Acta Math. Univ. Comenianae
Vol. LXXVII, 1(2008), pp. 141–145

SOLVABLE LIE ALGEBRAS AND MAXIMAL ABELIAN DIMENSIONS

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Abstract. In this paper some results on the structure of finite-dimensional Lie algebras are obtained by means of the concept of maximal abelian dimension. More concretely, a sufficient condition is given for the solvability in finite-dimensional Lie algebras by using maximal abelian dimensions. Besides, a necessary condition for the nilpotency is also stated for such Lie algebras. Finally, the maximal abelian dimension is applied to characterize the n-dimensional nilpotent Lie algebras with maximal abelian dimension equal to their codimension.

1. Introduction

Given a finite-dimensional Lie algebra $\mathfrak{g}$ over the complex number field $\mathbb{C}$, several Lie subalgebras can be found in it. In this paper, we are interested in knowing how many abelian Lie subalgebras are contained in $\mathfrak{g}$. As there is a unique non-isomorphic abelian algebra in each dimension, the number of non-isomorphic abelian subalgebras in $\mathfrak{g}$ can be computed starting from the maximum among the dimensions of the abelian subalgebras in $\mathfrak{g}$. This maximum is called the maximal abelian dimension of the Lie algebra $\mathfrak{g}$.

Our main goal in this paper is to prove some general results on the structure of the Lie algebras whose maximal abelian dimension is the codimension of the Lie algebra. More concretely, we are going to study some conditions on the solvability and the nilpotency of these Lie algebras.

This paper extends other earlier papers in which the maximal abelian dimension of the nilpotent Lie algebras $\mathfrak{g}_n$, formed by $n \times n$ strictly upper triangular matrices, were studied (see [**1, 2**]). In those papers, an algorithm was constructed to find abelian Lie subalgebras in $\mathfrak{g}_n$ up to a certain dimension which could not be improved by using that algorithm. Then the authors proved that the dimension of the obtained abelian Lie subalgebra was the maximal one and they called the maximal abelian dimension of $\mathfrak{g}_n$ to that value.

After this introduction, the structure of this paper is the following: in Section 2 we remind the definitions and results on solvable and nilpotent Lie algebras used

Received January 11, 2007; revised November 2, 2007.
2000 *Mathematics Subject Classification.* Primary 17B30; Secondary 17B05.
Key words and phrases. solvable Lie algebra; nilpotent Lie algebra; maximal abelian dimension.

later in the paper. The concept of maximal abelian dimension is also explained in this section. In the last section, we state and prove some general results which relate the structure of a Lie algebra to its maximal abelian dimension.

2. Solvable and Nilpotent Lie Algebras

For a general overview on Lie algebras, the reader can consult [**5**], for instance. We will consider several classes of Lie algebras over the complex number field $\mathbb{C}$ in this paper: solvable, nilpotent and filiform Lie algebras.

Given a Lie algebra $\mathfrak{g}$, its *lower central series* is given by:

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}^2(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}],\ \mathcal{C}^3(\mathfrak{g}) = [\mathcal{C}^2(\mathfrak{g}),\mathfrak{g}],\ \dots,\ \mathcal{C}^k(\mathfrak{g}) = [\mathcal{C}^{k-1}(\mathfrak{g}),\mathfrak{g}],\ \dots$$

and its *commutator central series*, by:

$$\mathcal{C}_1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}_2(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}],\ \mathcal{C}_3(\mathfrak{g}) = [\mathcal{C}_2(\mathfrak{g}),\mathcal{C}_2(\mathfrak{g})],\ \dots,\ \mathcal{C}_k(\mathfrak{g}) = [\mathcal{C}_{k-1}(\mathfrak{g}),\mathcal{C}_{k-1}(\mathfrak{g})],\ \dots$$

The Lie algebra $\mathfrak{g}$ is called *nilpotent* if there exists a natural number m such that $\mathcal{C}^m(\mathfrak{g}) \equiv 0$. Analogously, the Lie algebra $\mathfrak{g}$ is said to be *solvable* if there exists a natural number m such that $\mathcal{C}_m(\mathfrak{g}) \equiv 0$.

The third class of Lie algebras considered in this paper is a particular subclass of nilpotent Lie algebras: filiform Lie algebras. An n-dimensional *filiform Lie algebra* is an n-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that the dimensions of the ideals $\mathcal{C}^2(\mathfrak{g}), \dots, \mathcal{C}^k(\mathfrak{g}), \dots, \mathcal{C}^n(\mathfrak{g})$ are, respectively, $n-2, \dots, n-k, \dots, 0$.

For each dimension, there exists a particular filiform Lie algebra which is called the *model filiform Lie algebra* and whose law is the following:

$$[e_1, e_2] = 0; \quad [e_1, e_j] = e_{j-1}, \quad j = 3, \dots, n.$$

The main properties of nilpotent Lie algebras and filiform ones can be checked in [**3**] and [**6**], respectively.

Given a finite dimensional complex Lie algebra $\mathfrak{g}$, its *maximal abelian dimension* is the maximum among the dimensions of all the abelian Lie subalgebras of $\mathfrak{g}$. This natural number is denoted by $\mathcal{M}(\mathfrak{g})$. This definition generalizes the one given in [**2**] for a particular class of nilpotent Lie algebras.

As every Lie algebra $\mathfrak{g}$ contains abelian Lie subalgebras, we ask ourselves what is the largest dimension of such subalgebras. This is equivalent to determine how many non-isomorphic abelian Lie algebras are contained in $\mathfrak{g}$, since there exists only one non-isomorphic abelian Lie algebra in each dimension.

An abelian Lie subalgebra of $\mathfrak{g}$ is said to be *maximal* if the dimension of this subalgebra is equal to the maximal abelian dimension of $\mathfrak{g}$.

3. General Results

First, a sufficient condition is given for the solvability of a finite-dimensional complex Lie algebra starting from its maximal abelian dimension.

Proposition 3.1. *Given an n-dimensional complex Lie algebra $\mathfrak{g}$ with maximal abelian dimension $\mathcal{M}(\mathfrak{g}) = n-1$, the Lie algebra $\mathfrak{g}$ is solvable.*

Proof. Let $\mathfrak{g}$ be an n-dimensional complex Lie algebra such that $\mathcal{M}(\mathfrak{g}) = n-1$. Let $\mathfrak{h}$ be a maximal abelian subalgebra of dimension $n-1$. If $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ is the Levi decomposition of $\mathfrak{g}$, then $\mathfrak{s} \cap \mathfrak{h}$ is a subspace of dimension $\dim(\mathfrak{s}) - 1$ or $\dim(\mathfrak{s})$. This subspace is an abelian subalgebra of $\mathfrak{s}$. As $\mathfrak{s}$ is semi-simple, this is impossible. Then $\mathfrak{s} = \{0\}$ and $\mathfrak{g} = \mathfrak{r}$. This shows that $\mathfrak{g}$ is solvable. $\square$

The next proposition gives a necessary condition for the nilpotency in Lie algebras under the same hypotheses of Proposition 3.1. The condition can be expressed as follows:

Proposition 3.2. *Let $\mathfrak{g}$ be an n-dimensional complex nilpotent Lie algebra satisfying $\mathcal{M}(\mathfrak{g}) = n-1$. Then $\mathfrak{g}$ is a one-dimensional extension by derivation of an $(n-1)$-dimensional abelian Lie subalgebra $\mathfrak{a}$. In particular, the derived subalgebra $\mathcal{D}(\mathfrak{g})$ is contained in $\mathfrak{a}$ and it is abelian.*

Proof. Let $\mathfrak{g}$ be nilpotent and let $\mathfrak{h}$ be an abelian subalgebra of dimension $n-1$. If $\{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathfrak{g}$ such that $\{e_2, \cdots, e_n\}$ is a basis of $\mathfrak{h}$, we have:

$$[e_1, e_i] = \lambda_i e_1 + \sum_{j=2}^{n} a_i^j e_j,$$

where $\lambda_i \in \mathbb{C}$ and $a_i^j \in \mathbb{C}$, for $i, j = 2, \ldots, n$. Then, as $\mathfrak{h}$ is abelian, it holds

$$(\mathrm{ad}\ e_i)^p(e_1) = -\lambda_i^p e_1 - \lambda_i \left(\sum_{j=2}^{n} a_i^j e_j \right), \qquad \forall p \in \mathbb{N}.$$

As ad e_i is a nilpotent operator, $\lambda_i = 0$ for $i = 2, \ldots, n$ and, therefore, ad e_1 is an endomorphism of $\mathfrak{h}$. In consequence, the operator ad e_1 is a derivation of the abelian Lie algebra $\mathfrak{h}$ and the derived subalgebra $\mathcal{D}(\mathfrak{g})$ is contained in $\mathfrak{h}$. $\square$

Note that the reciprocal of Proposition 3.2 is false as can be seen in the following:

Example 3.3. Let $\mathfrak{g}$ be the 2-dimensional complex Lie algebra whose law is given by the bracket $[e_1, e_2] = e_2$. This Lie algebra is solvable since $\mathcal{C}_3(\mathfrak{g}) \equiv 0$; but it is not nilpotent since $\mathcal{C}^k(\mathfrak{g}) \equiv \langle e_2 \rangle$, for all $k \in \mathbb{N} \setminus \{1\}$. However, a maximal abelian subalgebra is $\langle e_2 \rangle$ and, hence, it is satisfied $\mathcal{D}(\mathfrak{g}) = \langle e_2 \rangle$.

Proposition 3.2 can be used to determine whether the maximal abelian dimension of an n-dimensional complex nilpotent Lie algebra is equal to $n-1$ or not.

Example 3.4. Let $\mathfrak{g}$ be the 6-dimensional complex nilpotent Lie algebra defined by the following brackets:

$$\begin{array}{l} [e_1, e_6] = e_5,\ [e_1, e_5] = e_4,\ [e_1, e_4] = e_3,\ [e_1, e_3] = e_2; \\ [e_4, e_5] = e_2,\ [e_4, e_6] = e_3,\ [e_5, e_6] = e_4. \end{array}$$

Since the derived algebra $\mathcal{D}(\mathfrak{g}) = \langle e_2,\ e_3,\ e_4,\ e_5 \rangle$ is not abelian, the maximal abelian dimension $\mathcal{M}(\mathfrak{g})$ is not equal to 5. Indeed, $\mathcal{M}(\mathfrak{g}) \leq 4$.

We conclude this section giving a sufficient and necessary condition for nilpotent Lie algebras satisfying $\mathcal{M}(\mathfrak{g}) = \operatorname{codim}(\mathfrak{g})$. This result allows us to classify the full class of nilpotent Lie algebras with that property.

Theorem 3.5. *Let $\mathfrak{g}$ be an n-dimensional complex nilpotent Lie algebra satisfying $\mathcal{M}(\mathfrak{g}) = n-1$. Then there exists an ordered sequence $(s_1, \ldots, s_p)$ such that $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}_{s_1,\ldots,s_p}$ defined by the following law:*

$$\begin{cases} [Y, X_i^1] = X_{i+1}^1, \ \textit{with } i = 1, \ldots, s_1 - 1, \ [Y, X_{s_1}^1] = 0 \\ [Y, X_i^2] = X_{i+1}^2, \ \textit{with } i = 1, \ldots, s_2 - 1, \ [Y, X_{s_2}^2] = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [Y, X_i^p] = X_{i+1}^p, \ \textit{with } i = 1, \ldots, s_p - 1, \ [Y, X_{s_p}^p] = 0 \end{cases}$$

Proof. As $\mathfrak{g}$ is an n-dimensional complex nilpotent Lie algebra such that $\mathcal{M}(\mathfrak{g}) = n-1$, then $\mathfrak{g}$ is an extension by derivation of an $(n-1)$-dimensional abelian Lie algebra $\mathfrak{h}$ in virtue of Proposition 3.2. Since any derivation of $\mathfrak{h}$ is given by an endomorphism of $\mathfrak{h}$, the one-dimensional extensions of $\mathfrak{h}$ are classified by the characteristic sequence of nilpotent endomorphisms. Recall that the characteristic sequence is the ordered sequence of the dimensions of Jordan blocks. If such a sequence is denoted by $(s_1, \ldots, s_p)$, the corresponding one-dimensional extension by derivation of $\mathfrak{h}$ is the Lie algebra $\mathfrak{g}_{s_1,\ldots,s_p}$. □

As an immediate application of Theorem 3.5, we can prove that the model filiform Lie algebras are those filiform Lie algebras whose maximal abelian dimension is the largest one among the filiform Lie algebras of a fixed dimension. Indeed, they are the only filiform ones satisfying $\mathcal{M}(\mathfrak{g}) = \operatorname{codim}(\mathfrak{g})$.

Corollary 3.6. *Let $\mathfrak{g}$ be an n-dimensional complex filiform Lie algebra satisfying $\mathcal{M}(\mathfrak{g}) = n-1$. Then $\mathfrak{g}$ is isomorphic to the filiform model Lie algebra.*

Proof. If $\mathfrak{g}$ is filiform, then we have the ordered sequence $(s_1, \ldots, s_p) = (n-1)$ and, in virtue of Theorem 3.5, $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}_{n-1}$, which is precisely the n-dimensional model filiform Lie algebra. □

Filiform Lie algebras do not usually appear expressed with respect to an adapted basis. Then it is not trivial to set if such algebras are isomorphic to the model one or not.

Example 3.7. Let $\mathfrak{g}$ be the 5-dimensional complex Lie algebra defined by the following brackets:

$$\begin{array}{l} [e_1, e_2] = -[e_1, e_4] = -[e_2, e_3] = -[e_3, e_4] = 1/2 \cdot (e_3 - e_1); \\ [e_1, e_5] = [e_3, e_5] = 1/2 \cdot (e_4 - e_2); \quad [e_1, e_3] = -(e_2 + e_4). \end{array}$$

By computing its lower central series, we can prove that this Lie algebra is filiform. But it is not possible to answer whether it is the model one or not. To assert that this algebra as the model filiform one in dimension 5, we prove that

$\mathcal{M}(\mathfrak{g}) = 4$. But this is true because the following 4-dimensional subalgebra is abelian:

$$\langle e_1 - e_3,\ e_2 + e_4,\ e_2 - e_4,\ e_5 \rangle.$$

Proposition 3.2 and Corollary 3.6 can be also used to prove that a given n-dimensional filiform Lie algebra is not the model one in that dimension as can be seen in the following:

Example 3.8. Let $\mathfrak{g}$ be the 6-dimensional complex Lie algebra considered in Example 3.4. In that example, we have proved that the maximal abelian dimension $\mathcal{M}(\mathfrak{g})$ is less than 5 in virtue of Proposition 3.2.

By computing the lower central series of $\mathfrak{g}$, we can prove that this algebra is filiform. According to Corollary 3.6, $\mathfrak{g}$ cannot be the 6-dimensional model filiform Lie algebra.

Acknowledgment. I would like to thank the referee for his/her valuable suggestions and comments which helped me to improve the quality of this paper.

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