}^p, Z)$ bijectively and this implies the statement of the proposition. $\square$

2.6. Evidently we have the following bijections

$$J_0^q(\mathbb{C}^p, V) = \mathrm{Hom}(\mathbb{C}[V], \mathfrak{J}_p^q) = \mathrm{Lin}(V^*, \mathfrak{J}_p^q) = V \otimes \mathfrak{J}_p^q,$$

where V^* is the dual vector space for V, Hom means the set of morphisms in the category of $\mathbb{C}$-algebras, and Lin means the set of linear mappings. So each $h \in J_0^q(\mathbb{C}^p, V) = V \otimes (\mathfrak{J}_p^q)$ can be written uniquely as $h = \sum_{A \in \mathfrak{A}_{p,q}} h_A \otimes X_A$, where $h_A \in V$. Similarly, $J_0^\infty(\mathbb{C}^p, V) = V \otimes \mathbb{C}[[X_1, \dots, X_p]]$ and, for $h \in J_0^\infty(\mathbb{C}^p, V)$, we have $h = \sum_{A \in \mathfrak{A}_p} h_A \otimes X_A$, where $h_A \in V$.

Proposition. *The space of jets $J_0^q(\mathbb{C}^p, V)$ is isomorphic to the affine space $V^{\mathfrak{A}_{p,q}}$. Moreover, there are isomorphisms of G-modules*

$$J_0^q(\mathbb{C}^p, V) = V^{\mathfrak{A}_{p,q}}, \quad \text{and} \quad J_0^\infty(\mathbb{C}^p, V) = V^{\mathfrak{A}_p}$$

where the G-action on the products is the diagonal one.

Proof. The first statement follows from the definition of the structure of the affine variety on $J_0^q(\mathbb{C}^p, V)$. The maps $J_0^q(\mathbb{C}^p, V) \ni h \mapsto (h_A)_{A \in \mathfrak{A}_{p,q}}$ and $J_0^\infty(\mathbb{C}^p, V) \ni h \mapsto (h_A)_{A \in \mathfrak{A}_p}$ give the required isomorphisms of G-modules. $\square$

Note that for

$$A = (\underbrace{1, \dots, 1}_{r_1 \text{ times}}, \dots, \underbrace{p, \dots, p}_{r_p \text{ times}}),$$

where $r_1, \dots, r_p \geq 0$, and for a germ of a holomorphic map $F : \mathbb{C}^p, 0 \to V$ we have $h = j_0^\infty F = \sum_A \frac{1}{A!} \partial_A F(0) \otimes X_A$, where $A! = r_1! \dots r_p!$. A similar formula is true for the q-jet $j_0^q F$.

2.7. The problem of lifting. We consider the following problem. Let $f : \mathbb{C}^p \to Z$ be a rational morphism which is regular on a Zariski-open subset U of $\mathbb{C}^p$. A rational morphism $F : \mathbb{C}^p \to V$ which is regular on U is called a *regular lift* of f if $\pi \circ F = f$.

$$\begin{tikzcd} & V \arrow[d, "\pi"] \\ \mathbb{C}^p \arrow[ur, "F"] \arrow[r, "f"'] & Z \end{tikzcd}$$

Similarly, let U be a connected classically open subset U of $\mathbb{C}^p$ and let $U \to Z$ be a holomorphic map, i.e., a morphism in the category of complex analytic sets. A holomorphic map $F : U \to V$ is called a *holomorphic lift* of f if $\pi \circ F = f$. If $f : \mathbb{C}^p, x \to Z$ is a germ at $x \in \mathbb{C}^p$ of a holomorphic map from $\mathbb{C}^p$ to Z, a germ $F : \mathbb{C}^p, x \to V$ at x of a holomorphic map from $\mathbb{C}^p$ to V is called a *local lift* of f if $\pi \circ F = f$.

Let $f \in J_0^\infty(\mathbb{C}^p, Z)$ be a formal morphism. A formal morphism $F \in J_0^\infty(\mathbb{C}^p, V)$ is called a *formal lift* of f if $J_0^\infty(\mathbb{C}^p, \pi) \circ F = f$.

The aim of this paper is to find conditions for the existence of all these lifts.

A germ $f : \mathbb{C}^p, x \to Z$ at $x \in \mathbb{C}^p$ of a holomorphic map from $\mathbb{C}^p$ to Z is called *quasiregular* if $f^{-1}(Z_0)$ meets any neighborhood of x.

By 2.3, if a germ f is quasiregular, there is a choice of invariant coordinates y_i such that for the corresponding $\widetilde{\Delta}$ the composition $\widetilde{\Delta} \circ f$ does not vanish identically.

A formal morphism $f : \mathbb{C}[Z] \to \mathbb{C}[[X_1, \dots, X_p]]$ from $\mathbb{C}^p$ to Z is called *quasiregular* if there is no stratum S of Z of codimension ≥ 1 such that f factors through a morphism $\mathbb{C}[\bar{S}] \to \mathbb{C}[[X_1, \dots, X_p]]$, where $\bar{S}$ is the closure of S.

We claim that *for a quasiregular formal morphism f and for any choice of invariant coordinates y_i, for the corresponding $\widetilde{\Delta}$ we have $f(\widetilde{\Delta}) \neq 0$.* By 2.3 the set of zeros of all Δ obtained from any choice of invariant coordinates coincides with the Zariski-closed subset $Z_{>0}$ of Z. If our claim is wrong, by Hilbert's Nullstellensatz the formal morphism f vanishes on the ideal $I(Z_{>0}) = \sqrt{I(Z_{>0})}$ of $\mathbb{C}[W_1, \dots, W_m]$ which defines $Z_{>0}$. Consider the standard presentation of $I(Z_{>0})$ as an intersection of a finite set of prime ideals of $\mathbb{C}[Z]$ corresponding to the decomposition of $Z_{>0}$ into irreducible components. Since $\mathbb{C}[[X_1, \dots, X_p]]$ is an integral domain, f vanishes on at least one of these prime ideals. But each of these prime ideals defines a component of $Z_{>0}$ and such components are the closures of strata of Z of codimension ≥ 1. This contradicts our assumption.

Let K be a subgroup of G and let V^K be a subspace of V of fixed points of the action of K on V. Let $H = N_G(K)$ be the normalizer of K in G consisting of all elements of G preserving V^K, and let $W = H/K$ be the corresponding quotient group acting naturally on V^K. By definition, $\pi(V^K)$ is the closure $\bar{S}$ of a stratum S of Z.

Consider the natural map $\kappa : V^K/W \to \pi(V^K) = \bar{S}$. It is evidently bijective and regular since the natural map $V/W \to V/G$ is regular and $\pi(V^K) = \bar{S}$ is a Zariski-closed subset of Z as the projection π is a finite morphism. Then the morphism κ is birational. Denote by $\bar{S}_{\mathrm{nor}}$ the set of all normal points of $\bar{S}$, i.e., points $x \in \bar{S}$ such that the local ring $\mathcal{O}_x(\bar{S})$ is integrally closed. It is known that $\bar{S}_{\mathrm{nor}}$ is a Zariski-open subset of $\bar{S}$ and $S \subseteq \bar{S}_{\mathrm{nor}}$ since S is smooth. Since the affine variety V^K/W is normal, by Zariski's main theorem, the restriction $\kappa|_{\bar{S}_{\mathrm{nor}}}$ of κ to $\bar{S}_{\mathrm{nor}}$ induces an isomorphism between the algebraic varieties $\pi^{-1}(\bar{S}_{\mathrm{nor}})$ and $\bar{S}_{\mathrm{nor}}$.

Assume that, for a holomorphic map f as above, $f(U)$ is contained in $Z_{>0}$. Then $f(U)$ is contained in the closure $\bar{S}$ of a stratum S of Z of codimension ≥ 1. Namely, let $f(U) \subseteq Z_{>i-1}$ for maximal i. Then there exists $x \in U$ such that $f(x)$ is a point of some stratum S of codimension i; otherwise $f(U) \subseteq Z_{>i}$. If a regular function $h \in \mathbb{C}[Z]$ vanishes on S then $h \circ f$ vanishes on an open neighborhood of x in U and thus on the whole of U. So $f(U) \subseteq \bar{S}$ and there is a subgroup K of G distinct from G such that $f(U) \subseteq \bar{S} = \pi(V^K)$.

It is clear that if each morphism f of the above type (regular, holomorphic, or formal) from $\mathbb{C}^p$ to V^K/W has a lift F (regular, holomorphic, local, or formal), then the composition of f with the morphism κ has the corresponding lift to V which is the composition of F with the inclusion $V^K \to V$.

Conversely, if $f : \mathbb{C}, x \to Z$ is a germ of a holomorphic map at $x \in \mathbb{C}^p$ such that $f(x) \in \bar{S}_{\mathrm{nor}}$, there is a unique germ of a holomorphic map $f' : \mathbb{C}^p, x \to V^K/W$ such that $\kappa \circ f' = f$. Similarly, let $f : \mathbb{C}[Z] \to \mathbb{C}[[X_1, \dots, X_p]]$ be a formal morphism from $\mathbb{C}^p$ to Z which can be extended to the morphism $\widetilde{\mathcal{O}}_z(Z) \to \mathbb{C}[[X_1, \dots, X_p]]$ for some $z \in \bar{S}_{\mathrm{nor}}$. There is a unique formal morphism $f' : \mathbb{C}[V/W] \to \mathbb{C}[[X_1, \dots, X_p]]$ such that $J_0^\infty(\mathbb{C}^p, \kappa)(f') = f$. In both cases the lifting problem for $\pi : V \to V/G$ reduces to the corresponding one for $\pi_{V^K} : V^K \to V^K/W$.

Although these arguments give nothing if the above points $f(x)$ or z do not belong to $\bar{S}_{\mathrm{nor}}$, we have the following theorem.

Theorem.

(1) *Let f be a holomorphic map from a classically open subset U of $\mathbb{C}^p$ to $Z = V/G$, S a stratum of maximal codimension such that $f(U) \subseteq \bar{S}$, and $f : \mathbb{C}^p, x \to Z$ be a germ of f at some $x \in U$. Let K be a subgroup of G such that $\pi(V^K) = \bar{S}$, and let $W = N_G(K)/K$. If the germ $f : \mathbb{C}^p, x \to Z$ at $x \in U$ has a local lift to V, then the germ $f' : \mathbb{C}^p, x \to V^K/W$ of the map defined by f is a quasiregular germ of a holomorphic map and this germ has a local lift to V^K.*

$$\begin{tikzcd}
 & & V^K \arrow[r, hook] \arrow[dl, two heads] \arrow[dd, "\pi"] & V \arrow[dd, "\pi"] & \\
 & V^K/W \arrow[dr, "\kappa"] & & & \\
\mathbb{C}^p, x \arrow[ur, "f'"] \arrow[rr, "f"'] & & \bar{S} \arrow[r, hook] & Z \arrow[r, equal] & V/G
\end{tikzcd}$$

(2) *Let f be a formal morphism from $\mathbb{C}^p$ to Z and let S be a stratum of the maximal codimension such that f factors through a formal morphism f'*

from $\mathbb{C}^p$ to $\bar{S}$. If f has a formal lift to V, then the formal morphism f' is quasiregular and has a formal lift to V^K such that $\pi(V^K) = \bar{S}$.

(3) *If F_1 and F_2 are holomorphic lifts of a holomorphic map $f : U \to Z$, then there is a $g \in G$ such that $F_2 = g \circ F_1$. The same is true for local lifts of germs of holomorphic maps, and for lifts of formal morphisms.*

Proof. (1) Consider a local lift of f which is a germ of a holomorphic map $F : U' \to V$, where $U' \subseteq U$ is an open subset. By assumption, $F(U') \subseteq \pi^{-1}(\bar{S}) = \cup_{g\in G} gV^K$ and there is a point $x \in U'$ such that the stabilizer $G_{F(x)} = gKg^{-1}$. Then $F(U') \subseteq g(V^K)$ and $F' = g^{-1} \circ F$ is a local lift of f such that $F'(U') \subseteq V^K$. Then $\pi_{V^K} \circ F'$ is a germ of a holomorphic map which by construction coincides with the germ of f'. By definition, the germ f' is quasiregular and F' is its local lift.

(2) Let F be a formal lift of f to V. Let $I(\bar{S})$ be the prime ideal of $\mathbb{C}[Z]$ defining $\bar{S}$. Consider the pullback $\pi^*(I(\bar{S}))$ of $I(\bar{S})$. By the definition of V^K we have $\pi^{-1}(\bar{S}) = \cup_{g\in G} gV^K$. By definition of a formal lift the formal lift F vanishes on $\pi^*(I(\bar{S}))$ and then, by Hilbert's Nullstellensatz, on the ideal $I(\cup_{g\in G} gV^K) = \sqrt{I(\cup_{g\in G} gV^K)}$ of $\mathbb{C}[V]$ defining a Zariski-closed subset $\cup_{g\in G} gV^K$ of V. Evidently the ideal $I(\cup_{g\in G} gV^K)$ equals the intersection of prime ideals $I(gV^K)$. Since $\mathbb{C}[[X_1, \dots, X_p]]$ is an integral domain, there is a $g \in G$ such that the formal morphism F vanishes on $I(gV^K)$ and then $F \circ g^{-1}$ is a formal lift of f which factors through the formal morphism $F' : \mathbb{C}[V^K] \to \mathbb{C}[[X_1, \dots, X_p]]$. Thus the formal morphism f factors through the formal morphism $f' = J_0^\infty(\mathbb{C}^p, \pi_{V^K})(F')$ from $\mathbb{C}^p$ to V^K/W, F' is a formal lift of f', and, by assumption, the formal morphism f' is quasiregular.

(3) First assume that the germ of f is quasiregular at $x \in U$.

Let F_1 and F_2 be holomorphic lifts of f. By assumption, there is a point y in a neighborhood of x such that $F_1(y)$ and $F_2(y)$ are regular points of V. Since $(\pi \circ F_1)(y) = (\pi \circ F_2)(y)$, there is a unique $g \in G$ such that $F_2(y) = (g \circ F_1)(y)$. As the projection π is étale at $F_2(y)$, the lift F_2 coincides with $g \circ F_1$ in a neighborhood of y and then on the whole of U.

Let K be the maximal subgroup of G such that $f(U) \subseteq \pi(V^K)$, let $N_G(K)$ be the normalizer of K in G, and $W = N_G(K)/K$. By the proof of (1) the germ $f : \mathbb{C}^p, x \to Z$ for $x \in U$ can be considered as a quasiregular germ of a holomorphic map $f' : \mathbb{C}^p, x \to \bar{S} = \pi(V^K)$ and there are $g_1, g_2 \in G$ such that the germs $g_1 \circ F_1 : \mathbb{C}^p, x \to V$ and $g_2 \circ F_2 : \mathbb{C}^p, x \to V$ are local lifts of the above germ f' to V^K. Then, for some $g \in H = N_G(K)$ we have $g_2 \circ F_2 = (gg_1) \circ F_1$ in a neighborhood of x and then in the whole of U. Thus we have $F_2 = (g_2^{-1} g g_1) F_1$.

For local lifts the proof is the same. For lifts of formal morphisms the proof follows from 5.4 below. □

The above theorem shows that the problem of lifting for local and formal lifts is reduced in some sense to one for the quasiregular case.

Namely, let the conditions of Theorem 2.7(1) be satisfied. Since the morphism κ is birational, for each basic invariant τ of $\mathbb{C}[V^K]$, the composition $\kappa^{-1} \circ \tau$ is a rational function on V^K/W and, in general, the function $\kappa^{-1} \circ \tau \circ f$ is a meromorphic function on U. First we have to check that this function is analytic near x. If $f(x) \in \bar{S}_{\text{nor}}$ this is always true, because κ^{-1} is an isomorphism near $f(x)$. Then f has a local lift at x iff the germ $f' : \mathbb{C}^p, x \to V^K/W$ has a local lift to V^K.

The analogous statement for formal lifts is true whenever the conditions of Theorem 2.7(2) are satisfied.

2.8. An algebraic interpretation of the problem of lifting. The results of this section are not used in the rest of the paper.

The above geometric problem of lifting has the following algebraic interpretation. For instance, suppose that $f : \mathbb{C}^p \to Z$ is a regular morphism and F is its regular lift. Consider the morphism $f^* : \mathbb{C}[Z] = \mathbb{C}[V]^G \to \mathbb{C}[\mathbb{C}^p]$ induced by f and the morphism $F^* : \mathbb{C}[V] \to \mathbb{C}[\mathbb{C}^p]$ induced by F. Since, by definition, $\mathbb{C}[Z] \subseteq \mathbb{C}[V]$, the morphism F^* is an extension of the morphism f^* to $\mathbb{C}[V]$.

$$\begin{tikzcd} & V \arrow[d, two heads] \\ \mathbb{C}^p \arrow[ur, dashed, "F"] \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C}[V] \arrow[dl, dashed, "F^*"'] \\ \mathbb{C}[\mathbb{C}^p] & \mathbb{C}[Z] \arrow[l, "f^*"] \arrow[u, hook] \end{tikzcd}$$

Similarly, consider a germ of a holomorphic morphism $f : \mathbb{C}^p, 0 \to Z, O$, where $O = \pi(0)$ and its local lift $F : \mathbb{C}^p, 0 \to V, 0$. We have the morphisms $f^* : \mathfrak{F}_{Z,O} \to \mathfrak{F}_{\mathbb{C}^p,0}$ and $F^* : \mathfrak{F}_{V,0} \to \mathfrak{F}_{\mathbb{C}^p,0}$ induced by f and F respectively. Since the projection π induces the inclusion $\mathfrak{F}_{Z,O} \subseteq \mathfrak{F}_{V,0}$, the morphism F^* is an extension of the morphism f^* to $\mathfrak{F}_{V,0}$.

Finally, let $f : \mathbb{C}[Z] \to \mathbb{C}[[X_1, \ldots, X_p]]$ be a formal morphism from $\mathbb{C}^p$ to Z and let $F : \mathbb{C}[V] \to \mathbb{C}[[X_1, \ldots, X_p]]$ be its formal lift. Since the projection π induces the inclusion $\mathbb{C}[Z] \subseteq \mathbb{C}[V]$ the lift F is an extension of the morphism f to $\mathbb{C}[V]$.

2.9. Let τ be a homogeneous G-invariant polynomial of degree d on V and let τ^s be the corresponding symmetric d-linear form on V. For each germ $F : \mathbb{C}^p, 0 \to V$ of a holomorphic map and each system of multi-indices $(A_1, \ldots, A_d)$ we put

$$T(A_1, \ldots, A_d)(j_0^q F) := \tau^s\Big(\partial_{A_1} F(0), \ldots, \partial_{A_d} F(0)\Big).$$

By 2.4, $T(A_1, \ldots, A_d)$ is a function on $J_0^q(\mathbb{C}^p, V)$ for $q \geq |A_1|, \ldots, |A_d|$. From 2.2 and Proposition 2.6 it follows that the function

$$T(A_1, \ldots, A_d) : J_0^q(\mathbb{C}^p, V) = V^{\mathfrak{A}_{p,q}} \to \mathbb{C}$$

is regular, G-invariant, and equal to a polarization of τ up to a nonzero factor. It is also symmetric in $A_1, \ldots, A_d$.

By Proposition 2.5, define a rational function $\widetilde{T}(A_1, \ldots, A_d)$ on $\bar{J}_0^q(\mathbb{C}^p, Z)$ by the condition $T(A_1, \ldots, A_d) = \widetilde{T}(A_1, \ldots, A_d) \circ J_0^q(\mathbb{C}^p, \pi)$. By definition, we have

$$\widetilde{T}(\emptyset, \ldots, \emptyset) \circ \pi = T(\emptyset, \ldots, \emptyset) = \tau. \tag{1}$$

Now extend the d-linear form τ^s on V to a d-linear form $\mathfrak{T}$ on $J_0^\infty(\mathbb{C}^p, V) = V \otimes \mathbb{C}[[X_1, \dots, X_p]]$ with values in $\mathbb{C}[[X_1, \dots, X_p]]$ which is defined by the following condition. For $i = 1, \dots, d$, $v_i \in V$, and $F_i \in \mathbb{C}[[X_1, \dots, X_p]]$

$$\mathfrak{T}(v_1 \otimes F_1, \dots, v_d \otimes F_d) := \tau^s(v_1, \dots, v_d)\, F_1 \cdots F_d.$$

For $h \in J_0^\infty(\mathbb{C}^p, V) = V \otimes \mathbb{C}[[X_1, \dots, X_p]]$ and a system of multi-indices $A_1, \dots, A_d$ put

$$\mathfrak{T}(A_1, \dots, A_d)(h) := \mathfrak{T}(\partial_{A_1} h, \dots, \partial_{A_d} h).$$

By definition, the function $\mathfrak{T}(A_1, \dots, A_d) : J_0^\infty(\mathbb{C}^p, V) \to \mathbb{C}[[X_1, \dots, X_p]]$ is G-invariant and symmetric in $A_1, \dots, A_d$.

$$J_0^\infty(\mathbb{C}^p, V) = V \otimes \mathbb{C}[[X_1, \dots, X_p]] \xrightarrow{\mathfrak{T}(A_1,\dots,A_d)} \mathbb{C}[[X_1, \dots, X_p]]$$

$$\begin{tikzcd}
J_0^q(\mathbb{C}^p, V) \arrow[r, "T(A_1,\dots,A_d)"] \arrow[d, "J_0^q(\mathbb{C}^p,\pi)"'] & \mathbb{C} \\
\bar{J}_0^q(\mathbb{C}^p, Z) \arrow[ur, "\widetilde{T}(A_1,\dots,A_d)"'] &
\end{tikzcd}$$

3. The functions $\widetilde{T}(A_1, \dots, A_d)$ and $\widetilde{\mathfrak{T}}(A_1, \dots, A_d)$

3.1. The q-jet of the identity map on V in invariant coordinates. Let $v : V \to V$ be the identity map. Let $v_0 \in V_0$ be a regular point of V and let y_i be invariant coordinates in a neighborhood U of v_0 in V introduced in 2.3. Then in U the map v is defined by a holomorphic function $v(y_i)$ with values in V.

Let $I = (i_1, \dots, i_s)$ be a (unordered) multi-index with $i_1, \dots, i_s \in \{1, \dots, n\}$. In particular, for $s = 0$ we put $I := \emptyset$. Then the q-jet of the identity map v at each point $x \in U$ is defined by the set of partial derivatives $\partial_I v = \frac{\partial^s v}{\partial y_{i_1} \dots \partial y_{i_s}}$ for $|I| \le q$ at x.

Let e_a be a basis of V, let u_a be the corresponding coordinates, and let $J = \det\left(\frac{\partial y_i}{\partial u_j}\right)$ be the Jacobian.

Lemma. *Let $I = (i_1, \dots, i_s)$, where $s > 0$, be a multi-index. Then $\widetilde{\partial_I v} := J^{2s-1} \partial_I v$ is a regular map from U to V.*

Proof. We prove this lemma by induction with respect to s. We use that

$$\frac{\partial u_a}{\partial y_i} = \frac{1}{J} J_i^a,$$

where $J_i^a \in \mathbb{C}[V]$ is the algebraic complement of the entry $\partial y_i / \partial u_a$ in the Jacobi matrix.

For $s = 1$, $J \partial_i v = \sum_{a=1}^n J_i^a e_a$ is a regular map from U to V.

Let $I = (i_1, \dots, i_s)$ with $\widetilde{\partial_I v}$ being regular. Then for $I' = (i_1, \dots, i_{s+1})$ we have

$$\partial_{I'} v = \frac{\partial}{\partial y_{i_{s+1}}} \Big(\frac{\widetilde{\partial_I v}}{J^{2s-1}} \Big) = \frac{1}{J^{2s+1}} \sum_{a=1}^{n} J^a_{i_{s+1}} \Big(J \frac{\partial(\widetilde{\partial_I v})}{\partial u_a} - (2s-1) \frac{\partial J}{\partial u_a} \widetilde{\partial_I v} \Big),$$

where $\sum_{a=1}^{n} J^a_{i_{s+1}} \Big(J \frac{\partial(\widetilde{\partial_I v})}{\partial u_a} - (2s-1) \frac{\partial J}{\partial u_a} \widetilde{\partial_I v} \Big)$ is a regular map from U to V. $\square$

3.2. Let $h = j^q_0 F \in J^q_0(\mathbb{C}^p, V)$, where $F : \mathbb{C}^p, 0 \to V$ is a germ of a holomorphic map such that $F(0) \in V_0$. Put $F_i := y_i \circ F$, where y_i are the invariant coordinates on V. We need to express the q-jet $j^q_0 F$ in terms of the q-jet of the identity map v, i.e., we have to find the explicit formula for each $h_A = \partial_A F(0)$ with $A \in \mathfrak{A}_{p,q}$ in terms of $\partial_B F_i$ and $\partial_I v$ with $|B|, |I| \le |A|$. We can extract this formula from the following expression (see the classical Faà di Bruno formula for $p = 1$ and [**18**] or [**19**] for arbitrary p):

$$\begin{aligned} d^q(v \circ F) = \; & q! \sum_{k=1}^{q} \frac{1}{k!} \sum_{i_1,\dots,i_k=1}^{n} \Big(\frac{\partial^k v}{\partial y_{i_1} \dots \partial y_{i_k}} \circ F \Big) \\ & \cdot \sum_{q_1+\dots+q_k=q} \frac{d^{q_1} F_{i_1}}{q_1!} \cdots \frac{d^{q_k} F_{i_k}}{q_k!}, \end{aligned} \tag{2}$$

where y_i are arbitrary local coordinates in V. Note that the formula (2) is true whenever $F : \mathbb{C}[V] \to \mathbb{C}[[X_1, \dots, X_p]]$ is a formal morphism from $\mathbb{C}^p$ to V and $F_i = F(y_i)$.

The formula (2) implies the following

Lemma. *For each multi-index $A = (a_1, \dots, a_s) \neq \emptyset$, where $a_1, \dots, a_s \in \{1, \dots, p\}$, there is a well defined function*

$$\Psi_A : (X, Y) \mapsto \sum_{1 \le |I| \le |A|} a_{A,I}(Y)\, X_I,$$

where $X = (X_I)_{1 \le |I| \le |A|}$ and where the coefficients $a_{A,I}$ are polynomials in $Y = (y_{i,B})_{1 \le i \le n, 1 \le |B| \le |A|}$, such that for each germ of a holomorphic map $F : \mathbb{C}^p, 0 \to V, v$ with v regular and for the local coordinates y_i from above we have

$$\partial_A F = \Psi_A \Big((\partial_I v \circ F)_I, (\partial_B F_i)_{i,B} \Big).$$

For example,

$$\begin{aligned} \partial_a F &= \sum_{i=1}^{n} (\partial_i v \circ F)\, \partial_a F_i, \\ \partial_{(a_1,a_2)} F &= \sum_{i,j=1}^{n} (\partial_{(i,j)} v \circ F)\, \partial_{a_1} F_i\, \partial_{a_2} F_j + \sum_{i=1}^{n} (\partial_i v \circ F)\, \partial_{(a_1,a_2)} F_i, \end{aligned}$$

and so on.

3.3. We consider $T(A_1,\dots,A_d)$ which is a regular function on $J_0^q(\mathbb{C}^p,V)$, and $\widetilde{T}(A_1,\dots,A_d)$ which is a rational function on $\bar{J}_0^q(\mathbb{C}^p,Z)$, both defined in 2.9.

Let z_i be the regular function on Z for $i=1,\dots,n$, used in 2.3 for the construction of the invariant coordinates on V. Let $A_1,\dots,A_{d'}\neq\emptyset$ and $A_{d'+1}=\dots=A_d=\emptyset$. Put $M:=2(|A_1|+\dots+|A_d|)-d'$. By Lemma 3.1, for any system of multi-indices $I_1,\dots,I_d$ such that $1\le|I_1|\le|A_1|,\dots,1\le|I_{d'}|\le|A_{d'}|$ and $I_{d'+1}=\dots=I_d=\emptyset$, the expression $\Delta_M\cdot T\circ(\partial_{I_1}v,\dots,\partial_{I_d}v)$ is a G-invariant and regular function on V. Thus there is a unique rational function $\widetilde{T}(I_1,\dots,I_d)$ on Z such that $\widetilde{T}(I_1,\dots,I_d)\circ\pi=T(\partial_{I_1}v,\dots,\partial_{I_d}v)$ and $\widetilde{\Delta}_M\cdot\widetilde{T}(I_1,\dots,I_d)$ is a regular function on Z.

Let q be the maximal order of the multi-indices $A_1,\dots,A_d$. For $k=1,\dots,d'$ we may consider the a_{A_k,I_k} of 3.2 as polynomials in $Y=(y_{i,B})_{1\le i\le n,1\le|B|\le q}$. Put

$$a_{A_1,\dots,A_{d'},I_1,\dots,I_{d'}}(Y):=a_{A_1,I_1}(Y)\cdots a_{A_{d'},I_{d'}}(Y).$$

Theorem. *Let $A_1,\dots,A_{d'}\neq\emptyset$, $A_{d'+1},\dots,A_d=\emptyset$ and $A_1,\dots,A_d\in\mathfrak{A}_{p,q}$. Then:*

(1) *The following is a rational function $\widetilde{T}(A_1,\dots,A_d)$ on $\bar{J}_0^q(\mathbb{C}^p,Z)$:*

$$\widetilde{T}(A_1,\dots,A_d):=\sum_{\substack{1\le|I_1|\le|A_1|,\\ \dots\\ 1\le|I_{d'}|\le|A_{d'}|}} a_{A_1,\dots,A_{d'},I_1,\dots,I_{d'}}\Big((\partial_B z_i)_{i,B}\Big)\,\widetilde{T}(I_1,\dots,I_{d'},\emptyset,\dots,\emptyset);$$

(2) *$\widetilde{T}(A_1,\dots,A_d)\circ J_0^q(\mathbb{C}^p,\pi)=T(A_1,\dots,A_d)$;*

(3) *$\widetilde{\Delta}_M\cdot\widetilde{T}(A_1,\dots,A_d)$ is a regular function on $J_0^q(\mathbb{C}^p,Z)$.*

$$\begin{tikzcd}
J_0^q(\mathbb{C}^p,V) \arrow[r, "T(A_1,\dots,A_d)"] \arrow[d, "J_0^q(\mathbb{C}^p,\pi)"'] & \mathbb{C} \\
\bar{J}_0^q(\mathbb{C}^p,Z) \arrow[ur, dashed, "\widetilde{T}(A_1,\dots,A_d)"'] &
\end{tikzcd}$$

Proof. (1) By Proposition 2.5, it suffices to check that the condition

$$\widetilde{T}(A_1,\dots,A_d)\circ\pi^q=T(A_1,\dots,A_d)$$

is satisfied for the above expression of $\widetilde{T}(A_1,\dots,A_d)$. By Lemma 3.2, this follows from

$$T(A_1,\dots,A_d)(h)=\sum_{\substack{1\le|I_1|\le|A_1|,\\ \dots\\ 1\le|I_{d'}|\le|A_{d'}|}}\Big(a_{A_1,\dots,A_{d'},I_1,\dots,I_{d'}}\big((\partial_B F_i)_{i,B}\big)\,T(\partial_{I_1}v,\dots,\partial_{I_{d'}}v,v,\dots,v)\circ F\Big)(0),$$

where $h=j_0^qF\in J_0^q(\mathbb{C}^p,V)$.

(2) This statement follows from Proposition 2.5.

(3) This statement follows from (1) and Lemma 3.1. $\square$

3.4. Let $F \in V \otimes \mathbb{C}[[X_1, \ldots, X_p]]$ be a formal morphism from $\mathbb{C}^p$ to V and $F_i = y_i(F)$. Lemma 3.2 implies the following

Lemma. *For each multi-index A such that $|A| \geq 1$ and the invariant coordinates y_i on V we have*

$$J^{2|A|-1}(F)\, \partial_A F = \Psi_A\Big(\Big((J^{2(|A|-|I|)}\widetilde{\partial_I v})(F)\Big)_I, \Big(\partial_B F_i\Big)_{i,B}\Big).$$

3.5. The function $\widetilde{\mathfrak{T}}(A_1, \ldots, A_d)$. Consider the presentation of Z as an irreducible Zariski-closed subset of $\mathbb{C}^m$ defined in 2.1. Denote by $I(Z)$ the prime ideal of the ring of polynomials $\mathbb{C}[W_1, \ldots, W_m]$ defining $Z \subseteq \mathbb{C}^m$. By 2.4, each formal morphism $f \in J_0^\infty(\mathbb{C}^p, Z)$ is defined by the equations $f(W_j) = f_j$ for $(j = 1, \ldots, m)$, where $f_j \in \mathbb{C}[[X_1, \ldots, X_p]]$ and $\Phi(f_j) = 0$ for each $\Phi \in I(Z)$.

Let ψ be a regular function on Z which is the restriction to Z of a polynomial $\Psi \in \mathbb{C}[W_1, \ldots, W_m]$. For each $f \in J_0^\infty(\mathbb{C}^p, Z)$ put $\psi(f) := \Psi(f_j)$. By definition we have $\psi(f) = f(\psi)$, where f is considered as a morphism $\mathbb{C}[Z] \to \mathbb{C}[[X_1, \ldots, X_p]]$. Then $\psi(f)$ defines a unique function $J_0^\infty(\mathbb{C}^p, Z) \to \mathbb{C}[[X_1, \ldots, X_p]]$ which is independent of the choice of the polynomial Ψ.

Similarly, consider a rational function ψ on Z such that $\psi = \frac{\psi_1}{\psi_2}$, where ψ_1 and ψ_2 are regular functions on Z and put $\psi(f) := \frac{\psi_1(f)}{\psi_2(f)}$ whenever $\psi_2(f) \neq 0$. It is clear that $\psi(f)$ is a function on $J_0^\infty(\mathbb{C}^p, Z)$ with values in the field $\mathbb{C}((X_1, \ldots, X_p))$ of fractions of the ring $\mathbb{C}[[X_1, \ldots, X_p]]$ which is independent of the choice of the presentation $\psi = \frac{\psi_1}{\psi_2}$.

Let z_i be the regular functions on Z used in 2.3 for the construction of the invariant coordinates y_i on V. Let $f \in J_0^\infty(\mathbb{C}^p, Z)$ be a quasiregular formal morphism from $\mathbb{C}^p$ to Z such that $\widetilde{\Delta}(f) \neq 0$. For $A_1, \ldots, A_{d'} \neq \emptyset$, $A_{d'+1}, \ldots, A_d = \emptyset$ and $M = 2(|A_1| + \cdots + |A_d|) - d'$ put

$$\begin{aligned}&\widetilde{\mathfrak{T}}(A_1, \ldots, A_{d'}, \emptyset, \ldots, \emptyset)(f) :=\\ &\qquad = \sum_{\substack{1 \leq |I_1| \leq |A_1|, \\ \cdots \\ 1 \leq |I_{d'}| \leq |A_{d'}|}} \Big(a_{A_1, \ldots, A_{d'}, I_1, \ldots, I_{d'}}\big((\partial_B z_i)_{i,B}\big) \cdot \widetilde{T}(I_1, \ldots, I_{d'}, \emptyset, \ldots, \emptyset)\Big)(f).\end{aligned}$$

By definition, $\widetilde{\mathfrak{T}}(A_1, \ldots, A_d)$ is a function with values in the field $\mathbb{C}((X_1, \ldots, X_p))$ on the set $\bar{J}_0^\infty(\mathbb{C}^p, Z)$ consisting of all $f \in J_0^\infty(\mathbb{C}^p, Z)$ such that $\widetilde{\Delta}(f) \neq 0$.

Theorem. *Let $A_1, \ldots, A_{d'} \neq \emptyset$ and $A_{d'+1}, \ldots, A_d = \emptyset$ and $M = 2(|A_1| + \cdots + |A_d|) - d'$. Then the function $\widetilde{\mathfrak{T}}(A_1, \ldots, A_d)$ satisfies the following conditions:*

(1) $\widetilde{\mathfrak{T}}(A_1, \ldots, A_d) \circ J_0^\infty(\mathbb{C}^p, \pi) = \mathfrak{T}(A_1, \ldots, A_d)$, *where $\mathfrak{T}$ is from 2.9.*
(2) $\widetilde{\Delta}_M \cdot \widetilde{\mathfrak{T}}(A_1, \ldots, A_d)$, *where $\widetilde{\Delta}_M$ is regarded as a function on $J_0^\infty(\mathbb{C}^p, Z)$, is a function on $J_0^\infty(\mathbb{C}^p, Z)$ with values in $\mathbb{C}[[X_1, \ldots, X_p]]$.*

$$\begin{tikzcd}
J_0^\infty(\mathbb{C}^p, V) \arrow[r, "\mathfrak{T}(A_1,\dots,A_d)"] \arrow[d, "J_0^\infty(\mathbb{C}^p,\pi)"'] & \mathbb{C}[[X_1,\dots,X_p]] \\
\bar{J}_0^\infty(\mathbb{C}^p, Z) \arrow[ur, dashed, "\widetilde{\mathfrak{T}}(A_1,\dots,A_d)"'] &
\end{tikzcd}$$

Proof. The proof follows from the definition of the function $\widetilde{\mathfrak{T}}(A_1,\dots,A_d)$ and Lemma 3.4. $\square$

4. The conditions of local and global lifting

4.1. First we consider local lifts at regular points.

Proposition. *Let $f : \mathbb{C}^p, x \to Z, z$ be a germ at $x \in \mathbb{C}^p$ of a holomorphic map with z regular. Then for each $v \in \pi^{-1}(z)$ there is a unique local holomorphic lift $F : \mathbb{C}^p, x \to V, v$ of f.*

Proof. By Proposition 2.1, the map π is étale on V_0. Thus for each point $v \in \pi^{-1}(z)$ there is a unique local holomorphic lift $F : \mathbb{C}^p, x \to V_0, v$ of f. $\square$

4.2. Lifts of quasiregular holomorphic germs. Let X be an affine variety and let f be either a rational morphism from $\mathbb{C}^p$ to X or a holomorphic map defined on a classically open connected subset $U \subseteq \mathbb{C}^p$ to X. Consider the morphism $j^q f$ from $\mathbb{C}^p$ or from U to $J_0^q(\mathbb{C}^p, X)$, which for $x \in U$ is given by $j^q f(x) = j_0^q f(\quad + x)$. The morphism $j^q f$ is rational and is regular wherever f is regular; or holomorphic if f is holomorphic.

Let $\sigma : V \to \sigma(V) \subseteq \mathbb{C}^m$ be the morphism defined by the system of basic generators $\sigma_1, \dots, \sigma_m$. Recall that $\sigma(V)$ and $Z = V/G$ are isomorphic as affine varieties and, for this presentation of Z, σ equals π.

Denote by $w_1, \dots, w_m$ the standard coordinates in $\mathbb{C}^m$ and let $I(Z)$ be the prime ideal of the ring $\mathbb{C}[W_1, \dots, W_m]$ defining Z. Consider $\mathbb{C}[W_1, \dots, W_m]$ as a graded ring with a grading defined by $\deg W_j = \deg \sigma_j$ for $j = 1, \dots, m$. Then $I(Z)$ is a homogeneous ideal.

For $\tau = \sigma_j$ and a system $A_1, \dots, A_{d_j}$ of multi-indices denote by $\widetilde{S}_j(A_1, \dots, A_{d_j})$ the rational function $\widetilde{T}(A_1, \dots, A_{d_j})$ from 2.9 on $\bar{J}_0^q(\mathbb{C}^p, Z)$.

Recall that by 2.7 for a quasiregular germ $f : \mathbb{C}^p, 0 \to Z$ of a holomorphic map there is a choice of invariant coordinates such that for the corresponding function $\widetilde{\Delta}$ we have $\widetilde{\Delta} \circ f \neq 0$.

Theorem. *Consider a quasiregular germ $f : \mathbb{C}^p, 0 \to Z = V/G$ of a holomorphic map described by $w_j \circ f = f_j$ for $j = 1, \dots, m$. Assume that, for some choice of the invariant coordinates such that $\widetilde{\Delta} \circ f \neq 0$, q is the minimal order of nonzero terms of the Taylor expansion of $\widetilde{\Delta} \circ f$ at 0.*

Then the lift F of f at 0 exists if and only if for $j = 1, \dots, m$ and for each system of multi-indices $A_1, \dots, A_{d_j} \in \mathfrak{A}_{p,q}$ the functions $\widetilde{S}_j(A_1, \dots, A_{d_j}) \circ j^q f$ have holomorphic extensions to a neighborhood of 0.

$$\begin{tikzcd}
 & J_0^q(\mathbb{C}^p, V) \arrow[r, "S(A_1,\ldots,A_d)"] \arrow[d, "J_0^q(\mathbb{C}^p,\pi)"'] & \mathbb{C} \\
\mathbb{C}^p, 0 \arrow[ur, "j^q F"] \arrow[r, "j^q f"] & \bar{J}_0^q(\mathbb{C}^p, Z) \arrow[ur, "\widetilde{S}(A_1,\ldots,A_d)"'] &
\end{tikzcd}$$

Proof. Let F be a lift of f. Then by the definition of the function $\widetilde{S}_j(A_1,\ldots,A_{d_j})$ for each $q \geq 0$ we have

$$\widetilde{S}_j(A_1,\ldots,A_{d_j}) \circ j^q f = S_j(A_1,\ldots,A_{d_j}) \circ j^q F : \mathbb{C}^p, 0 \to \mathbb{C},$$

where the right hand side defines a holomorphic germ.

Conversely, let the assumptions of the theorem be satisfied. Let us now use a representative $f : U \to Z$ of the germ, where U is a connected open neighborhood of 0. Let q be the minimal order of nonzero terms of the Taylor expansion of $\widetilde{\Delta} \circ f$ at 0. For each $j = 1,\ldots,m$ consider the function

$$f_j^q(x,t) := \sum_{A_1,\ldots,A_{d_j} \in \mathfrak{A}_{p,q}} \Big(\widetilde{S}(A_1,\ldots,A_{d_j}) \circ j^q f\Big)(x)\, t_{A_1} \cdots t_{A_{d_j}}, \tag{3}$$

where $x \in U$ and $t = (t_A)_{A \in \mathfrak{A}_{p,q}} \in \mathbb{C}^{\mathfrak{A}_{p,q}}$. By assumption, the function f_j^q is a polynomial in t whose coefficients are holomorphic near 0. By definition, the map $f^q = (f_j^q)_{j=1,\ldots,m} : \mathbb{C}^p \times \mathbb{C}^{\mathfrak{A}_{p,q}} \to \mathbb{C}^m$ is holomorphic near 0.

Since $Z_{>0}$ is a Zariski-closed subset of Z of codimension ≥ 1, the inverse image $f^{-1}(Z_{>0})$ is a complex analytic subset of U of codimension ≥ 1 and $f^{-1}(Z_0)$ is a dense open subset of U.

Let, for $y \in U$, $f(y)$ be a regular point and let F_y be a local lift of f defined in a neighborhood U_y of y, which exists by Proposition 4.1. For each q consider the holomorphic map $F_y^q : U_y \times \mathbb{C}^{\mathfrak{A}_{p,q}} \to V$ given by:

$$F_y^q(x,t) := \sum_{A \in \mathfrak{A}_{p,q}} \partial_A F_y(x)\, t_A. \tag{4}$$

By Theorem 3.3, we have

$$\begin{aligned}
(\sigma_j \circ F_y^q)(x,t) &= \sum_{A_1,\ldots,A_{d_j} \in \mathfrak{A}_{p,q}} S_j\Big(\partial_{A_1} F_y(x), \ldots, \partial_{A_{d_j}} F_y(x)\Big)\, t_{A_1} \cdots t_{A_{d_j}} \\
&= \sum_{A_1,\ldots,A_{d_j} \in \mathfrak{A}_{p,q}} \Big(S_j(A_1,\ldots,A_{d_j}) \circ j^q F_y\Big)(x)\, t_{A_1} \cdots t_{A_{d_j}} = f_j^q(x,t).
\end{aligned}$$

Therefore for each polynomial $\Phi \in I(Z)$ we have $\Phi \circ f^q = 0$ on $U_y \times \mathbb{C}^{\mathfrak{A}_{p,q}}$ and thus also on $U \times \mathbb{C}^{\mathfrak{A}_{p,q}}$. So f^q is a holomorphic map from $U \times \mathbb{C}^{\mathfrak{A}_{p,q}}$ to Z and F_y^q is a lift of f^q.

For each germ of a holomorphic function $\varphi \in \mathfrak{F}_{\mathbb{C}^p,x}$, denote by $\mathrm{Tay}_x^q\, \varphi$ the sum of terms of the Taylor expansion at x of φ of orders $\leq q$. For each germ $\varphi = (\varphi_j)_j$ of a holomorphic map $\mathbb{C}^p, x \to \mathbb{C}^m$, put $\mathrm{Tay}_x^q\, \varphi := (\mathrm{Tay}_x^q\, \varphi_j)_j$.

By assumption, there is a multi-index $A \in \mathfrak{A}_{p,q}$ such that $\partial_A(\widetilde{\Delta} \circ f)(0) = \partial_A(\widetilde{\Delta} \circ \operatorname{Tay}_0^q f)(0) \neq 0$. This implies that there is a point $x_0 = (x_{0,1}, \ldots, x_{0,p}) \in \mathbb{C}^p$ such that $(\widetilde{\Delta} \circ \operatorname{Tay}_0^q f)(x_0) \neq 0$.

For

$$A = (\underbrace{1, \ldots, 1}_{r_1 \text{ times}}, \ldots, \underbrace{p, \ldots, p}_{r_p \text{ times}}),$$

put

$$t_A(x) := \frac{1}{r_1! \ldots r_p!} (x_1)^{r_1} \ldots (x_p)^{r_p}, \qquad t(x) := (t_A(x))_{A \in \mathfrak{A}_{p,q}}$$

where $x = (x_1, \ldots, x_p) \in \mathbb{C}^p$.

By definition, we have $F_y^q(y, t(x-y)) = \operatorname{Tay}_y^q F_y(x)$ and then $f_j^q(y, t(x-y)) = (\sigma_j \circ \operatorname{Tay}_y^q F_y)(x)$. On the other hand, since σ_j is homogeneous, for a fixed y we have $\operatorname{Tay}_y^q f_j = \operatorname{Tay}_y^q(\sigma_j \circ F_y) = \operatorname{Tay}_y^q(\sigma_j \circ \operatorname{Tay}_y^q F_y)$. Thus, we have

$$\operatorname{Tay}_y^q f_j^q(y, t(x-y)) = \operatorname{Tay}_y^q f_j(x). \tag{5}$$

By assumption, the function $\widetilde{S}_j(A_1, \ldots, A_{d_j}) \circ j^q f$ has a holomorphic extension to a neighborhood of 0 and we may suppose that the point y belongs to this neighborhood. Letting $y \to 0$ in (5) we get $\operatorname{Tay}_0^q f_j^q(0, t(x)) = \operatorname{Tay}_0^q f_j(x)$. Then we have $(\widetilde{\Delta} \circ f^q)(0, t(x)) = \widetilde{\Delta}(\operatorname{Tay}_0^q f)(x) \neq 0$ and, for the point $x_0 \in \mathbb{C}^p$ chosen above, we have $(\widetilde{\Delta} \circ f^q)(0, t(x_0)) \neq 0$, i.e., $f^q(0, t(x_0))$ is a regular point of Z.

Now we will construct a local lift of f. Consider a local holomorphic lift F^q of f^q near $(0, t(x_0))$ in $U \times \mathbb{C}^{\mathfrak{A}_{p,q}}$ which exists by Proposition 4.1. We can choose y near 0 so that $f(y) \in Z_0$, so there exists a local holomorphic lift F_y of f near y, and still $(\widetilde{\Delta} \circ f^q)(y, t(x_0)) \neq 0$. Consider the map F_y^q defined by formula (4). Both F_y^q and F^q are local lifts of f^q at $(y, t(x_0))$. By Theorem 2.7, there exists $g \in G$ such that $F_y^q = gF^q$ near $(y, t(x_0))$.

Since $F_y^q(x,t)$ is linear in $t \in \mathbb{C}^{\mathfrak{A}_{p,q}}$, also $F^q(x,t)$ is linear in t and thus is defined for all t. Put $t_1 := (t_{1,A})_A$, where $t_{1,\emptyset} = 1$ and $t_{1,A} = 0$ for $A \neq \emptyset$. Then near $0 \in \mathbb{C}^p$ we have by 2.9(1),

$$\sigma_j(F^q(x, t_1)) = f_j^q(x, t_1) = \Big(\widetilde{S}_j(\emptyset, \ldots, \emptyset) \circ j^q f\Big)(x) = f_j(x),$$

i.e., $F^q(\quad, t_1)$ is a local lift of f at 0. $\square$

Remark. Consider the grading of the ring $\mathbb{C}[Z] = \mathbb{C}[V]^G$ induced by the natural grading of the polynomial ring $\mathbb{C}[V]$ and denote by r the order of the homogeneous function $\widetilde{\Delta}$. Let $f : \mathbb{C}^p, 0 \to Z$ be a germ of a holomorphic map satisfying for some positive integer q and for each $j = 1, \ldots, m$ the following conditions:

(1) The function $\widetilde{S}_j(A_1, \ldots, A_{d_j}) \circ j^q f$ has a holomorphic extension to a neighborhood of 0 for each system of multi-indices $A_1, \ldots, A_{d_j} \in \mathfrak{A}_{p,q}$ such that, $\Big(\widetilde{S}_j(A_1, \ldots, A_{d_j}) \circ j^{q-1} f\Big)(0) = 0$ for all $A_1, \ldots, A_{d_j} \in \mathfrak{A}_{p,q-1}$;

(2) $\operatorname{Tay}_0^{rq}(\widetilde{\Delta}(f)) \neq 0$.

Then f has a local lift at 0.

Actually, since $\mathrm{Tay}_0^{rq}(\widetilde{\Delta}(f)) = \mathrm{Tay}_0^{rq}(\widetilde{\Delta}(\mathrm{Tay}_0^q f))$, the proof of Theorem 4.2 is valid for this q.

4.3. The conditions for lifting of first order. Next we use the notion of rational tensor fields on affine varieties (see [**7**]).

For each $0 \le s \le d$ consider the rational symmetric tensor field τ_s of type $\binom{0}{s}$ on Z defined as follows:

$$\tau_s := \sum_{i_1,\dots,i_s=1}^{n} \widetilde{T}\Big((i_1),\dots,(i_s),\emptyset,\dots,\emptyset\Big)\, dz_{i_1}\otimes\cdots\otimes dz_{i_s}$$

where $\widetilde{T}\Big((i_1),\dots,(i_s),\emptyset,\dots,\emptyset\Big)$ is a partial case of the function $\widetilde{T}(I_1,\dots,I_d)$ defined in 3.3.

Consider the pull back $\pi^*\tau_s$ of τ_s. By definition, we have

$$\pi^*\tau_s = \sum_{i_1,\dots,i_s=1}^{n} T(v_{i_1},\dots,v_{i_s},v,\dots,v)\, dy_{i_1}\otimes\cdots\otimes dy_{i_s} = T(dv,\dots,dv,v,\dots,v)$$

and then $\tau_s = \pi_* T(dv,\dots,dv,v,\dots,v)$, where dv occurs precisely s times. Since the projection π is étale on V_0 and by the above formula for $\pi^*\tau_s$ the tensor field $\pi^*\tau_s$ is independent of the choice of the invariant coordinates y_i, the tensor field τ_s is independent of the choice of the invariant coordinates as well. Note that $\tau_1 = \frac{1}{d}d\tau$ induces a regular differential 1-form on Z.

By Lemma 3.2, for each germ $f : \mathbb{C}^p, 0 \to Z$ of a holomorphic map we have

$$f^*\tau_s = \sum_{a_1,\dots,a_s}^{p} \Big(\widetilde{T}\big((a_1),\dots,(a_s),\emptyset,\dots,\emptyset\big)\circ j^1 f\Big)\cdot dx_{a_1}\otimes\cdots\otimes dx_{a_s}.$$

Denote by $\sigma_{j,s}$, where $(1 \le s \le d_j)$, the tensor field τ_s for $\tau = \sigma_j$. Then the conditions for lifting of Theorem 4.2 for each $j = 1,\dots,m$ and $A_1,\dots,A_{d_j} \in \mathfrak{A}_{p,1}$ are equivalent to the following statement: For each $1 \le s_j \le d_j$ the pull back $f^*\sigma_{j,s_j}$ is a holomorphic germ of a symmetric covariant tensor field on $\mathbb{C}^p$. We call these conditions the conditions of first order. For $s = 1$ these conditions are satisfied automatically.

Note that the conditions of the remark at the end of 4.2 for $q = 1$ use the conditions of first order only. For example, it suffices to consider only these conditions if we need to have lifts which are linear maps from $\mathbb{C}^p$ to V.

4.4. Global holomorphic lifts. The following theorem shows that the problem of global holomorphic lifting can be described topologically.

Theorem. *Let $U \subseteq \mathbb{C}^p$ be a classically open connected subset of $\mathbb{C}^p$ and let $f : U \to Z = V/G$ be a holomorphic map such that $f^{-1}(Z_0) \ne \emptyset$. Then a holomorphic lift $F : U \to V$ exists iff the image of the fundamental group $\pi_1(f^{-1}(Z_0))$ under f is contained in the image of the fundamental group $\pi_1(V_0)$ under the projection π.*

Proof. Since by Proposition 4.1, the local holomorphic lift of f exists for each $x \in f^{-1}(Z_0)$, the condition of the theorem is equivalent to the existence of a holomorphic lift for the restriction of f to $f^{-1}(Z_0)$. Actually, let F be such a lift. Since $f^{-1}(Z_0)$ is an open dense subset of U and π is a finite morphism, the lift F is bounded on bounded subsets of $U \cap f^{-1}(Z_0)$. Then by Riemann's extension theorem F has a holomorphic extension to U which is a holomorphic lift of f. $\square$

4.5. We indicate that the problem of the existence of a global regular lift reduces to the one for a holomorphic lift.

Theorem. *Let $U \subseteq \mathbb{C}^p$ be a Zariski-open subset of $\mathbb{C}^p$ and let $f : \mathbb{C}^p \to Z = V/G$ be a rational morphism which is regular in U and such that $f^{-1}(Z_0) \neq \emptyset$.*
If a global holomorphic lift of f on U exists then it is regular.

Proof. The proof follows from Lemma 5.1(1) of [**7**]. $\square$

4.6. Global regular lifts. Now we indicate conditions for the existence of a global regular lift.

Theorem. *Let $f : \mathbb{C}^p \to Z = V/G$ be a regular morphism such that $f(\mathbb{C}^p) \cap Z_0 \neq \emptyset$. Then f has a regular lift iff there is an integer $q > 0$ such that, for each $j = 1, \dots m$ and each multi-index $A = (a_1, \dots, a_q)$, the mapping $\widetilde{S}_j(A, \dots, A) \circ j^q f$ is constant.*

Proof. Let u_i be linear coordinates in V and let $F = (F_1, \dots, F_n)$ be the expression of a regular lift of f in these coordinates. Suppose q is the maximal degree of the polynomials F_i. Then for each $A = (a_1, \dots, a_q)$ the map $S_j(A, \dots, A) \circ j^q F$ is constant. Theorem 3.3 implies that $\widetilde{S}_j(A, \dots, A) \circ j^q f$ is constant as well.

Let the condition of the theorem be satisfied. Let $x \in \mathbb{C}^p$ be a point such that $f(x) \in Z_0$. Then there is a local lift F of f at x. By Theorem 4.2, the condition of the theorem implies that $\sigma_j(\partial_A F)$ is constant for each $A = (a_1, \dots, a_q)$ and each j. But this means that $\partial_A F$ is constant also and, therefore, F is a polynomial map of degree $\leq q$ in a neighborhood of x. Thus F has a polynomial extension to the whole of $\mathbb{C}^p$ and this extension is a lift of f. $\square$

4.7. We consider a special case when the existence of a local lift implies the existence of a global lift.

Theorem. *Let $f = (f_j) : \mathbb{C}^p \to \mathbb{C}^m$ be a regular morphism from $\mathbb{C}^p$ to $Z = V/G$ such that each function f_j is homogeneous of degree rd_j for some positive integer r. Then a global regular lift of f exists iff f has a local holomorphic lift at $0 \in \mathbb{C}^p$.*

Proof. Consider the action of the group $\mathbb{C}^*$ on Z induced by the action of the homothety group on V. This action induces a homotopy equivalence between the open subset $f^{-1}(Z_0)$ and the open subset $f^{-1}(Z_0) \cap B$, where B is an open ball in $\mathbb{C}^p$ centered at 0. Then the statement of the theorem follows from Theorems 4.4 and 4.5. $\square$

5. Formal lifts

In this section we find the conditions for lifts of quasiregular formal morphisms from $\mathbb{C}^p$ to Z. Note first that Proposition 4.1 about the existence of lifts at regular points also holds for formal morphisms.

Let $\sigma_1, \dots, \sigma_m$ be the basic generators of $\mathbb{C}[V]^G$, $\deg \sigma_j = d_j$ for $j = 1, \dots, m$, and $\sigma : V \to \sigma(V) \subseteq \mathbb{C}^m$ the corresponding morphism. Consider some invariant coordinates y_i on V and the corresponding function $\widetilde{\Delta}$ on Z. Recall that we consider Z as a Zariski-closed subset $\sigma(V)$ of $\mathbb{C}^m$ defined by the ideal $I(Z)$ of the ring $\mathbb{C}[W_1, \dots, W_m]$ and, for this presentation of Z, the projection π equals the map $\sigma : V \to \sigma(V) \subseteq \mathbb{C}^m$.

5.1. The functions $P_q(\tau)$ and $\widetilde{P}_q(\tau)$. For a homogeneous G-invariant polynomial τ on V, consider the function $P_q(\tau) : J_0^\infty(\mathbb{C}^p, V) \to \mathbb{C}[(t_A)_A] \otimes \mathbb{C}[[X_1, \dots, X_p]]$ and the function $\widetilde{P}_q(\tau)$ on the set of quasiregular formal morphisms $f \in J_0^\infty(\mathbb{C}^p, Z)$ such that $\widetilde{\Delta}(f) \neq 0$ with values in $\mathbb{C}[(t_A)_A] \otimes \mathbb{C}((X_1, \dots, X_p))$, where $(t_A)_A = (t_A)_{A \in \mathfrak{A}_{p,q}}$ and $\mathbb{C}[(t_A)_A]$ is the ring of polynomials in $(t_A)_A$ with complex coefficients, defined as follows:

$$\begin{aligned} P_q(\tau)(F) &:= \sum_{A_1, \dots, A_d \in \mathfrak{A}_{p,q}} \mathfrak{T}(A_1, \dots, A_d)(F)\, t_{A_1} \dots t_{A_d}, \\ \widetilde{P}_q(\tau)(f) &:= \sum_{A_1, \dots, A_d \in \mathfrak{A}_{p,q}} \widetilde{\mathfrak{T}}(A_1, \dots, A_d)(f)\, t_{A_1} \dots t_{A_d}, \end{aligned}$$

where $F \in J_0^\infty(\mathbb{C}^p, V)$, $f \in J_0^\infty(\mathbb{C}^p, Z)$, and where $\mathfrak{T}(A_1, \dots, A_d)$ and $\widetilde{\mathfrak{T}}(A_1, \dots, A_d)$ are the functions defined in 2.9 and 3.5.

The following lemma follows from the definitions of $P_q(\tau)$, $\widetilde{P}_q(\tau)$, $\widetilde{\mathfrak{T}}(A_1, \dots, A_d)$, and Theorem 3.5.

Lemma.

(1) *We have*

$$P_q(\tau) = \widetilde{P}_q(\tau) \circ J_0^\infty(\mathbb{C}^p, \pi) : J_0^\infty(\mathbb{C}^p, V) \to \mathbb{C}[(t_A)_A] \otimes \mathbb{C}[[X_1, \dots, X_p]].$$

$$\begin{tikzcd} J_0^\infty(\mathbb{C}^p, V) \arrow[r, "P_q(\tau)"] \arrow[d, "J_0^\infty(\mathbb{C}^p, \pi)"'] & \mathbb{C}[(t_A)_A] \otimes \mathbb{C}[[X_1, \dots, X_p]] \\ \bar{J}_0^\infty(\mathbb{C}^p, Z) \arrow[ur, "\widetilde{P}_q(\tau)"'] & \end{tikzcd}$$

(2) *If $\tau_1, \tau_2 \in \mathbb{C}[V]^G$ are homogeneous polynomials of the same degree, then $\widetilde{P}_q(\tau_1 + \tau_2) = \widetilde{P}_q(\tau_1) + \widetilde{P}_q(\tau_2)$;*

(3) *Let $\tau_1, \tau_2 \in \mathbb{C}[V]^G$ be homogeneous polynomials. Then we have $\widetilde{P}_q(\tau_1 \tau_2) = \widetilde{P}_q(\tau_1)\widetilde{P}_q(\tau_2)$.*

(4) *Let f be a polynomial in the graded variables $T_1,\dots,T_r$ of degrees $d_1,\dots,d_r$ which is homogeneous with respect to this grading, and let $\tau_1,\dots,\tau_r \in \mathbb{C}[V]^G$ be homogeneous polynomials of degrees $d_1,\dots,d_r$. Then we have*

$$P_q\Big(f(\tau_1,\dots,\tau_r)\Big) = f\Big(P_q(\tau_1),\dots,P_q(\tau_r)\Big),$$
$$\widetilde{P}_q\Big(f(\tau_1,\dots,\tau_r)\Big) = f\Big(\widetilde{P}_q(\tau_1),\dots,\widetilde{P}_q(\tau_r)\Big).$$

5.2. For $\sigma = (\sigma_1,\dots,\sigma_m)$ and a formal morphism $F : \mathbb{C}[V] \to \mathbb{C}[[X_1,\dots,X_p]]$ from $\mathbb{C}^p$ to V put $\sigma(F) := (F(\sigma_1),\dots,F(\sigma_m))$.

Lemma. *Let $F : \mathbb{C}[V] \to \mathbb{C}[[X_1,\dots,X_p]]$ be a formal morphism from $\mathbb{C}^p$ to V. If $\sigma(F) = 0$, then F vanishes on the set of all regular functions on V with zero constant terms; or, $F = 0$ as an element of $V \otimes \mathbb{C}[[X_1,\dots,X_p]]$.*

Proof. Let (e_i) be a basis of V and u_i the corresponding coordinates. It is sufficient to prove that $F(u_i) = 0$. Since the group G is finite the ring $\mathbb{C}[V]$ is integral over its subalgebra $\mathbb{C}[V]^G$. Then for each $i = 1,\dots,n$ there is a polynomial $p(x) = x^N + \sum_{j=1}^N a_{N-j}x^{N-j}$, whose coefficients a_{N-i} belong to $\mathbb{C}[V]^G$, such that $p(u_i) = 0$. Consider the natural grading of the ring $\mathbb{C}[V]$. Since $\deg((u_i)^N) = N$ we may assume that $\deg a_{N-j} = j$. This implies that the coefficients a_{N-j} as polynomials in σ_j have no constant terms. Then we have

$$0 = F(p(u_i)) = F(u_i)^N + \sum_{j=1}^n F(a_{N-j})F(u_i)^{N-j}.$$

Since $\sigma(F) = 0$, this equation implies $F(u_i)^N = 0$ and therefore $F(u_i) = 0$. $\square$

5.3. The conditions for formal lifts. For $F \in J_0^\infty(\mathbb{C}^p, V)$ consider $P_q(\tau)(F)$ as a polynomial in $(t_A)_A$ with coefficients in $\mathbb{C}[[X_1,\dots,X_p]]$. Denote by $P_q(\tau)(F)_0(t)$ the polynomial in $(t_A)_A$ which is obtained by the evaluation of the coefficients of the polynomial $P_q(\tau)(F)$ at $X = (X_1,\dots,X_p) = 0$. Similarly, for $f \in J_0^\infty(\mathbb{C}^p, Z)$ consider $\widetilde{P}_q(\tau)(F)$ as a polynomial in $(t_A)_A$ with coefficients in $\mathbb{C}((X_1,\dots,X_p))$ and denote by $\widetilde{P}_q(\tau)(f)_0(t)$ the polynomial in t which is obtained by the evaluation of the coefficients of the polynomial $\widetilde{P}_q(\tau)(f)$ at $X = (X_1,\dots,X_p) = 0$ whenever their values at $X = 0$ are defined.

For

$$A = (\underbrace{1,\dots,1}_{r_1 \text{ times}},\dots,\underbrace{p,\dots,p}_{r_p \text{ times}},$$

put

$$t_A(X) := \frac{1}{r_1!\dots r_p!}\,(X_1)^{r_1}\dots(X_p)^{r_p}, \quad t(X) := (t_A(X))_{A\in\mathfrak{A}_{p,q}}.$$

For a formal power series $\varphi \in \mathbb{C}[[X_1,\dots,X_p]]$, denote by $\operatorname{Tay}^q \varphi$ the sum of the terms of φ of orders $\le q$. Denote by $\widetilde{\mathfrak{S}}_j(A_1,\dots,A_{d_j})$ the function $\widetilde{\mathfrak{T}}(A_1,\dots,A_{d_j})$ for $\tau = \sigma_j$.

Recall that by 2.7 for a quasiregular formal morphism $f \in J_0^\infty(\mathbb{C}^p, Z)$ there is a choice of invariant coordinates such that for the corresponding function $\widetilde{\Delta}$ we have $f(\widetilde{\Delta}) \neq 0$ which here we write also as $\widetilde{\Delta}(f) \neq 0$.

Theorem. *Let $f \in J_0^\infty(\mathbb{C}^p, Z)$ be a quasiregular formal morphism given by the equations $f(w_j) = f_j \in \mathbb{C}[[X_1, \dots, X_p]]$ for $j = 1, \dots, m$, where w_j are the standard coordinate functions on $\mathbb{C}^m \supseteq Z$. Let y_i be invariant coordinates on V such that for the corresponding function $\widetilde{\Delta}$ we have $\widetilde{\Delta}(f) \neq 0$. Assume q is the minimal order of nonzero terms of $\widetilde{\Delta}(f)$.*

Then a formal lift F of f exists iff for $j = 1, \dots, m$ and for each system of multi-indices $A_1, \dots, A_{d_j} \in \mathfrak{A}_{p,q}$ we have $\widetilde{\mathfrak{S}}_j(A_1, \dots, A_{d_j})(f) \in \mathbb{C}[[X_1, \dots, X_p]]$ and $\operatorname{Tay}^q f_j = \widetilde{P}_q(\sigma_j)(f)_0(t(X))$.

Proof. Let F be a formal lift of f. Then, by Theorem 3.5, we have

$$\widetilde{\mathfrak{S}}_j(A_1, \dots, A_{d_j})(f) = \mathfrak{S}_j(A_1, \dots, A_{d_j})(F) \in \mathbb{C}[[X_1, \dots, X_p]].$$

Moreover, by Lemma 5.1, we have

$$\operatorname{Tay}^q f_j = \operatorname{Tay}^q(\sigma_j(F)) = P_q(\sigma_j)(F)_0(t(X)) = \widetilde{P}_q(\sigma_j)(f)_0(t(X)).$$

Conversely, let the assumptions of the theorem be satisfied and let q be the minimal order of nonzero terms of $\widetilde{\Delta}(f)$. By assumption, $\operatorname{Tay}^q(\widetilde{\Delta}(f)) \neq 0$. Then there is a point $x_0 = (x_{0,1}, \dots, x_{0,p}) \in \mathbb{C}^p$ such that $\operatorname{Tay}^q(\widetilde{\Delta}(f))(x_0) \neq 0$.

For each $j = 1, \dots, m$ consider the function

$$f_j^q((t_A)_A) := \widetilde{P}_q(\sigma_j)(f) = \sum_{A_1, \dots, A_{d_j} \in \mathfrak{A}_{p,q}} \widetilde{\mathfrak{S}}_j(A_1, \dots, A_{d_j})(f)\, t_{A_1} \dots t_{A_{d_j}}.$$

We may consider $f^q = (f_j^q)$ as a formal morphism $\mathbb{C}^{\mathfrak{A}_{p,q}} \times \mathbb{C}^p, (t(x_0)), 0) \to \mathbb{C}^m$, i.e., as a morphism $\widetilde{\mathcal{O}}_{(\mathbb{C}^m, f^q(t(x_0)), 0)} \to \widetilde{\mathcal{O}}_{\mathbb{C}^{\mathfrak{A}_{p,q}} \times \mathbb{C}^p, (t(x_0)), 0)}$. We prove that $f^q = (f_j^q)$ is a formal morphism $\mathbb{C}^{\mathfrak{A}_{p,q}} \times \mathbb{C}^p, (t(x_0), 0) \to Z$ by the following arguments. Let $\Phi \in I(Z)$ be a homogeneous polynomial. Then $\Phi \circ \sigma = 0$ and, by Lemma 5.1, we have $\Phi(f_j^q) = \Phi(\widetilde{P}_q(\sigma_j)(f)) = \widetilde{P}_q(\Phi \circ \sigma_j)(f) = 0$.

By assumption, we have $\operatorname{Tay}^q f_j = \widetilde{P}_q(\sigma_j)(f)_0((t_A(X))_A) = f_j^q((t_A(X))_A, 0)$. Since the polynomial Δ is homogeneous, we have

$$\operatorname{Tay}^q\Big(\widetilde{\Delta}(f^q(t(X), 0))\Big)(x_0) = \operatorname{Tay}^q(\widetilde{\Delta}(\operatorname{Tay}^q f))(x_0) = \operatorname{Tay}^q(\widetilde{\Delta}(f))(x_0) \neq 0.$$

Thus, the formal morphism $f^q = (f_j^q)_{j=1,\dots,m}$ has a formal lift

$$F^q : \mathbb{C}^{\mathfrak{A}_{p,q}} \times \mathbb{C}^p, ((t(x_0), 0) \to V,$$

which can be written as follows: $F^q = \sum_A F_A^q\, t_A$, where A is a multi-index and $F_A^q \in V \otimes \mathbb{C}[[X_1, \dots, X_p]]$.

Since F^q is a formal lift of f^q, for each $j = 1, \dots, m$ we have

$$\begin{aligned}\sigma_j(F^q) &= \sum_{A_1,\dots,A_{d_j}} \mathfrak{S}_j(F^q_{A_1}, \dots, F^q_{A_{d_j}})\, t_{A_1} \dots t_{A_{d_j}} \\ &= \sum_{A_1,\dots,A_{d_j} \in \mathfrak{A}_{p,q}} \widetilde{\mathfrak{S}}_j(A_1, \dots, A_{d_j})(f^q)\, (t_{A_1} + t_{A_1}(x_0)) \dots (t_{A_{d_j}} + t_{A_{d_j}}(x_0)),\end{aligned}$$

This implies that $\mathfrak{S}_j(F^q_{A_1}, \dots, F^q_{A_{d_j}}) = 0$ whenever for some $k = 1, \dots, d_j$ we have $|A_k| > q$. In particular, for a multi-index A such that $|A| > q$ and for each $j = 1, \dots, m$ we have $\sigma_j(F^q_A) = 0$. By Lemma 5.2, we have $F^q_A = 0$ and, therefore, the formal lift F^q is a polynomial in $(t_A)_A$ with coefficients in $V \otimes \mathbb{C}[[X_1, \dots, X_p]]$. Put $t_1 := (t_1^A)_A$ where $t_1^\emptyset = 1$, and $t_1^A = 0$ for $A \neq \emptyset$. Denote by $F^q(t_1)$ the value of F^q as a polynomial in t at $t = t_1$. Then we have

$$\sigma_j(F^q(t_1)) = f_j^q(t_1) = \widetilde{\mathfrak{S}}_j(\emptyset, \dots, \emptyset)(f) = f_j,$$

i.e., $F^q(t_1)$ is a formal lift of f. □

5.4. Theorem 5.3 implies the following

Corollary. *The map $\pi^\infty : J_0^\infty(\mathbb{C}^p, V)/G \to J_0^\infty(\mathbb{C}^p, Z)$ is injective.*

Proof. Let $f \in J_0^\infty(\mathbb{C}^p, Z)$ be a formal morphism which has a lift to V.

First assume that the morphism f is quasiregular. Consider a formal morphism $f^q = (f_j^q)_j$ from $\mathbb{C}^{\mathfrak{A}_{p,q}} \times \mathbb{C}^p, (t(x_0), 0)$ to Z constructed for f in the proof of Theorem 5.3 and one of its lifts $F^q : \mathbb{C}^{\mathfrak{A}_{p,q}} \times \mathbb{C}^p, (t(x_0), 0) \to V$. Since $F^q(t(x_0), 0)$ is a regular point of V, the lift F^q is unique up to the action of some $g \in G$. On the other hand, for each lift F of f, $F^q = \sum_A \frac{1}{A!} \partial_A F\ t_A$ is a lift of f^q. This implies that the lift F of f is unique up to the action of some $g \in G$.

For an arbitrary formal morphism $f \in J_0^\infty(\mathbb{C}^p, Z)$, there is a subgroup K of G such that we can consider f as a quasiregular formal morphism to $V^K/(N_G(K)/K)$. Then one can prove our statement using the same arguments as in the proof of Theorem 2.7. □

6. Complex reflection groups

In this section we consider the case when G is a finite group generated by complex reflections.

6.1. Relative invariants. Recall some standard facts about finite complex reflection groups (see [**15**], [**16**], and [**17**]).

First note that in this case the basic generators $\sigma_1, \dots, \sigma_m$ are algebraically independent, $m = n = \dim V$, and then $Z = \mathbb{C}^n$. Therefore we have a unique (up to permutation) choice of invariant coordinates y_i .

Suppose $\mathfrak{H}$ is the set of reflection hyperplanes of G and, for each $H \in \mathfrak{H}$, denote by e_H the order of the cyclic subgroup G_H of G fixing H pointwise, by s_H a generator of G_H, and by l_H a linear functional with the kernel H. Then the Jacobian J equals $\prod_{H \in \mathfrak{H}} l_H^{e_H - 1}$ up to some nonzero constant factor c and one can take $\prod_{H\in\mathfrak{H}} l_H^{e_H}$ for Δ. We may choose the functionals l_H such that $J = \prod_{H\in\mathfrak{H}} l_H^{e_H-1}$.

Denote by E_G the set of the orders e_H of the cyclic subgroups G_H fixing pointwise the reflection hyperplanes and, for each $e \in E_G$, set $\mathfrak{H}_e := \{H \in \mathfrak{H} | e_H = e\}$. Then $\Delta_e := \prod_{H\in\mathfrak{H}_e} l_H^e$ is G-invariant polynomial. Denote by $\widetilde{\Delta}_e$ the regular function on Z such that $\widetilde{\Delta}_e \circ \pi = \Delta_e$.

Consider the standard action of the group G on $\mathbb{C}[V]$ given by $g \cdot f = f \circ g^{-1}$ for every $f \in \mathbb{C}[V]$ and $g \in G$.

Let $\chi : G \to \mathbb{C} \setminus 0$ be a character of G. A *χ-relative invariant* is a polynomial $f \in \mathbb{C}[V]$ such that $g \cdot f = \chi(g) f$ for each $g \in G$.

Theorem. *Let G be a finite group generated by complex reflections and χ a character of G such that $\chi(s_H) = \det^{-k_s}(s_H)$ for some k_s. Then the polynomial $f_\chi := \prod_{H\in\mathfrak{H}} l_H^{k_s}$ is a χ-relative invariant and the space of χ-relative invariants equals the space $\mathbb{C}[V]^G f_\chi$.*

In particular, the polynomial $J_e = \prod_{H\in\mathfrak{H}_e} l_H^{e-1}$ is a χ_e-relative invariant for a character χ_e such that $\chi_e(s_H) = \det^{1-e}(s_H)$, when $H \in \mathfrak{H}_e$, and $\chi_e(s_H) = 1$, when $H \notin \mathfrak{H}_e$. Then the jacobian J is a χ-relative invariant for a character χ such that $\chi(s_H) = \det^{1-e_H}(s_H)$.

6.2. The functions $\widetilde{T}(A_1, \dots, A_d)$ and $\widetilde{\mathfrak{T}}(A_1, \dots, A_d)$ for complex reflection groups. For a complex reflection group G one can improve the statement (3) of Theorem 3.3 and the statement (2) of Theorem 3.5 as follows.

Theorem. *Let τ be a homogeneous G-invariant polynomial of degree d on V and let $A_1, \dots, A_d$ be a system of multi-indices such that $A_1, \dots, A_{d'} \neq \emptyset$ and $A_{d'+1} = \cdots = A_d = \emptyset$. Put $M := 2(|A_1| + \cdots + |A_d|) - d'$ and let, for each $e \in E_G$, both μ_e and ν_e be positive integers such that $M = \mu_e e - \nu_e$, where $0 \le \nu_e \le e$. Then*

(1) *$\prod_{e\in E_G} \widetilde{\Delta}_e^{M-\mu_e} \widetilde{T}(A_1, \dots, A_d)$ is a regular function on Z;*
(2) *Regard $\widetilde{\Delta}_e$ as a function on $J_0^\infty(\mathbb{C}^p, Z)$. Then $\prod_{e\in E_G} \widetilde{\Delta}_e^{M-\mu_e} \widetilde{\mathfrak{T}}(A_1, \dots, A_d)$ is a function on $J_0^\infty(\mathbb{C}^p, Z)$ with values in $\mathbb{C}[[X_1, \dots, X_p]]$.*

Proof. (1) Consider the G-invariant rational function $T(\partial_{I_1} v, \dots, \partial_{I_d} v)$ on V, where $I_1, \dots, I_d$ be a system of multi-indices such that $1 \le |I_1| \le |A_1|, \dots, 1 \le |I_{d'}| \le |A_{d'}|$ and $I_{d'+1} = \cdots = I_d = \emptyset$. It is easily checked that

$$J^M = \prod_{e\in E_G} \Delta_e^{M-\mu_e} f_\chi,$$

where χ is a character defined by the condition $\chi(s_H) = \det(s_H)^{-\nu_e}$ for $H \in \mathfrak{H}_e$. Since by Lemma 3.1, $J^M\ T(\partial_{I_1} v, \dots, \partial_{I_d} v)$ is a regular function on V, $J^M\ T(\partial_{I_1} v, \dots, \partial_{I_d} v)$ is a χ-relative invariant for the same character χ and then by Theorem 6.1 we have $J^M\ T(\partial_{I_1} v, \dots, \partial_{I_d} v) \in \mathbb{C}[V]^G f_\chi$. Therefore $\prod_{e\in E_G} \Delta_e^{M-\mu_e}\ T(I_1, \dots, I_d)$ is a G-invariant polynomial on V and (1) follows from the definition of the function $\widetilde{T}(I_1, \dots, I_d)$ and Theorem 3.3(1).

One can prove (2) similarly using the definition of the function $\widetilde{\mathfrak{T}}(A_1, \dots, A_d)$ and Lemma 3.4. □

To apply the above results on lifting we need to find explicit expressions for the functions $\widetilde{S}_j(A_1, \dots, A_{d_j})$ and $\widetilde{\mathfrak{S}}_j(A_1, \dots, A_{d_j})$ for particular G-modules V. Although this problem is purely technical the computations are rather complicated. Therefore we consider only simple examples.

6.3. First Example. Consider the simplest case when $V = \mathbb{C}$ and G is the complex reflection group generated by a generalized reflection $z \mapsto \exp(\frac{2\pi i}{n})z$ for some fixed $n \geq 2$. There is one basic invariant $\sigma : z \mapsto z^n$ and $Z = \mathbb{C}$. Consider, for example, the lifting problem for $f \in \mathfrak{F}_{\mathbb{C}^p,0}$, i.e., the problem of solving the equation $z^n = f$ in the ring $\mathfrak{F}_{\mathbb{C}^p,0}$. Note that in this case f is either quasiregular, or equals 0, and it suffices to assume that f is quasiregular.

Let $y = z^n$ be the invariant coordinate on V. It is clear that the symmetric n-linear form S on V corresponding to σ equals $z_1 \cdots z_n$. Consider the system of multi-indices $A_1, \dots, A_n$ where $A_1 = (a^1_1, \dots, a^1_{q_1}), \dots, A_r = (a^r_1, \dots, a^r_{q_r})$ and $A_{r+1} = \dots = A_n = \emptyset$. Put

$$f^r_{a^1_1 \dots a^1_{q_1}, \dots, a^r_1 \dots a^r_{q_r}} := S(\partial_{A_1} z, \dots, \partial_{A_n} z) = z^{n-r} \partial_{A_1} z \cdots \partial_{A_r} z.$$

By the general procedure we need to express $f^r_{a^1_1 \dots a^1_{q_1}, \dots, a^r_1 \dots a^r_{q_r}}$ via $y = z^n$ and its partial derivatives $\partial_A y$. We do this by the following recurrence relations.

$$\begin{aligned} f^1_a &= \frac{1}{n} \partial_a y, \\ y^{r-1} f^r_{a^1_1 \dots a^1_{q_1}, \dots, a^r_1 \dots a^r_{q_r}} &= f^1_{a^1_1 \dots a^1_{q_1}} \cdots f^1_{a^r_1 \dots a^r_{q_r}}, \\ f^1_{a_1 \dots a_q} &= \partial_{a_q} f^1_{a_1 \dots a_{q-1}} - (n-1) f^2_{a_q, a_1 \dots a_{q-1}}, \end{aligned} \tag{6}$$

which give the required formulas by induction with respect to $q = \max\{q_1, \dots, q_r\}$.

Thus, for the above system of multi-indices $A_1, \dots, A_n$ we have

$$\widetilde{S}(A_1, \dots, A_n) \circ j^q f = f^r_{a^1_1 \dots a^1_{q_1}, \dots, a^r_1 \dots a^r_{q_r}},$$

for $f \in J^q_0(\mathbb{C}, Z)$, and

$$\widetilde{\mathfrak{S}}(A_1, \dots, A_n)(f) = f^r_{a^1_1 \dots a^1_{q_1}, \dots, a^r_1 \dots a^r_{q_r}}(f),$$

for $f \in J^\infty_0(\mathbb{C}^p, Z)$.

There is the following stronger form of Theorem 4.2 for the case under consideration.

Theorem. *Let $f : \mathbb{C}^p, 0 \to Z = V/G$ be a germ of a holomorphic map. Then f has a local lift at 0 iff either $f = 0$, or there is a system of indices $(a_1, \dots, a_r)$ such that the functions $f^1_{a_1 \dots a_r} \circ j^r f$ and $f^n_{a_1 \dots a_r, \dots, a_1 \dots a_r} \circ j^r f$ have holomorphic extensions to a neighborhood of 0 and $\left(f^n_{a_1 \dots a_r, \dots, a_1 \dots a_r} \circ j^r f\right)(0) \neq 0$.*

Proof. We may suppose that $f(0) = 0$ and $f \neq 0$. Let $F : \mathbb{C}^p \to V$ be a local lift of f at 0. Since $F \neq 0$ there is a multi-index $A = (a_1, \dots, a_r)$ such that $\partial_A F(0) \neq 0$. Then the functions $f^1_{a_1\dots a_r} \circ j^r f = F^{n-1}\partial_A F$ and $f^n_{a_1\dots a_r,\dots,a_1\dots a_r} \circ j^r f = (\partial_A F)^n$ are holomorphic near 0 and $f^n_{a_1\dots a_r,\dots,a_1\dots a_r} \circ j^r f \neq 0$.

Suppose the conditions of the theorem are satisfied. By (6), the equality

$$f^{n-1} f^n_{a_1\dots a_r,\dots,a_1\dots a_r} \circ j^r f = (f^1_{a_1\dots a_r} \circ j^r f)^n \tag{7}$$

is satisfied in a neighborhood of 0 outside the complex analytic set $f = 0$. Thus it is satisfied near 0. By assumption, the germ $f^n_{a_1\dots a_r,\dots,a_1\dots a_r} \circ j^r f$ is invertible and the germs f and $f^1_{a_1\dots a_r} \circ j^r f$ are not invertible in the ring $\mathfrak{F}_{\mathbb{C}^p,0}$. Since this ring is factorial (see, for example, [**3**]), (7) implies that $f^1_{a_1\dots a_r} \circ j^r f$ divides f in this ring. Then, by (7),

$$F := \frac{f \sqrt[n]{f^n_{a_1\dots a_r,\dots,a_1\dots a_r} \circ j^r f}}{f^1_{a_1\dots a_r} \circ j^r f} \tag{8}$$

is a germ of a holomorphic function at 0 and a local lift of f at 0. By definition, F is defined up to multiplication by $\exp \frac{2\pi i}{n}$.

Note that for $n = 2$ we have, instead of (8), the following simpler formula for the local lift F:

$$F := \frac{f^1_{a_1\dots a_r} \circ j^r f}{\sqrt[n]{f^n_{a_1\dots a_r,\dots,a_1\dots a_r} \circ j^r f}}.$$

□

We leave it to the reader to formulate the corresponding results for formal and regular lifts.

6.4. Second Example. Now we consider the dihedral groups. Since the computations in this case are more complicated, we treat only the tensor fields which are used in the conditions of first order.

Let $G = \mathfrak{D}_l$ ($l \geq 3$) be the dihedral group acting on the real Euclidean plane $\mathbb{R}^2$. We consider the corresponding complexification of this action on $V = \mathbb{C}^2$.

Let $v = (x, y) \in \mathbb{C}^2$. We put $z := x + \mathrm{i}\, y$, $\widehat{z} := x - \mathrm{i}\, y$, and for each integer $k > 0$

$$\mathrm{R}\, z^k = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \binom{k}{2j} x^{k-2j} y^{2j} \quad \text{and} \quad \mathrm{I}\, z^k = \sum_{j=0}^{\left[\frac{k+1}{2}\right]-1} (-1)^j \binom{k}{2j+1} x^{k-2j-1} y^{2j+1}.$$

In particular, we have $\mathrm{R}\, z = x$ and $\mathrm{I}\, z = y$. It is easily checked that $z^k = \mathrm{R}\, z^k + \mathrm{i}\, \mathrm{I}\, z^k$ and $\widehat{z}^k = \mathrm{R}\, z^k - \mathrm{i}\, \mathrm{I}\, z^k$.

Choose the generators σ_1 and σ_2 of the ring $\mathbb{C}[V]^{\mathfrak{D}_l}$ as follows: $\sigma_1 = \frac{1}{2} z\widehat{z} = \frac{1}{2}(x^2 + y^2)$ and $\sigma_2 = \frac{1}{l}\, \mathrm{R}\, z^l = \frac{1}{2l}(z^l + \widehat{z}^l)$.

By definition, $y_1 = \sigma_1$ and $y_2 = \sigma_2$ are the invariant coordinates on V which we will consider also as the coordinates on the orbit space $Z = V/G$.

Denote by $\sigma_{1,s}$ and $\sigma_{2,s}$ the tensor fields τ_s from 4.3 for $\tau = \sigma_1$ and $\tau = \sigma_2$, respectively. To write the conditions of first order it suffices to calculate the tensor fields $\sigma_{1,2}$ and $\sigma_{2,s}$ for $s = 2, \dots, l$.

It is easily checked that for the coordinates z and $\widehat{z}$ on V we have

$$
\begin{aligned}
&J = -\frac{\mathrm{i}}{2}\,\mathrm{I}\, z^l \quad \sigma_1^s = \frac{1}{4}(z_1\widehat{z}_2 + z_2\widehat{z}_1), \quad \sigma_2^s = \frac{1}{2l}(z_1\cdots z_l + \widehat{z}_1\cdots\widehat{z}_l),\\
&dv = -\frac{1}{\mathrm{i}\,\mathrm{I}\, z^l}\Big((\widehat{z}^{l-1}dy_1 - zdy_2)\frac{\partial}{\partial z} + (-z^{l-1}dy_1 + \widehat{z}dy_2)\frac{\partial}{\partial \widehat{z}}\Big).
\end{aligned}
\tag{9}
$$

One can put $\Delta := (\mathrm{I}\, z^l)^2 = 2^l y_1^l - l^2 y_2^2$.

Furthermore we denote by $(dy_1)^p(dy_2)^q$ the symmetrized tensor product of p factors which are equal to dy_1 and q factors which are equal to dy_2.

Using (9) we get

$$\sigma_{1,s}(dv,dv) = \frac{1}{(\mathrm{I}\, z^l)^2}\Big(2^{l-1}y_1^{l-1}dy_1^2 - 2ly_2dy_1dy_2 + 2y_1dy_2^2\Big).$$

By the definition of $\sigma_{1,2}$, we have

$$\sigma_{1,2} = \frac{1}{\widetilde{\Delta}}\Big(2^{l-1}y_1^{l-1}dy_1^2 - 2ly_2dy_1dy_2 + 2y_1dy_2^2\Big).$$

Similarly, using (9) we get

$$
\begin{aligned}
&\sigma_{2,s}(\underbrace{dv,\dots,dv}_{s\text{ times}},\underbrace{v,\dots,v}_{l-s\text{ times}})\\
&= \frac{i^s}{2l(\mathrm{I}\, z^l)^s}\sum_{t=0}^{s}\binom{s}{t}\Big((-1)^{s-t}\widehat{z}^{(l-1)t}z^{l-t} + (-1)^t z^{(l-1)t}\widehat{z}^{l-t}\Big)dy_1^t dy_2^{s-t} =\\
&= \frac{(-1)^s i^s}{2l(\mathrm{I}\, z^l)^s}\Bigg((z^l + (-1)^s\widehat{z}^l)dy_2^s\\
&\qquad + \sum_{t=1}^{s}(-1)^t 2^{l-t}\binom{s}{t}y_1^{l-t}\Big(\widehat{z}^{l(t-1)} + (-1)^s z^{l(t-1)}\Big)dy_1^t dy_2^{s-t}\Bigg).
\end{aligned}
$$

Using the equality

$$\widehat{z}^{l(t-1)} + (-1)^s z^{l(t-1)} = (\mathrm{R}\, z^l - \mathrm{i}\,\mathrm{I}\, z^l)^{t-1} + (-1)^s(\mathrm{R}\, z^l + \mathrm{i}\,\mathrm{I}\, z^l)^{t-1}$$

we get

$$
\begin{aligned}
&\sigma_{2,s}(\underbrace{dv,\dots,dv}_{s\text{ times}},\underbrace{v,\dots,v}_{l-s\text{ times}})\\
&=\frac{(-1)^s i^s}{2l(\mathrm{I}\, z^l)^s}\Bigg((z^l + (-1)^s\widehat{z}^l)dy_2^s + \sum_{t=1}^{s}(-1)^t 2^{l-t}\binom{s}{t}y_1^{l-t}\\
&\qquad\cdot\sum_{j=0}^{t-1}\frac{1}{i^j}\binom{t-1}{j}(\mathrm{R}\, z^l)^{t-j-1}(\mathrm{I}\, z^l)^j(1+(-1)^{s+j})dy_1^t dy_2^{s-t}\Bigg).
\end{aligned}
\tag{10}
$$

Let $s = 2r$. Then (10) implies

$$\sigma_{2,2r} = \frac{(-1)^r}{\widetilde{\Delta}^r}\left(y_2 dy_2^{2r} + \sum_{t=1}^{2r} (-1)^t 2^{l-t} \binom{2r}{t} y_1^{l-t} \cdot \sum_{u=0}^{\left[\frac{t-1}{2}\right]} (-1)^u l^{t-2u-2} \binom{t-1}{2u} \widetilde{\Delta}^u y_2^{t-2u-1} dy_1^t dy_2^{2r-t} \right).$$

Let $s = 2r+1$. Then (10) implies

$$\sigma_{2,2r+1} = \frac{(-1)^r}{l\widetilde{\Delta}^r}\left(dy_2^{2r+1} + \sum_{t=2}^{2r+1} (-1)^t 2^{l-t} \binom{2r+1}{t} y_1^{l-t} \cdot \sum_{u=1}^{\left[\frac{t}{2}\right]} (-1)^u l^{t-2u} \binom{t-1}{2u-1} \widetilde{\Delta}^{u-1} y_2^{t-2u} dy_1^t dy_2^{2r+1-t} \right).$$

By 4.3 we can use the above tensor fields $\sigma_{1,2}$ and $\sigma_{2,s}$ to write down the conditions of first order for lifting for the dihedral groups $\mathfrak{D}_l$.

Remark. *The conditions of lifting for the $\mathfrak{D}_l$-module $\mathbb{C}^2$ could be reduced to those of the first example 6.3. This follows from the following formulas:*

$$(\mathrm{I}\, z^l)^2 = \widetilde{\Delta}, \quad z^l = \mathrm{R}\, z^l + \mathrm{i}\,\mathrm{I}\, z^l, \quad \widehat{z}^l = \mathrm{R}\, z^l - \mathrm{i}\,\mathrm{I}\, z^l.$$

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