CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Let $E$ be a uniformly convex Banach space, and let $K$ be a nonempty convex closed subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and let $F(T)$ be nonempty, where $F(T)$ denotes the fixed points set of $T$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}, \{\alpha''_n\}, \{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\varepsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$, starting with arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by setting

$$
\begin{align*}
  z_n &= P(\alpha''_n T(PT)^{n-1} x_n + \beta''_n x_n + \gamma''_n w_n), \\
  y_n &= P(\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n v_n), \\
  x_{n+1} &= P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n u_n),
\end{align*}
$$

with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma''_n < \infty$, where $\{w_n\}, \{v_n\}$ and $\{u_n\}$ are bounded sequences in $K$. (i) If $E$ is real uniformly convex Banach space satisfying Opial’s condition, then weak convergence of $\{x_n\}$ to some $p \in F(T)$ is obtained; (ii) If $T$ satisfies condition (A), then $\{x_n\}$ convergence strongly to some $p \in F(T)$.

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1. Introduction and Preliminaries

Let $E$ be a real Banach space, $K$ be a nonempty subset of $X$ and $F(T)$ denote the set of fixed points of $T$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence \( \{k_n\} \) of positive real numbers with \( k_n \to 1 \) as \( n \to \infty \) such that

\[
\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \text{for all } x, y \in K.
\]

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. They proved that, if $K$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $E$, then every asymptotically nonexpansive self-mapping $T$ of $K$ has a fixed point. Moreover, the fixed point set $F(T)$ of $T$ is closed and convex.

Recently, Chidume et al. have introduced another new concept about asymptotically nonexpansive mappings.

**Definition 1.1 ([1]).** Let $E$ be a real normed linear space, $K$ a nonempty subset of $E$. Let $P : E \to K$ be the nonexpansive retraction of $E$ onto $K$. A map $T : K \to E$ is said to be an asymptotically nonexpansive if there exists a sequence \( \{k_n\} \subset [1, \infty) \) and \( k_n \to 1 \) as \( n \to \infty \) such that the following inequality holds:

\[
\|T^n(P)x - T^n(P)y\| \leq k_n \|x - y\|, \quad \forall x, y \in K, \ n \geq 1.
\]

$T$ is called uniformly $L$-lipschitzian if there exists $L > 0$ such that

\[
\|T^n(P)x - T^n(P)y\| \leq L \|x - y\|, \quad \forall x, y \in K, \ n \geq 1.
\]

Many authors have contributed by their efforts to investigate the problem of finding a fixed point of asymptotically nonexpansive mappings and non-self asymptotically nonexpansive mappings. In
Schu introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space $H$. More precisely, he proved the following theorems.

**Theorem JS1 ([5])**. Let $H$ be a Hilbert space, $K$ a nonempty closed convex and bounded subset of $H$, and $T : K \to K$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1,$$

converges strongly to a fixed point of $T$.

**Theorem JS2 ([6])**. Let $E$ be a uniformly convex Banach space satisfying Opial’s condition, $K$ a nonempty closed convex and bounded subset of $E$, and $T : K \to K$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, for all $n \geq 1$ and some $a, b \in (0,1)$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1,$$

converges weakly to a fixed point of $T$.

In [4], Rhoades extended Theorem JS1 to a uniformly convex Banach space using a modified Ishikawa iteration method. In [3], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on $K$, provided that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. 
In [9], Tan and Xu introduced a modified Ishikawa processes to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space $E$. More precisely, they proved the following theorem.

**Theorem TX ([9]).** Let $E$ be a uniformly convex Banach space which satisfies Opial’s condition or has a Frechet differentiable norm. Let $C$ be a nonempty closed convex bounded subset of $E$, $T : C \to C$ a nonexpansive mapping and $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1 - \alpha_n) = \infty$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_nTx_n], \quad n \geq 1$$

converges weakly to a fixed point of $T$.

In the above results, $T$ remains a self-mapping of a nonempty closed convex subset $K$ of a uniformly convex Banach space, however if, the domain $K$ of $T$ is a proper subset of $E$ (and this is the case in several applications), and $T$ maps $K$ into $E$, then iteration processes of Mann and Ishikawa may fail to be well defined.

In 2003, Chidume et al. [1] studied the iteration scheme defined by

$$x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1.$$

In the framework of a uniformly convex Banach space, where $K$ is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. $T : K \to E$ is an asymptotically nonexpansive non-self map with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonself-maps.
Recently, Naseer Shahzad [7] studied the sequence \( \{ x_n \} \) defined by

\[
x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_nTP[(1 - \beta_n)x_n + \beta_nTx_n]),
\]

where \( K \) is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space \( E \) with \( P \) as a nonexpansive retraction. He proved weak and strong convergence theorems for non-self nonexpansive mappings in Banach spaces.

Motivated by the Chidume et al. [1], Nasser Shahzad [7] and some others, the purpose of this paper is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive non-self maps (provided that such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \). In this paper, the following iteration scheme is studied

\[
\begin{cases}
  x_1 \in K \\
  z_n = P(\alpha''_nT(PT)^{n-1}x_n + \beta''_nx_n + \gamma''_nw_n) \\
  y_n = P(\alpha'_nT(PT)^{n-1}z_n + \beta'_nx_n + \gamma'_nv_n) \\
  x_{n+1} = P(\alpha_nT(PT)^{n-1}y_n + \beta_nx_n + \gamma_nu_n)
\end{cases}
\]

(1.2)

where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \}, \{ \alpha'_n \}, \{ \beta'_n \}, \{ \gamma'_n \}, \{ \alpha''_n \}, \{ \beta''_n \} \) and \( \{ \gamma''_n \} \) are real sequences in \((0, 1)\) such that \( \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1 \).

Our theorems improve and generalize some previous results to some extent.

Let \( E \) be a real Banach space. A subset \( K \) of \( E \) is said to be a retract of \( E \) if there exists a continuous map \( P : E \to E \) such that \( Px = x \) for all \( x \in K \). A map \( P : E \to E \) is said to be a retraction if \( P^2 = P \). It follows that if a map \( P \) is a retraction, then \( Py = y \) for all \( y \) in the range of \( P \).
A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demiclosed at $p$ if whenever \( \{x_n\} \) is a sequence in $D(T)$ such that \( \{x_n\} \) converges weakly to $x^* \in D(T)$ and \( \{Tx_n\} \) converges strongly to $p$, then $Tx^* = p$.

Recall that the mapping $T : K \to E$ with $F(T) \neq \emptyset$ where $K$ is a subset of $E$, is said to satisfy condition $A$ [8] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$

$$
\|x - Tx\| \geq f(d(x, F(T)),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

In order to prove our main results, we shall make use of the following Lemmas.

**Lemma 1.1** (Schu [6]). Suppose that $E$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose further that \( \{x_n\} \) and \( \{y_n\} \) are sequences of $E$ such that

$$
\limsup_{n \to \infty} \|x_n\| \leq r, \quad \limsup_{n \to \infty} \|y_n\| \leq r
$$

and

$$
\lim_{n \to \infty} \|t_n x_n + (1 - t_n)y_n\| = r
$$

hold for some $r \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 1.2** ([1] Demiclosed principle for nonself-map). Let $E$ be a uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$. Let $T : K \to E$ be an asymptotically nonexpansive mapping with \( \{k_n\} \subset [1, \infty) \) and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed with respect to zero.

**Lemma 1.3** (Tan and Xu [9]). Let \( \{r_n\} \), \( \{s_n\} \) and \( \{t_n\} \) be three nonnegative sequences satisfying the following condition

$$
r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \geq 1.
$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} r_n$ exists.
2. Main results

Lemma 2.1. Let $E$ be a uniformly convex Banach space and $K$ a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by the recursion (1.2) taking arbitrary $x_1 \in K$, with the restrictions $\sum_{n=1}^{\infty} \gamma''_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n \to \infty} \|x_n - p\|$ exists, for any $p \in F(T)$, where $F(T)$ denotes the nonempty fixed point set of $T$.

Proof. Since $\{w_n\}$, $\{v_n\}$ and $\{u_n\}$ are bounded sequences in $C$, for any given $p \in F(T)$, we can set

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \quad M_2 = \sup\{\|v_n - p\| : n \geq 1\},$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_i : i = 1, 2, 3\}.$$

It follows from (1.2) that

$$\|z_n - p\| = \|P(\alpha''_nT(PT)^{n-1}x_n + \beta''_n x_n + \gamma''_n w_n) - p\|
\leq \|\alpha''_nT(PT)^{n-1}x_n + \beta''_n x_n + \gamma''_n w_n - p\|
\leq \alpha''_n \|T(PT)^{n-1}x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\|
= \alpha''_n \|T(PT)^{n-1}x_n - T(PT)^{n-1}p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\|
\leq \alpha''_n k_n \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\|
\leq \alpha''_n k_n \|x_n - p\| + (1 - \alpha''_n) \|x_n - p\| + \gamma''_n \|w_n - p\|
\leq k_n \|x_n - p\| + \gamma''_n M,$$

which implies that

$$(2.1) \quad \|z_n - p\| \leq k_n \|x_n - p\| + \gamma''_n M.$$
From (1.2) and (2.1) we get
\[ \|y_n - p\| = \|P(\alpha'_n T(PT)^{n-1}z_n + \beta'_n x_n + \gamma'_n v_n) - p\| \]
\[ \leq \|\alpha'_n T(PT)^{n-1}z_n + \beta'_n x_n + \gamma'_n v_n - p\| \]
\[ \leq \alpha'_n \|T(PT)^{n-1}z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \]
\[ = \alpha'_n \|T(PT)^{n-1}z_n - T(PT)^{n-1}p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \]
\[ \leq \alpha'_n k_n \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \]
\[ \leq \alpha'_n k_n \|z_n - p\| + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \]
\[ \leq \alpha'_n k_n (k_n \|x_n - p\| + \gamma'' M) + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \]
\[ \leq k^2_n \|x_n - p\| + k_n \gamma'' M + \gamma'_n M, \]

which implies that
\[ (2.2) \quad \|y_n - p\| \leq k^2_n \|x_n - p\| + k_n \gamma'' M + \gamma'_n M. \]

Again, from (1.2) and (2.2) we have
\[ \|x_{n+1} - p\| = \|P(\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n) - p\| \]
\[ = \|\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n - p\| \]
\[ \leq \alpha_n \|T(PT)^{n-1}y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \]
\[ \leq \alpha_n \|T(PT)^{n-1}y_n - T(PT)^{n-1}p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \]
\[ \leq \alpha_n k_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \]
\[ \leq \alpha_n k_n \|y_n - p\| + (1 - \alpha_n) \|x_n - p\| + \gamma_n \|u_n - p\| \]
\[ \leq \alpha_n k_n (k_n \|x_n - p\| + k_n \gamma'' M + \gamma'_n M) + (1 - \alpha'_n) \|x_n - p\| + \gamma_n M \]
\[ \leq k^3_n \|x_n - p\| + k^2_n \gamma'' M + k_n \gamma'_n M + \gamma_n M. \]
Therefore
\[
\|x_{n+1} - p\| \leq (1 + (k_n^3 - 1))\|x_n - p\| + (k_n\gamma_n'' + k_n\gamma_n' + \gamma_n)M.
\]
Note that \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\) is equivalent to \(\sum_{n=1}^{\infty} (k_n^3 - 1) < \infty\), therefore by Lemma 1.3, \(\lim_{n \to \infty} \|x_n - p\|\) exists for all \(p \in F(T)\). This completes the proof. \(\square\)

**Lemma 2.2.** Let \(E\) be a normed linear space, \(K\) a nonempty closed convex subset which is also a nonexpansive retract of \(E\), \(T : K \to E\) a uniformly \(L\)-Lipschitzian mapping. Let \(\{x_n\}\) be the sequence defined by the recursion (1.2) taking arbitrary \(x_1 \in K\), with the restrictions \(\sum_{n=1}^{\infty} \gamma_n' < \infty\), \(\sum_{n=1}^{\infty} \gamma_n < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\) and set \(C_n = \|x_n - T(PT)^{n-1}x_n\|\), \(\forall n \geq 1\). If \(\lim_{n \to \infty} C_n = 0\), then \(\lim_{n \to \infty} \|x_n - Tx_n\| = 0\).

**Proof.** Since \(\{u_n\}, \{v_n\}\) and \(\{w_n\}\) are bounded, it follows from Lemma 2.1 that \(\{u_n - x_n\}, \{v_n - x_n\}, \{w_n - x_n\}\) are all bounded, now, we set
\[
\begin{align*}
  r_1 &= \sup\{\|u_n - x_n\| : n \geq 1\}, \\
  r_2 &= \sup\{\|v_n - x_n\| : n \geq 1\}, \\
  r_3 &= \sup\{\|w_n - x_n\| : n \geq 1\}, \\
  r_4 &= \sup\{\|v_{n-1} - x_n\| : n \geq 1\}, \\
  r_5 &= \sup\{\|u_{n-1} - T(PT)^{n-2}x_n\| : n \geq 1\}, \\
  r &= \max\{r_i : i = 1, 2, 3, 4, 5\}.
\end{align*}
\]
It follows from (1.2) that
\[
\begin{align*}
\|x_{n+1} - x_n\| &\leq \|\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n - x_n\| \\
&\leq \|T(PT)^{n-1}y_n - x_n\| + \gamma_n r \\
&\leq \|T(PT)^{n-1}x_n - x_n\| + \|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\| + \gamma_n r \\
&\leq C_n + L\|y_n - x_n\| + \gamma_n r \\
&\leq C_n + L\|\alpha'_n T(PT)^{n-1}z_n + \beta'_n x_n + \gamma' v_n - x_n\| + \gamma_n r
\end{align*}
\]
\[
\begin{align*}
\leq C_n + L\|T(PT)^{-1}z_n - x_n\| + \gamma_n Lr + \gamma_n r \\
\leq C_n + L\|T(PT)^{-1}x_n - x_n\| + L\|T(PT)^{-1}z_n - T(PT)^{-1}x_n\| \\
+ \gamma_n Lr + \gamma_n r \\
\leq C_n + LC_n + L^2\|z_n - x_n\| + \gamma_n Lr + \gamma_n r \\
\leq C_n + LC_n + L^2\|\alpha'_{n-1}T(PT)^{-1}x_n + \beta'_{n-1}x_n + \gamma'_{n-1}u_{n-1} - x_n\| \\
+ \gamma_n Lr + \gamma_n r \\
= C_n(1 + L + L^2) + \gamma''_{n-1}L^2r + \gamma_n Lr + \gamma_n r \\
\end{align*}
\]

(2.4)

and

\[
\begin{align*}
\|y_{n-1} - x_n\| &\leq \|\alpha'_{n-1}T(PT)^{-2}z_{n-1} - x_n\| + \|\beta'_{n-1}x_{n-1} + \gamma'_{n-1}u_{n-1} - x_n\| \\
&\leq \|T(PT)^{-2}z_{n-1} - x_n\| + \|x_{n-1} - x_n\| + \gamma'_{n-1}r \\
&\leq \|T(PT)^{-2}x_{n-1} - x_{n-1}\| + \|T(PT)^{-2}z_{n-1} - T(PT)^{-2}x_{n-1}\| \\
&\quad + 2\|x_{n-1} - x_n\| + \gamma'_{n-1}r \\
&\leq C_{n-1} + LC_{n-1} + L\gamma''_{n-1}r + 2\|x_{n-1} - x_n\| + \gamma'_{n-1}r. \\
\end{align*}
\]

(2.5)

Substituting (2.4) into (2.5) we obtain

\[
\|y_{n-1} - x_n\| \leq C_{n-1}(3 + 3L + 2L^2) + (1 + 2L)r(L\gamma''_{n-1} + \gamma'_{n-1}) \\
+ 2\gamma'_{n-1}r. \\
\]

(2.6)

On the other hand, from (2.4) and (2.6) we have

\[
\begin{align*}
\|x_n - (PT)^{-1}x_n\| &\leq \|\alpha_{n-1}T(PT)^{-2}y_{n-1} + \beta_{n-1}x_{n-1} + \gamma_{n-1}u_{n-1} - T(PT)^{-2}x_n\| \\
&\leq \|\alpha_{n-1}T(PT)^{-2}y_{n-1} + \beta_{n-1}x_{n-1} + \gamma_{n-1}u_{n-1} - T(PT)^{-2}x_n\| \\
&\quad + L\|T(PT)^{-2}x_{n-1} - x_{n-1}\| + \gamma_{n-1}r \\
&\quad + 2\|x_{n-1} - x_n\| + \gamma_{n-1}r. \\
&\leq C_{n-1}(3 + 3L + 2L^2) + (1 + 2L)r(L\gamma''_{n-1} + \gamma'_{n-1}) \\
&\quad + 2\gamma'_{n-1}r. \\
\end{align*}
\]
It follows from \((2.7)\) that

\[
\|x_n - T x_n\| \leq \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - T x_n\|
\]

\[
\leq C_n + L\|(PT)^{n-1}x_n - x_n\|
\]

\[
\leq C_n + L^2C_{n-1}(4 + 4L + 3L^2) + LC_{n-1} + L^3 r r'_{n-1}(1 + 3L)
\]

\[
+ 3L^2 r r'_{n-1}(1 + L) + L(1 + L)r r'_{n-1}.
\]

It follows from \(\lim_{n \to \infty} C_n = 0, \sum_{n=1}^{\infty} r''_{n} < \infty, \sum_{n=1}^{\infty} r'_{n} < \infty\) and \(\sum_{n=1}^{\infty} r_{n} < \infty\) that

\[
\lim_{n \to \infty} \|x_n - T x_n\| = 0.
\]

This completes the proof. \(\Box\)

**Theorem 2.1.** Let \(E\) be a uniformly convex Banach space and \(K\) a nonempty closed convex subset which is also a nonexpansive retract of \(E\). Let \(T: K \to E\) be an asymptotically non-expansive mapping with \(\{k_n\} \subset [1, \infty)\) such that \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\) and \(F(T) \neq \emptyset\). Let \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_{n}\}, \{\beta'_{n}\}, \{\gamma'_{n}\}, \{\alpha''_{n}\}, \{\beta''_{n}\}\) and \(\{\gamma''_{n}\}\) be real sequences in \([0, 1]\) such that \(\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1\) and \(\varepsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \varepsilon\) for all \(n \in N\) and
some \( \varepsilon > 0 \). Let \( \{x_n\} \) be the sequence defined by the recursion (1.2) taking arbitrary \( x_1 \in K \). Then \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \).

Proof. Take \( p \in F(T) \), by Lemma 2.1 we know, \( \lim_{n \to \infty} \|x_n - p\| \) exists. Let \( \lim_{n \to \infty} \|x_n - p\| = c \). If \( c = 0 \), then by the continuity of \( T \) the conclusion follows. Now suppose \( c > 0 \). We claim \( \lim_{n \to \infty} \|Tx_n - x_n\| = 0 \). Taking limsup on both the sides in the inequality (2.1), we have

\[
\limsup_{n \to \infty} \|z_n - p\| \leq c. 
\] (2.8)

Similarly, taking limsup on both sides of the inequality (2.2), we have

\[
\limsup_{n \to \infty} \|y_n - p\| \leq c. 
\] (2.9)

Next, we consider

\[
\|T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)\| \leq \|T(PT)^{n-1}y_n - p\| + \gamma_n\|u_n - x_n\|
\leq k_n\|y_n - p\| + \gamma_nr.
\]

Taking limsup on both the sides in the above inequality and using (2.9) we get

\[
\limsup_{n \to \infty} \|T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)\| \leq c.
\]

and

\[
\|x_n - p + \gamma_n(u_n - x_n)\| \leq \|x_n - p\| + \gamma_n\|u_n - x_n\|
\leq \|x_n - p\| + \gamma_nr,
\]

which imply that

\[
\limsup_{n \to \infty} \|x_n - p + \gamma_n(u_n - x_n)\| \leq c.
\]
Again, \( \lim_{n \to \infty} \|x_{n+1} - p\| = c \) means that

\[
\liminf_{n \to \infty} \|\alpha_n(T)\gamma_{n-1}z_n - p + \gamma_n(u_n - x_n)\|
+ (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \geq c.
\]  

(2.10)

On the other hand, using (2.1) yields

\[
\|\alpha_n(T)\gamma_{n-1}z_n - p + \gamma_n(u_n - x_n)\|
+ (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \leq c.
\]  

(2.11)

Combining (2.10) with (2.11) we obtain

\[
\lim_{n \to \infty} \|\alpha_n(T)\gamma_{n-1}z_n - p + \gamma_n(u_n - x_n)\|
+ (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| = c.
\]  

(2.12)

By applying Lemma 1.1, we have

\[
\lim_{n \to \infty} \|T\gamma_{n-1}z_n - x_n\| = 0.
\]
Notice that
\[ \|x_n - p\| \leq \|T(PT)^{n-1}y_n - x_n\| + \|T(PT)^{n-1}y_n - p\| \]
\[ \leq \|T(PT)^{n-1}y_n - x_n\| + k_n\|y_n - p\|, \]
which yields
\[ c \leq \liminf_{n \to \infty} \|y_n - p\| \leq \limsup_{n \to \infty} \|y_n - p\| \leq c. \]
This implies that
\[ \lim_{n \to \infty} \|y_n - p\| = c. \]
Again, \( \lim_{n \to \infty} \|y_n - p\| = c \) gives
\[ \liminf_{n \to \infty} \|\alpha_n'(Tz_n - p + \gamma_n'(v_n - x_n)) \]
\[ + (1 - \alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| \geq c. \]
(2.13)

Similarly, we have
\[ \|\alpha_n'(T(PT)^{n-1}z_n - p + \gamma_n'(v_n - x_n)) + (1 - \alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| \]
\[ \leq \alpha_n'\|T(PT)^{n-1}z_n - p\| + (1 - \alpha_n')\|x_n - p\| + \gamma_n'\|v_n - x_n\| \]
\[ \leq \alpha_n'k_n\|z_n - p\| + (1 - \alpha_n')\|x_n - p\| + \gamma_n'\|v_n - x_n\| \]
\[ \leq \alpha_n'k_n(\|x_n - p\| + \gamma_n''r) + (1 - \alpha_n')\|x_n - p\| + \gamma_n'\|v_n - x_n\| \]
\[ \leq k_n^2\|x_n - p\| + k_n\gamma_n''r + \gamma_n'r. \]
Therefore,
\[ \limsup_{n \to \infty} \|\alpha_n'(T(PT)^{n-1}z_n - p + \gamma_n'(v_n - x_n)) \]
\[ + (1 - \alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| \leq c. \]
(2.14)
Combining (2.13) with (2.14) yields that
\[
\lim_{n \to \infty} \| \alpha'_n (T(PT)^{n-1} z_n - p + \gamma'_n (v_n - x_n)) \\
+ (1 - \alpha'_n) (x_n - p + \gamma'_n (v_n - x_n)) \| = c.
\] (2.15)

On the other hand, we have
\[
\| T(PT)^{n-1} z_n - p + \gamma'_n (v_n - x_n) \| \leq \| T(PT)^{n-1} z_n - p \| + \gamma'_n \| v_n - x_n \|
\leq k_n \| z_n - p \| + \gamma'_n r
\]

Taking limsup on both sides of the above inequality and using (2.1), we have
\[
\limsup_{n \to \infty} \| T(PT)^{n-1} z_n - p + \gamma'_n (v_n - x_n) \| \leq c.
\] (2.16)

and
\[
\| x_n - p + \gamma'_n (v_n - x_n) \| \leq \| x_n - p \| + \gamma'_n \| v_n - x_n \|
\leq \| x_n - p \| + \gamma'_n r,
\]
which yields
\[
\limsup_{n \to \infty} \| x_n - p + \gamma'_n (v_n - x_n) \| \leq c.
\] (2.17)

Applying Lemma 1.1, it follows from (2.15), (2.16) and (2.17) that
\[
\lim_{n \to \infty} \| T(PT)^{n-1} z_n - x_n \| = 0.
\] (2.18)

Notice that
\[
\| x_n - p \| \leq \| T(PT)^{n-1} z_n - x_n \| + \| T(PT)^{n-1} z_n - p \|
\leq \| T(PT)^{n-1} z_n - x_n \| + k_n \| z_n - p \|.
\]
We have
\[ c \leq \liminf_{n \to \infty} \| z_n - p \| \leq \limsup_{n \to \infty} \| z_n - p \| \leq c. \]
That implies that
\[ \lim_{n \to \infty} \| z_n - p \| = c. \] (2.19)

By the same method, we have
\[ \lim_{n \to \infty} \| z_n - p \| = c. \] (2.20)

Moreover,
\[ \| T(PT)^{n-1} x_n - p + \gamma''(w_n - x_n) \| \leq \| T(PT)^{n-1} x_n - p \| + \gamma'' \| w_n - x_n \| \leq k_n \| x_n - p \| + \gamma'' r \]
which implies that
\[ \limsup_{n \to \infty} \| T(PT)^{n-1} x_n - p + \gamma''(w_n - x_n) \| \leq c. \] (2.21)

It follows from
\[ \| x_n - p + \gamma''(w_n - x_n) \| \leq \| x_n - p \| + \gamma'' \| w_n - x_n \| \leq \| x_n - p \| + \gamma'' r. \]
we obtain
\[ \limsup_{n \to \infty} \| x_n - p + \gamma''(w_n - x_n) \| \leq c. \] (2.22)

Combining (2.20), (2.21) with (2.22) yields
\[ \lim_{n \to \infty} \| T(PT)^{n-1} x_n - x_n \| = 0. \] (2.23)
Since $T$ is uniformly $L$-Lipschitzian for some $L > 0$, it follows form Lemma 2.2 that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$.

This completes the proof. \hfill \Box

**Theorem 2.2.** Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ satisfying Opial’s condition. Suppose that $T : K \to E$ is an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $k_n \to 1$ as $n \to \infty$. Let $\{x_n\}$ be defined by $(1.2)$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\varepsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \varepsilon$ for all $n \in N$ and some $\varepsilon > 0$. Then $\{x_n\}$ converges weakly to a fixed point of $F(T)$.

**Proof.** For any $p \in F(T)$, it follows from Lemma 2.1 that $\lim_{n \to \infty} \|x_n - p\|$ exists. We now prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Firstly, let $p_1$ and $p_2$ be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemmas 2.1 and 2.2, we know that $p \in F(T)$. Secondly, let us assume $p_1 \neq p_2$, then by Opial’s condition, we obtain

$$\lim_{n \to \infty} \|x_n - p_1\| = \lim_{k \to \infty} \|x_{n_k} - p_1\| < \lim_{k \to \infty} \|x_{n_k} - p_2\| = \lim_{j \to \infty} \|x_{n_j} - p_2\|$$

$$< \lim_{k \to \infty} \|x_{n_k} - p_1\| = \lim_{n \to \infty} \|x_n - p_1\|$$

which is a contradiction. Hence $p_1 = p_2$. Then $\{x_n\}$ converges weakly to a fixed point of $T$. The proof is complete. \hfill \Box

Next, we shall prove a strong convergence theorem.

**Theorem 2.3.** Let $E$ be a uniformly convex Banach space and $K$ a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be a nonexpansive mapping with $p \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$
be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and and $\varepsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Let $\{x_n\}$ be the sequence defined by the recursion (1.2) taking arbitrary $x_1 \in K$. Suppose $T$ satisfies condition (A). Then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** By Lemma 2.1, $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F = F(T)$. Let $\lim_{n \to \infty} \|x_n - p\| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Theorem 2.1, $\lim_{n \to \infty} \|Tx_n - x_n\| = 0$, and (2.5) give

$$\inf_{p \in F} \|x_{n+1} - p\| \leq \inf_{p \in F} \left(1 + (k^3_n - 1)\right) \|x_n - p\| + (k^{2\gamma''}_n + k\gamma'_n + \gamma_n)M.$$ 

This means that $d(x_{n+1}, F) \leq (1 + (k^3_n - 1))d(x_n, F) + (k^{2\gamma''}_n + k\gamma'_n + \gamma_n)M.$

Thus $\lim_{n \to \infty} d(x_n, F)$ exists by virtue of Lemma 1.3. Now by condition (A), $\lim_{n \to \infty} f(d(x_n, F)) = 0.$ Since $f$ is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n \to \infty} d(x_n, F) = 0.$ Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the method in the proof of Tan and Xu [9], we get that $\{y_j\}$ is a Cauchy sequence in $F$ and so it converges. Let $y_j \to y$. Since $F$ is closed, therefore $y \in F$ and then $x_{n_j} \to y$. As $\lim_{n \to \infty} \|x_n - p\|$ exists, $x_n \to y \in F = F(T)$ thereby completing the proof. □


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