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CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Let E be a uniformly convex Banach space, and let K be a nonempty convex closed subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and let $F(T)$ be nonempty, where $F(T)$ denotes the fixed points set of T. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in [0, 1] such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\varepsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1 - \varepsilon$ for all $n \in N$ and some $\varepsilon > 0$, starting with arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by setting

$$\begin{cases} z_n = P(\alpha''_n T(PT)^{n-1} x_n + \beta''_n x_n + \gamma''_n w_n), \\ y_n = P(\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n v_n), \\ x_{n+1} = P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n u_n), \end{cases}$$

with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma''_n < \infty$, where $\{w_n\}, \{v_n\}$ and $\{u_n\}$ are bounded sequences in K. (i) If E is real uniformly convex Banach space satisfying $Opial's$ condition, then weak convergence of $\{x_n\}$ to some $p \in F(T)$ is obtained; (ii) If T satisfies condition (A), then $\{x_n\}$ convergence strongly to some $p \in F(T)$.

1. Introduction and Preliminaries

Let E be a real Banach space, K be a nonempty subset of X and $F(T)$ denote the set of fixed points of T. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \qquad \text{for all } x, y \in K.$$

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [**2**] in 1972. They proved that, if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of K has a fixed point. Moreover, the fixed point set $F(T)$ of T is closed and convex.

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Recently, Chidume et al. have introduced another new concept about asymptotically nonexpansive mappings

Definition 1.1 ([**1**])**.** Let E be a real normed linear space, K a nonempty subset of E. Let $P: E \to K$ be the nonexpansive retraction of E onto K. A map $T: K \to E$ is said to be an asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$ such that the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \qquad \forall x, y \in K,\ n \geq 1.$$

T is called uniformly L-lipschitzian if there exists $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \qquad \forall x, y \in K,\ n \geq 1.$$

Many authors have contributed by their efforts to investigate the problem of finding a fixed point of asymptotically nonexpansive mappings and non-self asymptotically nonexpansive mappings. In [**5**], [**6**], Schu introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H. More precisely, he proved the following theorems.

Theorem JS1 ([**5**]). *Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H, and $T: K \to K$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in [0,1] satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1,$$

converges strongly to a fixed point of T.

Theorem JS2 ([**6**]). *Let E be a uniformly convex Banach space satisfying Opial's condition, K a nonempty closed convex and bounded subset of E, and $T: K \to K$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0, 1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, for all $n \geq 1$ and some $a, b \in (0, 1)$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1,$$

converges weakly to a fixed point of T.

In [**4**], Rhoades extended Theorem JS1 to a uniformly convex Banach space using a modified Ishikawa iteration method. In [**3**], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on K, provided that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$.

In [**9**], Tan and Xu introduced a modified Ishikawa processes to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. More precisely, they proved the following theorem.

Theorem TX ([**9**]). *Let E be a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable norm. Let C be a nonempty closed convex bounded subset of E, $T : C \to C$ a nonexpansive mapping and $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n],\ n \geq 1 \tag{1.1}$$

converges weakly to a fixed point of T.

In the above results, T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space, however if, the domain K of T is a proper subset of E (and this is the case in several applications), and T maps K into E, then iteration processes of Mann and Ishikawa may fail to be well defined.

In 2003, Chidume et al. [**1**] studied the iteration scheme defined by

$$x_1 \in K, \qquad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \qquad n \geq 1.$$

In the framework of a uniformly convex Banach space, where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. $T : K \to E$ is an asymptotically nonexpansive non-self map with sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1-\varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonself-maps.

Recently, Naseer Shahzad [**7**] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K, \qquad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n + \beta_n T x_n]),$$

where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. He proved weak and strong convergence theorems for non-self nonexpansive mappings in Banach spaces.

Motivated by the Chidume et al. [**1**], Nasser Shahzad [**7**] and some others, the purpose of this paper is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive non-self maps (provided that such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. In this paper, the following iteration scheme is studied

$$\begin{cases} x_1 \in K \\ z_n = P(\alpha_n'' T(PT)^{n-1}x_n + \beta_n'' x_n + \gamma_n'' w_n) \\ y_n = P(\alpha_n' T(PT)^{n-1}z_n + \beta_n' x_n + \gamma_n' v_n) \\ x_{n+1} = P(\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n) \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$, $\{\alpha_n''\}$, $\{\beta_n''\}$ and $\{\gamma_n''\}$ are real sequences in (0, 1) such that $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.

Our theorems improve and generalize some previous results to some extent.

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to E$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Recall that the mapping $T : K \to E$ with $F(T) \neq \emptyset$ where K is a subset of E, is said to satisfy condition A [**8**] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$

$$\|x - Tx\| \geq f(d(x, F(T)),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

In order to prove our main results, we shall make use of the following Lemmas.

Lemma 1.1 (Schu [**6**].)**.** *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.2 ([**1**] Demiclosed principle for nonself-map)**.** *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed with respect to zero.*

Lemma 1.3 (Tan and Xu [**9**])**.** *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition*

$$r_{n+1} \leq (1 + s_n)r_n + t_n, \qquad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

2. Main results

Lemma 2.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by the recursion* (1.2) *taking arbitrary $x_1 \in K$, with the restrictions $\sum_{n=1}^{\infty} \gamma''_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists, for any $p \in F(T)$, where $F(T)$ denotes the nonempty fixed point set of T.*

Proof. Since $\{w_n\}$, $\{v_n\}$ and $\{u_n\}$ are bounded sequences in C, for any given $p \in F(T)$, we can set

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \qquad M_2 = \sup\{\|v_n - p\| : n \geq 1\},$$
$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \qquad M = \max\{M_i : i = 1, 2, 3\}.$$

It follows from (1.2) that

$$\begin{aligned}
\|z_n - p\| &= \|P(\alpha''_n T(PT)^{n-1}x_n + \beta''_n x_n + \gamma''_n w_n) - p\| \\
&\leq \|\alpha''_n T(PT)^{n-1}x_n + \beta''_n x_n + \gamma''_n w_n - p\| \\
&\leq \alpha''_n \|T(PT)^{n-1}x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&= \alpha''_n \|T(PT)^{n-1}x_n - T(PT)^{n-1}p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\leq \alpha''_n k_n \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\leq \alpha''_n k_n \|x_n - p\| + (1 - \alpha''_n)\|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\leq k_n \|x_n - p\| + \gamma''_n M,
\end{aligned}$$

which implies that

$$\|z_n - p\| \leq k_n \|x_n - p\| + \gamma''_n M. \tag{2.1}$$

From (1.2) and (2.1) we get

$$\begin{aligned}
\|y_n - p\| &= \|P(\alpha'_n T(PT)^{n-1}z_n + \beta'_n x_n + \gamma'_n v_n) - p\| \\
&\leq \|\alpha'_n T(PT)^{n-1}z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\
&\leq \alpha'_n \|T(PT)^{n-1}z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&= \alpha'_n \|T(PT)^{n-1}z_n - T(PT)^{n-1}p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n k_n \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n k_n \|z_n - p\| + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n k_n (k_n \|x_n - p\| + \gamma''_n M) + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq k_n^2 \|x_n - p\| + k_n \gamma''_n M + \gamma'_n M,
\end{aligned}$$

which inplies that

$$\|y_n - p\| \leq k_n^2 \|x_n - p\| + k_n \gamma''_n M + \gamma'_n M. \tag{2.2}$$

Again, from (1.2) and (2.2) we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|P(\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n) - p\| \\
&= \|\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n - p\| \\
&\leq \alpha_n \|T(PT)^{n-1}y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|T(PT)^{n-1}y_n - T(PT)^{n-1}p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n k_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n k_n \|y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n k_n (k_n^2 \|x_n - p\| + k_n \gamma''_n M + \gamma'_n M) + (1 - \alpha'_n)\|x_n - p\| + \gamma_n M \\
&\leq k_n^3 \|x_n - p\| + k_n^2 \gamma''_n M + k_n \gamma'_n M + \gamma_n M.
\end{aligned}$$

Therefore

$$\|x_{n+1}-p\| \leq (1+(k_n^3-1))\|x_n-p\| + (k_n^2\gamma_n'' + k_n\gamma_n' + \gamma_n')M. \tag{2.3}$$

Note that $\sum_{n=1}^{\infty}(k_n-1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^3-1) < \infty$, therefore by Lemma 1.3, $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in F(T)$. This completes the proof. □

Lemma 2.2. *Let E be a normed linear space, K a nonempty closed convex subset which is also a nonexpansive retract of E, $T : K \to E$ a uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be the sequence defined by the recursion* (1.2) *taking arbitrary $x_1 \in K$, with the restrictions $\sum_{n=1}^{\infty}\gamma_n'' < \infty$, $\sum_{n=1}^{\infty}\gamma_n' < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$ and set $C_n = \|x_n - T(PT)^{n-1}x_n\|$, $\forall n \geq 1$. If $\lim_{n\to\infty} C_n = 0$, then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded, it follows from Lemma 2.1 that $\{u_n - x_n\}$, $\{v_n - x_n\}$, $\{w_n - x_n\}$ are all bounded, now, we set

$$\begin{aligned}
&r_1 = \sup\{\|u_n - x_n\| : n \geq 1\}, && r_2 = \sup\{\|v_n - x_n\| : n \geq 1\},\\
&r_3 = \sup\{\|w_n - x_n\| : n \geq 1\}, && r_4 = \sup\{\|v_{n-1} - x_n\| : n \geq 1\},\\
&r_5 = \sup\{\|u_{n-1} - T(PT)^{n-2}x_n\| : n \geq 1\}, && r = \max\{r_i : i = 1,2,3,4,5\}.
\end{aligned}$$

It follows from (1.2) that

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq \|\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n - x_n\|\\
&\leq \|T(PT)^{n-1}y_n - x_n\| + \gamma_n r\\
&\leq \|T(PT)^{n-1}x_n - x_n\| + \|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\| + \gamma_n r\\
&\leq C_n + L\|y_n - x_n\| + \gamma_n r\\
&\leq C_n + L\|\alpha_n' T(PT)^{n-1}z_n + \beta_n' x_n + \gamma_n' v_n - x_n\| + \gamma_n r\\
&\leq C_n + L\|T(PT)^{n-1}z_n - x_n\| + \gamma_n' Lr + \gamma_n r\\
&\leq C_n + L\|T(PT)^{n-1}x_n - x_n\| + L\|T(PT)^{n-1}z_n - T(PT)^{n-1}x_n\|\\
&\quad + \gamma_n' Lr + \gamma_n r\\
&\leq C_n + LC_n + L^2\|z_n - x_n\| + \gamma_n' Lr + \gamma_n r\\
&\leq C_n + LC_n + L^2\|\alpha_n'' T(PT)^{n-1}x_n + \beta_n'' x_n + \gamma_n'' w_n - x_n\|\\
&\quad + \gamma_n' Lr + \gamma_n r\\
&= C_n(1+L+L^2) + \gamma_n'' L^2 r + \gamma_n' Lr + \gamma_n r \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n-1}-x_n\| &\leq \|\alpha_{n-1}' T(PT)^{n-2}z_{n-1} + \beta_{n-1}' x_{n-1} + \gamma_{n-1}' v_{n-1} - x_n\|\\
&\leq \|T(PT)^{n-2}z_{n-1} - x_n\| + \|x_{n-1} - x_n\| + \gamma_{n-1}' r\\
&\leq \|T(PT)^{n-2}x_{n-1} - x_{n-1}\| + \|T(PT)^{n-2}z_{n-1} - T(PT)^{n-2}x_{n-1}\|\\
&\quad + 2\|x_{n-1} - x_n\| + \gamma_{n-1}' r\\
&\leq C_{n-1} + LC_{n-1} + L\gamma_{n-1}'' r + 2\|x_{n-1} - x_n\| + \gamma_{n-1}' r. \qquad (2.5)
\end{aligned}$$

Substituting (2.4) into (2.5) we obtain

$$\|y_{n-1}-x_n\| \le C_{n-1}(3+3L+2L^2)+(1+2L)r(L\gamma''_{n-1}+\gamma'_{n-1}) + 2\gamma'_{n-1}r. \tag{2.6}$$

On the other hand, from (2.4) and (2.6) we have

$$\begin{aligned}
&\|x_n-(PT)^{n-1}x_n\| \\
&\quad \le \|\alpha_{n-1}T(PT)^{n-2}y_{n-1}+\beta_{n-1}x_{n-1}+\gamma_{n-1}u_{n-1}-T(PT)^{n-2}x_n\| \\
&\quad \le \|T(PT)^{n-2}y_{n-1}-T(PT)^{n-2}x_n\|+\|x_{n-1}-T(PT)^{n-2}x_n\| \\
&\qquad +\gamma_{n-1}r \\
&\quad \le L\|y_{n-1}-x_n\|+\|x_{n-1}-T(PT)^{n-2}x_{n-1}\| \\
&\qquad +\|T(PT)^{n-2}x_{n-1}-T(PT)^{n-2}x_n\|+\gamma_{n-1}r \\
&\quad \le L\|y_{n-1}-x_n\|+C_{n-1}+L\|x_{n-1}-x_n\|+\gamma_{n-1}r \\
&\quad \le LC_{n-1}(4+4L+3L^2)+C_{n-1}+L^2r\gamma''_{n-1}(1+3L) \\
&\qquad +3Lr\gamma'_{n-1}(1+L)+(1+L)r\gamma_{n-1}.
\end{aligned} \tag{2.7}$$

It follows from (2.7) that

$$\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-T(PT)^{n-1}x_n\|+\|T(PT)^{n-1}x_n-Tx_n\| \\
&\le C_n+L\|(PT)^{n-1}x_n-x_n\| \\
&\le C_n+L^2C_{n-1}(4+4L+3L^2)+LC_{n-1}+L^3r\gamma''_{n-1}(1+3L) \\
&\quad +3L^2r\gamma'_{n-1}(1+L)+L(1+L)r\gamma_{n-1}.
\end{aligned}$$

It follows from $\lim_{n\to\infty} C_n=0$, $\sum_{n=1}^{\infty}\gamma''_n<\infty$, $\sum_{n=1}^{\infty}\gamma'_n<\infty$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$ that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

This completes the proof. □

Theorem 2.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T: K\to E$ be an asymptotically nonexpansive mapping with $\{k_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $F(T)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=\alpha''_n+\beta''_n+\gamma''_n=1$ and $\varepsilon\le\alpha_n,\alpha'_n,\alpha''_n\le 1-\varepsilon$ for all $n\in N$ and some $\varepsilon>0$. Let $\{x_n\}$ be the sequence defined by the recursion* (1.2) *taking arbitrary $x_1\in K$. Then $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.*

Proof. Take $p\in F(T)$, by Lemma 2.1 we know, $\lim_{n\to\infty}\|x_n-p\|$ exists. Let $\lim_{n\to\infty}\|x_n-p\|=c$. If $c=0$, then by the continuity of T the conclusion follows. Now suppose $c>0$. We claim $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Taking limsup on both the sides in the inequality (2.1), we have

$$\limsup_{n\to\infty}\|z_n-p\|\le c. \tag{2.8}$$

Similarly, taking limsup on both sides of the inequality (2.2), we have

$$\limsup_{n\to\infty} \|y_n - p\| \le c. \tag{2.9}$$

Next, we consider

$$\begin{aligned}\|T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)\| &\le \|T(PT)^{n-1}y_n - p\| + \gamma_n\|u_n - x_n\| \\ &\le k_n\|y_n - p\| + \gamma_n r.\end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.9) we get

$$\limsup_{n\to\infty} \|T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)\| \le c.$$

and

$$\begin{aligned}\|x_n - p + \gamma_n(u_n - x_n)\| &\le \|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\le \|x_n - p\| + \gamma_n r,\end{aligned}$$

which imply that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(u_n - x_n)\| \le c.$$

Again, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\begin{aligned}\liminf_{n\to\infty} \|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) \\ + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \ge c.\end{aligned} \tag{2.10}$$

On the other hand, using (2.1) yields

$$\begin{aligned}&\|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \\ &\quad \le \alpha_n\|T(PT)^{n-1}y_n - p\| + (1-\alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\quad \le \alpha_n k_n\|y_n - p\| + (1-\alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\quad \le \alpha_n k_n(k_n^2\|x_n - p\| + k_n\gamma_n'' r + \gamma_n' r) + (1-\alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\quad \le k_n^3\|x_n - p\| + k_n^2\gamma_n'' r + k_n\gamma_n' r + \gamma_n r.\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) \\ + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \le c.\end{aligned} \tag{2.11}$$

Combining (2.10) with (2.11) we obtain

$$\begin{aligned}\lim_{n\to\infty} \|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) \\ + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| = c.\end{aligned}$$

By applying Lemma 1.1, we have

$$\lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0. \tag{2.12}$$

Notice that

$$\begin{aligned}\|x_n - p\| &\le \|T(PT)^{n-1}y_n - x_n\| + \|T(PT)^{n-1}y_n - p\| \\ &\le \|T(PT)^{n-1}y_n - x_n\| + k_n\|y_n - p\|,\end{aligned}$$

which yields

$$c \leq \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq c.$$

This implies that

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Again, $\lim\limits_{n\to\infty} \|y_n - p\| = c$ gives

$$\begin{aligned} \liminf_{n\to\infty} \|\alpha'_n(Tz_n - p + \gamma'_n(v_n - x_n)) \\ + (1-\alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| \geq c. \end{aligned} \tag{2.13}$$

Similarly, we have

$$\begin{aligned} &\|\alpha'_n(T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n)) + (1-\alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| \\ &\quad \leq \alpha'_n\|T(PT)^{n-1}z_n - p\| + (1-\alpha'_n)\|x_n - p\| + \gamma'_n\|v_n - x_n\| \\ &\quad \leq \alpha'_n k_n\|z_n - p\| + (1-\alpha'_n)\|x_n - p\| + \gamma'_n\|v_n - x_n\| \\ &\quad \leq \alpha'_n k_n(k_n\|x_n - p\| + \gamma''_n r) + (1-\alpha'_n)\|x_n - p\| + \gamma'_n\|v_n - x_n\| \\ &\quad \leq k_n^2\|x_n - p\| + k_n\gamma''_n r + \gamma'_n r. \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \|\alpha'_n(T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n)) \\ + (1-\alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| \leq c. \end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14) yields that

$$\begin{aligned} \lim_{n\to\infty} \|\alpha'_n(T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n)) \\ + (1-\alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| = c. \end{aligned} \tag{2.15}$$

On the other hand, we have

$$\begin{aligned} \|T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n)\| &\leq \|T(PT)^{n-1}z_n - p\| + \gamma'_n\|v_n - x_n\| \\ &\leq k_n\|z_n - p\| + \gamma'_n r \end{aligned}$$

Taking limsup on both sides of the above inequality and using (2.1), we have

$$\limsup_{n\to\infty} \|T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n)\| \leq c \tag{2.16}$$

and

$$\begin{aligned} \|x_n - p + \gamma'_n(v_n - x_n)\| &\leq \|x_n - p\| + \gamma'_n\|v_n - x_n\| \\ &\leq \|x_n - p\| + \gamma'_n r, \end{aligned}$$

which yields

$$\limsup_{n\to\infty} \|x_n - p + \gamma'_n(v_n - x_n)\| \leq c. \tag{2.17}$$

Applying Lemma 1.1, it follows from (2.15), (2.16) and (2.17) that

$$\lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0. \tag{2.18}$$

Notice that

$$\begin{aligned}\|x_n - p\| &\le \|T(PT)^{n-1}z_n - x_n\| + \|T(PT)^{n-1}z_n - p\| \\ &\le \|T(PT)^{n-1}z_n - x_n\| + k_n\|z_n - p\|.\end{aligned}$$

We have

$$c \le \liminf_{n\to\infty} \|z_n - p\| \le \limsup_{n\to\infty} \|z_n - p\| \le c.$$

That implies that

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{2.19}$$

By the same method, we have

$$\begin{aligned}\lim_{n\to\infty} \|\alpha''_n(T(PT)^{n-1}x_n - p + \gamma''_n(w_n - x_n)) \\ + (1-\alpha''_n)(x_n - p + \gamma''_n(w_n - x_n)) - p\| = c.\end{aligned} \tag{2.20}$$

Moreover,

$$\begin{aligned}\|T(PT)^{n-1}x_n - p + \gamma''_n(w_n - x_n)\| &\le \|T(PT)^{n-1}x_n - p\| + \gamma''_n\|w_n - x_n\| \\ &\le k_n\|x_n - p\| + \gamma''_n r\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|T(PT)^{n-1}x_n - p + \gamma''_n(w_n - x_n)\| \le c. \tag{2.21}$$

It follows from

$$\begin{aligned}\|x_n - p + \gamma''_n(w_n - x_n)\| &\le \|x_n - p\| + \gamma''_n\|w_n - x_n\| \\ &\le \|x_n - p\| + \gamma''_n r.\end{aligned}$$

we obtain

$$\limsup_{n\to\infty} \|x_n - p + \gamma''_n(w_n - x_n)\| \le c. \tag{2.22}$$

Combining (2.20), (2.21) with (2.22) yields

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| = 0. \tag{2.23}$$

Since T is uniformly *L-Lipschitzian* for some $L > 0$, it follows form Lemma 2.2 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E satisfying Opial's condition. Suppose that $T : K \to E$ is an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $k_n \to 1$ as $n \to \infty$. Let $\{x_n\}$ be defined by (1.2), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n = \alpha'_n+\beta'_n+\gamma'_n = \alpha''_n+\beta''_n+\gamma''_n = 1$ and $\varepsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1-\varepsilon$ for all $n \in N$ and some $\varepsilon > 0$. Then $\{x_n\}$ converges weakly to a fixed point of $F(T)$.*

Proof. For any $p \in F(T)$, it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. We now prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Firstly, let p_1 and p_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemmas 2.1 and 2.2, we know that $p \in F(T)$. Secondly, let us assume $p_1 \neq p_2$, then by Opial's condition, we obtain

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{k\to\infty} \|x_{n_k} - p_1\| < \lim_{k\to\infty} \|x_{n_k} - p_2\| = \lim_{j\to\infty} \|x_{n_j} - p_2\| \\ &< \lim_{k\to\infty} \|x_{n_k} - p_1\| = \lim_{n\to\infty} \|x_n - p_1\|\end{aligned}$$

which is a contradiction. Hence $p_1 = p_2$. Then $\{x_n\}$ converges weakly to a fixed point of T. The proof is complete. □

Next, we shall prove a strong convergence theorem.

Theorem 2.3. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be a nonexpansive mapping with $p \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in* $[0, 1]$ *such that $\alpha_n+\beta_n+\gamma_n = \alpha'_n+\beta'_n+\gamma'_n = \alpha''_n+\beta''_n+\gamma''_n = 1$ and and $\varepsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1-\varepsilon$ for all $n \in N$ and some $\varepsilon > 0$. Let $\{x_n\}$ be the sequence defined by the recursion* (1.2) *taking arbitrary $x_1 \in K$. Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F = F(T)$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \ge 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Theorem 2.1, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, and (2.5) give

$$\inf_{p\in F} \|x_{n+1} - p\| \le \inf_{p\in F} (1 + (k_n^3 - 1))\|x_n - p\| + (k_n^2\gamma''_n + k_n\gamma'_n + \gamma_n)M.$$

This means that

$$d(x_{n+1}, F) \le (1 + (k_n^3 - 1))d(x_n, F) + (k_n^2\gamma''_n + k_n\gamma'_n + \gamma_n)M.$$

Thus $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 1.3. Now by condition (?A?), $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the method in the proof of Tan and Xu [**9**], we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. As $\lim_{n\to\infty} \|x_n - p\|$ exists, $x_n \to y \in F = F(T)$ thereby completing the proof. □

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