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FURTHER GENERALIZATION OF SOME DOUBLE INTEGRAL INEQUALITIES AND APPLICATIONS

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ABSTRACT. Further generalization or improvement of some double integral inequalities are obtained. Applications in numerical integration are also given.

1. INTRODUCTION

Recently, N. Ujević [**8**] obtained the following double integral inequalities, which gave upper and lower error bounds for the well-known mid-point and trapezoid quadrature rules:

$$\frac{3S-2\Gamma}{24}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right) \le \frac{3S-2\gamma}{24}(b-a)^2, \tag{1.1}$$

$$\frac{3S-\Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{3S-\gamma}{24}(b-a)^2, \tag{1.2}$$

by defining

$$p(t) = \begin{cases} \dfrac{1}{2}(t-a)^2, & t \in \left[a, \dfrac{a+b}{2}\right] \\ \dfrac{1}{2}(t-b)^2, & t \in \left(\dfrac{a+b}{2}, b\right] \end{cases} \quad \text{and} \quad q(t) = \frac{1}{2}(t-a)(b-t),$$

respectively, where $f : [a,b] \to \mathbb{R}$ is a twice differentiable mapping, $\gamma \le f''(t) \le \Gamma$ for all $t \in (a,b)$ and $S = \dfrac{f'(b)-f'(a)}{b-a}$.

In this paper, we will generalize the above mentioned integral inequalities with a parameter λ by defining $p(t)$ as in (2.3) and (2.15). Our result in special the case yields (1.1) and can be better than (1.2). The sharpness of (1.1) and (1.2) is also obtained. Finally, we give applications in numerical integration.

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2. Main Results

Theorem 2.1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is a twice differentiable function such that f'' is integrable and there exist constants $\gamma, \Gamma \in R$, with $\gamma \le f''(t) \le \Gamma$, $t \in [a, b], 0 \le \lambda \le 1$. Then we have*

$$\begin{aligned}
&\left(\frac{1-2\lambda}{8}S - \frac{2-3\lambda}{24}\Gamma\right)(b-a)^2 \\
&\le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] \\
&\le \left(\frac{1-2\lambda}{8}S - \frac{2-3\lambda}{24}\gamma\right)(b-a)^2, \qquad \text{for } \lambda \in [0, \sqrt{2}-1],
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\left(\frac{\lambda^2}{8}S - \frac{3\lambda^2+3\lambda-1}{24}\Gamma\right)(b-a)^2 \\
&\le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] \\
&\le \left(\frac{\lambda^2}{8}S - \frac{3\lambda^2+3\lambda-1}{24}\gamma\right)(b-a)^2, \qquad \text{for } \lambda \in (\sqrt{2}-1, 1],
\end{aligned} \tag{2.2}$$

where $S = \dfrac{f'(b)-f'(a)}{b-a}$. If γ, Γ are given by

$$\gamma = \min_{t\in[a,b]} f''(t), \qquad \Gamma = \max_{t\in[a,b]} f''(t)$$

then the inequalities given by (2.1) *and* (2.2) *are sharp in the usual sense provided that $\lambda \ne 1/3$.*

Proof. Let $p : [a, b] \to \mathbb{R}$ be given by

$$p(t) = \begin{cases} \dfrac{1}{2}(t-a)[t-(1-\lambda)a-\lambda b], & t \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{1}{2}(b-t)[\lambda a+(1-\lambda)b-t], & t \in \left(\dfrac{a+b}{2}, b\right]. \end{cases} \tag{2.3}$$

Then we have

$$\int_a^b p(t)\mathrm{d}t = \frac{1-3\lambda}{24}(b-a)^3. \tag{2.4}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_a^b p(t)f''(t)\mathrm{d}t \\
&= \int_a^b f(t)\mathrm{d}t - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right].
\end{aligned} \tag{2.5}$$

From (2.4)–(2.5) it follows

$$\begin{aligned}\int_a^b p(t)[f''(t)-\gamma]\mathrm{d}t \\ = \int_a^b f(t)\mathrm{d}t - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2}\right] \\ - \gamma\frac{1-3\lambda}{24}(b-a)^3.\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\int_a^b p(t)[\Gamma-f''(t)]\mathrm{d}t \\ = -\int_a^b f(t)\mathrm{d}t + (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2}\right] \\ + \Gamma\frac{1-3\lambda}{24}(b-a)^3.\end{aligned} \tag{2.7}$$

We also have

$$\begin{aligned}\int_a^b p(t)[f''(t)-\gamma]\mathrm{d}t \le \max_{t\in[a,b]}|p(t)|\int_a^b |f''(t)-\gamma|\mathrm{d}t \\ = \begin{cases}\frac{1-2\lambda}{8}(S-\gamma)(b-a)^3, & 0\le\lambda\le\sqrt{2}-1,\\ \frac{\lambda^2}{8}(S-\gamma)(b-a)^3, & \sqrt{2}-1<\lambda\le 1,\end{cases}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\int_a^b p(t)[\Gamma-f''(t)]\mathrm{d}t \le \max_{t\in[a,b]}|p(t)|\int_a^b |\Gamma-f''(t)|\mathrm{d}t \\ = \begin{cases}\frac{1-2\lambda}{8}(\Gamma-S)(b-a)^3, & 0\le\lambda\le\sqrt{2}-1,\\ \frac{\lambda^2}{8}(\Gamma-S)(b-a)^3, & \sqrt{2}-1<\lambda\le 1.\end{cases}\end{aligned} \tag{2.9}$$

From (2.6)–(2.9) we see that (2.1) and (2.2) hold.

If we now substitute $f(t)=(t-a)^2$ in the inequality (2.1) or (2.2) then we find that the left-hand side, middle term and right-hand side are all equal to $\frac{1-3\lambda}{12}(b-a)^2$. Thus, the inequalities (2.1) and (2.2) are sharp in the usual sense provided that $\lambda\neq 1/3$. $\square$

Remark. We note that in the special cases, if we take $\lambda=0$ in Theorem 2.1 we get (1.1). Furthermore, the inequality (1.1) is sharp in the usual sense.

Corollary 2.2. *Under the assumptions of Theorem 2.1 and with $\lambda = 1/3$, we have the following inequality*

$$\begin{aligned}
&\frac{1}{24}(S-\Gamma)(b-a)^2 \\
&\quad\le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
&\quad\le \frac{1}{24}(S-\gamma)(b-a)^2.
\end{aligned} \tag{2.10}$$

Corollary 2.3. *Under the assumptions of Theorem 2.1 and with $\lambda = 1/2$, we have the following sharp inequality*

$$\begin{aligned}
&\frac{3S-5\Gamma}{96}(b-a)^2 \\
&\quad\le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{2}f\left(\frac{a+b}{2}\right) - \frac{1}{2}\frac{f(a)+f(b)}{2} \\
&\quad\le \frac{3S-5\gamma}{96}(b-a)^2.
\end{aligned} \tag{2.11}$$

Corollary 2.4. *Under the assumptions of Theorem 2.1 and with $\lambda = 1$, we have the following sharp trapezoid inequality*

$$\begin{aligned}
\frac{5\gamma-3S}{24}(b-a)^2 &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\le \frac{5\Gamma-3S}{24}(b-a)^2.
\end{aligned} \tag{2.12}$$

We now show that (2.12) can be better than (1.2). For that purpose, we give the following examples.

Example 1. Let us choose $f(t) = t^k$, $k > 2$, $a = 0$, $b > 0$. Then we have

$$\begin{aligned}
&f'(t) = kt^{k-1}, \qquad f''(t) = k(k-1)t^{k-2}, \qquad \gamma = 0, \\
&\Gamma = k(k-1)b^{k-2}, \quad S = kb^{k-2}.
\end{aligned}$$

Thus, the left-hand sides of (2.12) and (1.2) become:

$$\text{L.H.S.}(2.12) = -\frac{k}{8}b^k \qquad \text{and} \qquad \text{L.H.S.}(1.2) = -\frac{k(k-4)}{24}b^k.$$

We easily find that L.H.S.(2.12) $>$ L.H.S.(1.2) if $k > 7$. In fact, if $k \gg 7$ then (2.12) is much better than (1.2).

Example 2. Let us choose $f(t) = -t^k$, $k > 2$, $a = 0$, $b > 0$. Then we have

$$\begin{aligned}
&f'(t) = -kt^{k-1}, \qquad f''(t) = -k(k-1)t^{k-2}, \qquad \Gamma = 0, \\
&\gamma = -k(k-1)b^{k-2}, \quad S = -kb^{k-2}.
\end{aligned}$$

Thus, the right-hand sides of (2.12) and (1.2) become:

$$\text{R.H.S.}(2.12) = \frac{k}{8}b^k \qquad \text{and} \qquad \text{R.H.S.}(1.2) = \frac{k(k-4)}{24}b^k.$$

We easily find that R.H.S.(2.12) $<$ R.H.S.(1.2) if $k > 7$. In fact, if $k \gg 7$ then (2.12) is also much better than (1.2).

Theorem 2.5. *Let the assumptions of Theorem 2.1 be satisfied. Then we have*

$$\begin{aligned}
&\left[\frac{1}{8}S - \frac{(1-2\lambda)^2}{8}\Gamma\right](b-a)^2 \\
&\qquad + \frac{\Gamma}{12}\left\{(a^2+4ab+b^2) - 6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\right\} \\
&\qquad \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\qquad \le \left[\frac{1}{8}S - \frac{(1-2\lambda)^2}{8}\gamma\right](b-a)^2 \\
&\qquad + \frac{\gamma}{12}\{(a^2+4ab+b^2) - 6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\},
\end{aligned} \tag{2.13}$$

for $\lambda \in \left[0, \frac{1}{2} - \frac{\sqrt{2}}{4}\right] \cup \left(\frac{1}{2} + \frac{\sqrt{2}}{4}, 1\right]$, *and*

$$\begin{aligned}
&\frac{\lambda(1-\lambda)}{2}(2S-\Gamma)(b-a)^2 \\
&\qquad + \frac{\Gamma}{12}\left\{(a^2+4ab+b^2) - 6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\right\} \\
&\qquad \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\qquad \le \frac{\lambda(1-\lambda)}{2}(2S-\gamma)(b-a)^2 \\
&\qquad + \frac{\gamma}{12}\{(a^2+4ab+b^2) - 6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\},
\end{aligned} \tag{2.14}$$

for $\lambda \in \left(\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{1}{2} + \frac{\sqrt{2}}{4}\right]$, *where* $S = \dfrac{f'(b) - f'(a)}{b-a}$. *If* γ, Γ *are given by*

$$\gamma = \min_{t\in[a,b]} f''(t), \qquad \Gamma = \max_{t\in[a,b]} f''(t)$$

then the inequalities given by (2.13) *and* (2.14) *are sharp in the usual sense.*

Proof. Let $q : [a,b] \to \mathbb{R}$ be given by

$$q(t) = \frac{1}{2}[t - (1-\lambda)a - \lambda b][\lambda a + (1-\lambda)b - t]. \tag{2.15}$$

We have

$$\int_a^b q(t)\mathrm{d}t = \frac{1}{12}(b-a)\left\{(a^2+4ab+b^2) - 6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\right\} \tag{2.16}$$

Integrating by parts, we obtain

$$\int_a^b q(t)f''(t)\mathrm{d}t = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\mathrm{d}t - \frac{\lambda(1-\lambda)}{2}S(b-a)^3. \tag{2.17}$$

From (2.16) and (2.17) it follows

$$\begin{aligned}
&\int_a^b q(t)[f''(t)-\gamma]\mathrm{d}t \\
&= -\int_a^b f(t)\mathrm{d}t + \frac{f(a)+f(b)}{2}(b-a) - \frac{\lambda(1-\lambda)}{2}S(b-a)^3 \\
&\quad -\frac{\gamma}{12}(b-a)\{(a^2+4ab+b^2)-6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\},
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&\int_a^b q(t)[\Gamma-f''(t)]\mathrm{d}t \\
&= \int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2}(b-a) + \frac{\lambda(1-\lambda)}{2}S(b-a)^3 \\
&\quad +\frac{\Gamma}{12}(b-a)\{(a^2+4ab+b^2)-6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\}.
\end{aligned} \tag{2.19}$$

We also have

$$\begin{aligned}
&\int_a^b q(t)[f''(t)-\gamma]\mathrm{d}t \\
&\le \max_{t\in[a,b]}|q(t)|\int_a^b |f''(t)-\gamma|\mathrm{d}t \\
&= \begin{cases} \dfrac{(1-2\lambda)^2}{8}(S-\gamma)(b-a)^3, & \lambda\in\left[0,\dfrac{1}{2}-\dfrac{\sqrt{2}}{4}\right]\cup\left(\dfrac{1}{2}+\dfrac{\sqrt{2}}{4},1\right], \\ \dfrac{\lambda(1-\lambda)}{2}(S-\gamma)(b-a)^3, & \lambda\in\left(\dfrac{1}{2}-\dfrac{\sqrt{2}}{4},\dfrac{1}{2}+\dfrac{\sqrt{2}}{4}\right], \end{cases}
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&\int_a^b q(t)[\Gamma-f''(t)]\mathrm{d}t \\
&\le \max_{t\in[a,b]}|q(t)|\int_a^b |\Gamma-f''(t)|\mathrm{d}t \\
&= \begin{cases} \dfrac{(1-2\lambda)^2}{8}(\Gamma-S)(b-a)^3, & \lambda\in\left[0,\dfrac{1}{2}-\dfrac{\sqrt{2}}{4}\right]\cup\left(\dfrac{1}{2}+\dfrac{\sqrt{2}}{4},1\right], \\ \dfrac{\lambda(1-\lambda)}{2}(\Gamma-S)(b-a)^3, & \lambda\in\left(\dfrac{1}{2}-\dfrac{\sqrt{2}}{4},\dfrac{1}{2}+\dfrac{\sqrt{2}}{4}\right]. \end{cases}
\end{aligned} \tag{2.21}$$

From (2.18)–(2.21) we see that (2.13) and (2.14) hold.

If we now substitute $f(t) = (t-a)^2$ in the inequality (2.13) or (2.14) then we find that the left-hand side, middle term and right-hand side are all equal to $\frac{1}{6}(b-a)^2$. Thus, the inequalities (2.13) and (2.14) are sharp in the usual sense. □

Remark. We note that in the special cases, if we take $\lambda = 0$ or $\lambda = 1$ in Theorem 2.5 , we can also get (1.2). Furthermore, the inequality (1.2) is also sharp in the usual sense.

Corollary 2.6. *Under the assumptions of Theorem 2.5 and with $\lambda = 1/2$, we have another sharp trapezoid inequality*

$$\begin{aligned} \frac{3S-2\Gamma}{12}(b-a)^2 &\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\leq \frac{3S-2\gamma}{12}(b-a)^2. \end{aligned} \tag{2.22}$$

3. Applications in numerical integration

We restrict considerations to the following quadrature rule with a parameter. We also emphasize that similar considerations can be done for all quadrature rules considered in the previous section.

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold. If $D = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a given division of the interval $[a, b], h_i = x_{i+1} - x_i$,*

$$S_i = \frac{f'(x_{i+1}) - f'(x_i)}{h_i}, \qquad i = 0, 1, 2, \cdots, n-1,$$

then we have

$$\int_a^b f(t)\mathrm{d}t = A_{MT}(f) + R_{MT}(f),$$

where

$$A_{MT}(f) = \sum_{i=0}^{n-1} h_i \left[(1-\lambda) f\left(\frac{x_i + x_{i+1}}{2}\right) + \lambda \frac{f(x_i) + f(x_{i+1})}{2}\right],$$

and

$$\sum_{i=0}^{n-1}\left(\frac{1-2\lambda}{8}S_i - \frac{2-3\lambda}{24}\Gamma\right)h^3 \leq R_{MT}(f) \leq \sum_{i=0}^{n-1}\left(\frac{1-2\lambda}{8}S_i - \frac{2-3\lambda}{24}\gamma\right)h^3,$$

for $\lambda \in [0, \sqrt{2}-1]$, while

$$\sum_{i=0}^{n-1}\left(\frac{\lambda^2}{8}S_i - \frac{3\lambda^2+3\lambda-1}{24}\Gamma\right)h^3 \leq R_{MT}(f) \leq \sum_{i=0}^{n-1}\left(\frac{\lambda^2}{8}S_i - \frac{3\lambda^2+3\lambda-1}{24}\gamma\right)h^3$$

for $\lambda \in (\sqrt{2}-1, 1]$.

Proof. Apply Theorem 2.1 to the interval $[x_i, x_{i+1}]$, $i = 0, 1, 2, \cdots, n-1$ and sum. Then use the triangle inequality to obtain the desired result. □

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