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ON SOME ORDINARY AND FUZZY HOMOGENITY TYPES

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Abstract. Finite SLH topological spaces are characterized as partition topological spaces. As a consequence, two partial answers for a question raised in [**3**] are obtained. Closed-homogeneous topological spaces are characterized. Having a minimal open set is proved to be a sufficient condition for a homogeneous topological space to be closed-homogeneous. Closed-homogeneity is extended to include fuzzy topological spaces as a "good extension" according to Lowen's sense of closed-homogeneity in ordinary topological spaces. It is proved that homogeneity and closed-homogeneity in fuzzy topological spaces are equivalent under some conditions. Various open questions are also given.

1. Introduction

Throughout this paper, we follow the notions and terminologies of [**7**]. Let X be a nonempty set, A be a nonempty subset of X, λ be a fuzzy set in X, and (X,τ), $(X,\Im)$ be two ordinary and fuzzy topological spaces, respectively. We denote the relative topology on A by τ_A, the discrete topology on X by τ_{disc}, the group of all homeomorphisms from (X,τ) into itself by $H(X,\tau)$, and the group of all fuzzy homeomorphisms from $(X,\Im)$ into itself by $FH(X,\Im)$. A fuzzy set p defined by

$$p(x) = \begin{cases} t & \text{if } x = x_p \\ 0 & \text{if } x \neq x_p \end{cases}$$

where $0 < t < 1$ is called a fuzzy point in X, $x_p \in X$ is called the support of p and $p(x_p) = t$ the value (level) of p [**12**]. Two fuzzy points p and q in X are said to be distinct iff their supports are distinct, i.e., $x_p \neq x_q$. For a bijection $h : X \to Y$ and any fuzzy points p, q of X and Y respectively, it is easy to see that $h(p) = q$ iff $p(x_p) = q(x_q)$ and $h(x_p) = x_q$.

Definition 1.1 ([**11**])**.** Associated with a given fuzzy topological space $(X,\Im)$ and arbitrary ordinary subset M of X, we define the induced topology on M or the relative topology on M by:

$$\Im_M = \left\{\lambda_{|M} : \lambda \in \Im\right\}$$

The corresponding pair $(M,\Im_M)$ is called a fuzzy open (closed) subspace iff the fuzzy set χ_M is fuzzy open (closed) in $(X,\Im)$.

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Definition 1.2 ([**4**]). Let (X, τ) be a topological space and let $\widetilde{\tau}$ be the equivalence relation on X defined by $x \,\widetilde{\tau}\, y$ if there exists $h \in H(X, \tau)$ such that $h(x) = y$. Then the equivalence class $C_x^\tau = \{y \in X : x \,\widetilde{\tau} y\}$ is called the homogeneous component of (X, τ) determined by x.

It is known that homogeneous components are invariant under homeomorphisms.

Definition 1.3 ([**8**]). Let (X, τ) be a topological space and let β be a base for τ. Then we say that β is a representable base for τ if for any nonempty $U \in \beta$ and for any $x, y \in U$, there exists $h \in H(X, \tau)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in X - U$. A topological space (X, τ) is SLH (strongly locally homogeneous) if it has a representable base.

Definition 1.4 ([**6**]). A topological space (X, τ) is called LH (locally homogeneous) at x in X provided that there exists an open set U in X containing x such that for any $y \in U$ there is $h \in H(X, \tau)$ such that $h(x) = y$. A topological space (X, τ) is called LH if it is LH at each $x \in X$.

Definition 1.5 ([**13**]). Let $(X, \Im)$ be a fuzzy topological space. A family β of fuzzy open sets is called a base for $\Im$ if each nonzero member of $\Im$ can be written as a union of members of β.

Definition 1.6 ([**3**]). Let $(X, \Im)$ be a fuzzy topological space and let β be a base for $\Im$. Then we say that β is a representable base for $\Im$ if for any nonzero $\lambda \in \beta$ and for any $x, y \in S(\lambda)$, there exists $h \in FH(X, \Im)$ such that $h(x) = y$ and $h(t) = t$ for all $t \notin S(\lambda)$.

Definition 1.7 ([**3**]). A fuzzy topological space $(X, \Im)$ is said to be SLH (strongly locally homogeneous) if $\Im$ has a representable base.

Definition 1.8 ([**6**]). A topological space (X, τ) is called closed-homogeneous provided that for any x, y in X and for any K closed subset of $X - \{x, y\}$ there exists $h \in H(X, \tau)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in K$.

Definition 1.9 ([**10**]). Let (X, τ) be a topological space. A nonempty open set A of X is called a minimal open set in X if any open set in X which is contained in A is $\emptyset$ or A, i.e., τ_A is the indiscrete topology on A.

It is known that the homeomorphic image of a minimal open set is a minimal open set.

Mathematicians extended many notions of general topology to the fuzzy set theory. Some of these notions are separation and countability axioms, compactness, connectedness, paracompactness, metric space. Recently some homogeneity types have been extended to include fuzzy topological spaces. In [**1**], the author extended homogeneity, n-homogeneity, weakly n-homogeneity, countable dense homogeneity, densely homogeneity and strong local homogeneity as known ordinary topological concepts to include fuzzy topological spaces. Results concerning homogeneity, n-homogeneity and weakly n-homogeneity appeared in [**7**], in which the authors introduced some open questions, Al Ghour in [**2**] solved two of them. The results in [**1**] concerning homogeneous components and strong local homogeneity appeared in [**3**].

In this paper, we characterize finite SLH topological spaces, which will be useful to introduce two partial answers for the question which raised in [**3**], and we extend closed-homogeneity to include fuzzy topological spaces as a "good extension" for closed-homogeneity in ordinary topological spaces. Closed-homogeneity in both ordinary and fuzzy topological spaces will be characterized. We give a sufficient condition for homogeneous ordinary topological spaces to be closed-homogeneous. Finally, in one of the main results, we provide conditions to insure that homogeneity implies closed-homogeneity in the fuzzy case.

The following lemmas will be needed in the sequel.

Lemma 1.1 ([**10**])**.** *Let (X,τ) be a topological space and let A be a minimal open set in X. Then*

$$A = \bigcap \{O : O \textit{ is open in } X \textit{ with } x \in O\}$$

for any element x of A.

Lemma 1.2 ([**9**])**.** *Let (X,τ) be a topological space and let $f : (X,\tau) \to (X,\tau)$ be a function. Then the following are equivalent.*

(i) *$f : (X,\tau) \to (X,\tau)$ is continuous.*
(ii) *$f : (X,\omega(\tau)) \to (X,\omega(\tau))$ is fuzzy continuous.*

Lemma 1.3 ([**6**])**.** *Every closed-homogeneous topological space is SLH, but not conversely.*

Lemma 1.4 ([**6**])**.** *Every closed-homogeneous topological space is homogeneous, but not conversely.*

Lemma 1.5 ([**3**])**.** *The SLH property in fuzzy topological spaces is a "good extension" of SLH property in ordinary topological spaces.*

Lemma 1.6 ([**7**])**.** *The homogeneity property in fuzzy topological spaces is a "good extension" of homogeneity property in ordinary topological spaces.*

Lemma 1.7 ([**5**])**.** *Let (X,τ) be a topological space which contains a minimal open set. Then the following are equivalent.*

(i) *(X,τ) is a homogeneous topological space.*
(ii) *(X,τ) has a partition base consisting of minimal open sets all of which is homeomorphic to one another.*

The following result follows easily.

Lemma 1.8. *Let $(X,\Im)$ be a fuzzy topological space. Then for each $a \in [0,1)$, $FH(X,\Im) \subseteq H(X,\Im_a)$.*

2. Finite SLH ordinary and fuzzy topological spaces

Let (X,τ) be a topological space. Denote by $m(X,\tau)$ the set of all minimal open subsets of (X,τ). For each $x \in X$, denote the intersection of all open sets in X containing x, by U_x i.e.,

$$U_x = \bigcap \{O : O \text{ is open in } X \text{ with } x \in O\}$$

Theorem 2.1. *Let (X, τ) be a topological space and $x \in X$. Then the following are equivalent.*

(i) $U_x \in m(X, \tau)$.
(ii) $U_x \subseteq C_x^\tau$ *and* C_x^τ *is a union of minimal open sets of* (X, τ) *all of which are homeomorphic to one another.*

Proof. (i) $\Longrightarrow$(ii) Suppose that $U_x \in m(X, \tau)$. To see that $U_x \subseteq C_x^\tau$, let $y \in U_x$ and $f : (X, \tau) \to (X, \tau)$ defined by $f(x) = y$, $f(y) = x$, and $f(t) = t$ elsewhere. Let $V \in \tau$. Then by Lemma 1.1, it follows that either $\{x, y\} \subseteq V$ or $\{x, y\} \cap V = \emptyset$. Therefore, $f(V) = V$ and $f^{-1}(V) = V$. Hence f is continuous and open. Since f is clearly bijective, it follows that $f \in H(X, \tau)$. Thus $y \in C_x^\tau$ and hence $U_x \subseteq C_x^\tau$. To complete the proof of this implication, let $y \in C_x^\tau$, then there exists $f \in H(X, \tau)$ such that $f(x) = y$ and so $y \in f(U_x) \subseteq f(C_x^\tau) = C_x^\tau$. Since $f(U_x) \in m(X, \tau)$ and $U_x \cong f(U_x)$, the proof of this implication is completed.

(ii) $\Longrightarrow$(i) Since C_x^τ is a union of minimal open sets of (X, τ), there exists $A \in m(X, \tau)$ such that $x \in A$. Therefore by Lemma 1.1, it follows that $A = U_x$ and hence $U_x \in m(X, \tau)$. □

Corollary 2.1. *Let (X, τ) be a topological space. If for some $x \in X$, $U_x \in m(X, \tau)$ then (X, τ) is LH at x.*

Lemma 2.1. *Let (X, τ) be a topological space such that $|\tau| < \infty$. Then for any nonempty $U \in \tau$, there exists a minimal open set contained in U.*

Proof. If possible, choose $A_1 \in \tau - \{\emptyset\}$ such that $A_1 \subset U$, otherwise, $U \in m(X, \tau)$. Since $|\tau| < \infty$, inductively, we can find a sequence $A_n \subset A_{n-1} \subset \cdots \subset A_1 \subset U$ such that $A_i \in \tau - \{\emptyset\}$ for $i = 1, 2, \ldots, n$ and there is no $A \in \tau - \{\emptyset\}$ such that $A \subset A_n$. Therefore, A_n is a minimal open set contained in U. □

Theorem 2.2. *Let (X, τ) be an LH topological space at x for which $|\tau| < \infty$. Then $U_x \in m(X, \tau)$.*

Proof. Since $|\tau| < \infty$, then $U_x \in \tau$. Since (X, τ) is LH at x, there exists an open set U in X such that $x \in U \subseteq C_x^\tau$, hence $U_x \subseteq C_x^\tau$. Now by Lemma 2.1, there exists $A \in m(X, \tau)$ such that $A \subseteq U_x$. Choose $y \in A$, then $y \in C_x^\tau$ and hence there exists $h \in H(X, \tau)$ such that $h(y) = x$. Therefore, $x \in h(A)$ and hence $U_x \subseteq h(A)$. But $h(A) \in m(X, \tau)$, thus, $U_x = h(A) \in m(X, \tau)$. □

Recall that a topological space (X, τ) is called a partition topological space if it has a partition base.

Theorem 2.3. *Let (X, τ) be a topological space such that $|\tau| < \infty$. Then the following are equivalent.*

(i) (X, τ) *is SLH.*
(ii) (X, τ) *is LH.*
(iii) *For every $x \in X$, $U_x \in m(X, \tau)$.*
(iv) (X, τ) *is a partition topological space.*

Proof. (i) $\Longrightarrow$(ii) Obvious.

(ii) $\Longrightarrow$ (iii) Theorem 2.2.

(iii) $\Longrightarrow$(iv) By (iii), it follows that there exists a subset $X_1 \subseteq X$ such that $\beta = \{U_x : x \in X_1\}$ is a partition on X. It is not difficult to see that β is a base for τ.

(iv) $\Longrightarrow$(i) Let β be a partition base for τ. To see that β is representable, let $B \in \beta$ and let $x, y \in B$. Define $h : (X, \tau) \to (X, \tau)$ by $h(x) = y$, $h(y) = x$, and $h(t) = t$ elsewhere. Then $h \in H(X, \tau)$, $h(x) = y$, and $h(t) = t$ for every $t \in X - B$. Therefore, β is a representable base for τ and hence (X, τ) is SLH. $\square$

Corollary 2.2. *Let (X, τ) be a topological space such that $|X| < \infty$. Then the following are equivalent.*

(i) *(X, τ) is SLH.*
(ii) *(X, τ) is LH.*
(iii) *For every $x \in X$, $U_x \in m(X, \tau)$*
(iv) *(X, τ) is a partition topological space.*

The following two Lemmas are useful in the proof of our next main result.

Lemma 2.2. *Let $(X, \Im)$ be a fuzzy topological space and let β be a base for $\Im$. Then for each $a \in [0, 1)$, $\beta_a = \left\{\lambda^{-1}(a, 1] : \lambda \in \beta\right\}$ is a base for $\Im_a$.*

Proof. Let $\emptyset \neq U \in \Im_a$ and let $x \in U$. Choose $\lambda \in \Im$ such that $U = \lambda^{-1}(a, 1]$. Then $\lambda(x) > a$. Consider the fuzzy point p with support $x_p = x$ and level $p(x_p) = (a + \lambda(x))/2$. Then $p \in \lambda$. Since β is a base for $\Im$, then there exists $\mu \in \beta$ such that $p \in \mu \subseteq \lambda$. Therefore, we have $\mu^{-1}(a, 1] \in \beta_a$ and $x \in \mu^{-1}(a, 1] \subseteq \lambda^{-1}(a, 1]$. Hence, β_a is a base for $\Im_a$. $\square$

Lemma 2.3. *Let (X, τ) be a topological space and let β be a base for τ. If for some $x \in X$, $U_x \in \tau$ then $U_x \in \beta$.*

Proof. Since β is a base for τ and $x \in U_x \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U_x$. Therefore, by the definition of U_x we must have $B = U_x$. $\square$

Theorem 2.4. *Let $(X, \Im)$ be an SLH fuzzy topological space. If for some $a \in [0, 1)$, $|\Im_a| < \infty$ then $(X, \Im_a)$ is SLH.*

Proof. For each $x \in X$, let

$$U_x = \bigcap \{O : O \in \Im_a \text{ and } x \in O\}.$$

Then by Theorem 2.3, it is sufficient to show that $U_x \in m(X, \Im_a)$ for each $x \in X$. Let $x \in X$. Since $|\Im_a| < \infty$ then $U_x \in \Im_a$. Thus by Lemma 2.1, it follows that there exists $A \in m(X, \Im_a)$ such that $A \subseteq U_x$. Let β be a representable base for $\Im$, then by Lemma 2.2, it follows that $\beta_a = \left\{\lambda^{-1}(a, 1] : \lambda \in \beta\right\}$ is a base for $\Im_a$. Since $U_x \in \Im_a$, then by Lemma 2.3, it follows that there exists $\lambda \in \beta$ such that $U_x = \lambda^{-1}(a, 1]$. Choose $y \in A$, then $x, y \in S(\lambda)$ and so there exists $h \in FH(X, \Im)$ such that $h(y) = x$. Now by Lemma 1.8, we have $h \in H(X, \Im_a)$. Since $x \in h(A)$ and $h(A) \in m(X, \Im_a)$, it follows that $U_x = h(A)$. Thus, $U_x \in m(X, \Im_a)$. $\square$

Corollary 2.3. *Let $(X, \Im)$ be an SLH fuzzy topological space for which $|X| < \infty$ or $|\Im| < \infty$. Then for every $a \in [0, 1)$, $(X, \Im_a)$ is SLH.*

Corollary 2.3 gives two partial answers to the following question, which raised by Al Ghour [**3**].

Question 2.1. *Let X be a set with $|X| > 2$ and let $(X, \Im)$ be an SLH fuzzy space. Is it true that $(X, \Im_a)$ is SLH for all $a \in [0, 1)$?*

This will leads us to propose the following question.

Question 2.2. *Let $(X, \Im)$ be an SLH fuzzy topological space such that both X and $\Im$ are infinite sets. Is it true that $(X, \Im_a)$ is SLH for all $a \in [0, 1)$?*

3. Closed homogeneous fuzzy topological spaces

We start this section by the following proposition which characterizes the closed-homogeneous topological spaces.

Proposition 3.1. *Let (X, τ) be a topological space. Then the following are equivalent.*

(i) *(X, τ) is closed-homogeneous.*
(ii) *τ is a representable base for τ.*
(iii) *Every base for τ is representable.*

Proof. (i) $\Longrightarrow$(ii) Suppose that (X, τ) is a closed-homogeneous topological space. It is clear that τ is a base for τ. To see that τ is representable, let $\emptyset \neq U \in \tau$ and let $x, y \in U$. Let $K = X - U$. Then K is a closed subset of $X - \{x, y\}$, hence there exists $h \in H(X, \tau)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in K = X - U$. Therefore, τ is a representable base for τ.

(ii) $\Longrightarrow$(iii) Obvious.

(iii) $\Longrightarrow$(ii) The proof of this direction is similar to that used in (i) $\Longrightarrow$(ii). □

Definition 3.1. A fuzzy topological space $(X, \Im)$ is called closed-homogeneous provided that for any x, y in X and for any C closed subset of $(X, \Im_0)$ with $C \subseteq X - \{x, y\}$, there exists $h \in FH(X, \Im)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in C$.

Theorem 3.1. *Let $(X, \Im)$ be a fuzzy topological space. Then the following are equivalent.*

(i) *$(X, \Im)$ is closed-homogeneous.*
(ii) *$\Im$ is a representable base for $\Im$.*
(iii) *Every base for $\Im$ is representable.*
(iv) *For any non zero $\lambda \in \Im$ and any two fuzzy points $p, q \in \lambda$ with $p(x_p) = q(x_q)$, there exists $h \in FH(X, \Im)$ such that $h(p) = q$ and $h(t) = t$ for all $t \in X - S(\lambda)$.*

Proof. (i) $\Longrightarrow$(ii) Suppose that $(X, \Im)$ is a closed-homogeneous fuzzy topological space. It is obvious that $\Im$ is a base for $\Im$. To show that $\Im$ is representable, let $0 \neq \lambda \in \Im$ and $x, y \in S(\lambda)$. Since $S(\lambda) \in \Im_0$, then $C = X - S(\lambda)$ is a closed subset of $(X, \Im_0)$ and $C \subseteq X - \{x, y\}$. Since $(X, \Im)$ is closed-homogeneous, it follows that there exists $h \in FH(X, \Im)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in C = X - S(\lambda)$. Therefore, $\Im$ is a representable base for $\Im$.

(ii) $\Longrightarrow$(iii) Obvious.

(iii) $\Longrightarrow$(iv) Similar to that used in [**3**, Theorem 3.4].

(iv) $\Longrightarrow$(i) Let C be any closed subset of $(X, \Im_0)$ with $C \subseteq X - \{x, y\}$. Then $X - C$ is open in $(X, \Im_0)$ and so there exists $\lambda \in \Im$ such that $X - C = S(\lambda)$. Let p, q be the fuzzy points such that $x_p = x$, $x_q = y$ and $p(x_p) = q(x_q) = \min\{\lambda(x), \lambda(y)\}/2$. Then $p, q \in \lambda$ and so by (iv), it follows that there exists $h \in FH(X, \Im)$ such that $h(p) = q$ and $h(t) = t$ for all $t \in X - S(\lambda) = C$. Therefore, $h(x) = y$. Hence, $(X, \Im)$ is closed-homogeneous. □

Corollary 3.1. *Every closed-homogeneous fuzzy topological space is SLH.*

Corollary 3.2. *Every closed-homogeneous fuzzy topological space is homogeneous.*

Proof. Let $(X, \Im)$ be a closed-homogeneous fuzzy topological space and let $x, y \in X$. Since $x, y \in S(1)$ and by Theorem 3.1 $\Im$ is representable, it follows that there exists $h \in FH(X, \Im)$ such that $h(x) = y$. Hence $(X, \Im)$ is homogeneous. □

Theorem 3.2. *Let (X, τ) be a topological space. Then (X, τ) is closed-homogeneous iff $(X, \omega(\tau))$ is closed-homogeneous.*

Proof. Suppose that (X, τ) is a closed-homogeneous topological space. Let $0 \neq \lambda \in \omega(\tau)$ and $x, y \in S(\lambda)$. Since $S(\lambda) \in \tau$, then by Proposition 3.1, it follows that there exists $h \in H(X, \tau)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in X - S(\lambda)$. Using Lemma 1.2 we conclude that $h \in FH(X, \omega(\tau))$. Therefore, by Theorem 3.1, it follows that $(X, \omega(\tau))$ is closed-homogeneous. Conversely, suppose that $(X, \omega(\tau))$ is a closed-homogeneous fuzzy topological space, and $\emptyset \neq U \in \tau$, then $\chi_U \in \omega(\tau)$. Since $(X, \omega(\tau))$ is closed-homogeneous and $x, y \in S(\chi_U) = U$, then by Theorem 3.1, it follows that there exists $h \in FH(X, \omega(\tau))$ such $h(x) = y$ and $h(t) = t$ for all $t \in X - U$. Again, by Lemma 1.2 we conclude that $h \in H(X, \tau)$. Now, Proposition 3.1 completes the proof of this direction. □

Corollary 3.3. *The closed-homogeneity property in fuzzy topological spaces is a "good extension" of closed homogeneity property in ordinary topological spaces.*

Remark 3.1. Lemmas 1.3, 1.6, and Corollary 3.3 show that the converse of Corollary 3.1 is not true in general.

Question 3.1. Is there a sufficient condition for which SLH fuzzy topological spaces are closed-homogeneous?

Question 3.2. Is there a sufficient condition for which SLH topological spaces are closed-homogeneous?

Remark 3.2. Lemmas 1.4, 1.6, and Corollary 3.3 show that the converse of Corollary 3.2 is not true in general.

Question 3.3. Is there a sufficient condition for which homogeneous fuzzy topological spaces are closed-homogeneous?

Question 3.4. Is there a sufficient condition for which homogeneous topological spaces are closed-homogeneous?

Concerning Question 3.4, the author in [**6**] has the following result.

Proposition 3.2. *Every zero-dimensional homogeneous space that is T_0 is a closed-homogeneous.*

The following theorem gives another answer to Question 3.4.

Theorem 3.3. *Let (X, τ) be a topological space which contains a minimal open set. Then the following are equivalent.*

(i) *(X, τ) is closed-homogeneous.*
(ii) *(X, τ) is homogeneous.*

Proof. (i) $\Longrightarrow$(ii) Corollary 3.2.
(ii) $\Longrightarrow$(i) Suppose that (X, τ) is a homogeneous topological space which contains a minimal open set. Then by Lemma 1.7, it follows that (X, τ) has a partition base β consisting of minimal open sets all of which are homeomorphic to one another. Let $\emptyset \neq U \in \tau$ and $x, y \in U$, then there exist $B_1, B_2 \in \beta$ such that $x \in B_1 \subseteq U$ and $y \in B_2 \subseteq U$. If $B_1 = B_2$, we define $h : (X, \tau) \to (X, \tau)$ by $h(x) = y$, $h(y) = x$, and $h(t) = t$ elsewhere to be the required homeomorphism. If $B_1 \cap B_2 = \emptyset$, choose a homeomorphism $f : (B_1, \tau_{B_1}) \to (B_2, \tau_{B_2})$. Let $z = f(x)$ and define $g : (B_2, \tau_{B_2}) \to (B_2, \tau_{B_2})$ by $g(y) = z, g(z) = y$, and $g(t) = t$ elsewhere. Then $g \in H(B_2, \tau_{B_2})$. Define $h : (X, \tau) \to (X, \tau)$ by

$$h(t) = \begin{cases} (g \circ f)(t) & \text{if } t \in B_1 \\ (g \circ f)^{-1}(t) & \text{if } t \in B_2 \\ t & \text{if } t \in X - (B_1 \cup B_2) \end{cases}$$

Then $h \in H(X, \tau)$, $h(x) = y$, and $h(t) = t$ for all $t \in X - U$. Therefore, by Proposition 3.1, it follows that (X, τ) is closed-homogeneous. □

The following result is a direct consequence of Lemma 2.1 and Theorem 3.3.

Corollary 3.4. *Let (X, τ) be a topological space such that $|\tau| < \infty$. Then the following are equivalent.*

(i) *(X, τ) is closed-homogeneous.*
(ii) *(X, τ) is homogeneous.*

Corollary 3.5. *Let (X, τ) be a topological space such that $|X| < \infty$. Then the following are equivalent.*

(i) *(X, τ) is closed-homogeneous.*
(ii) *(X, τ) is homogeneous.*

Theorem 3.4. *Let $(X, \Im)$ be a closed-homogeneous fuzzy topological space. If for some $a \in [0, 1)$, $|\Im_a| < \infty$ then $(X, \Im_a)$ is closed-homogeneous.*

Proof. Since $(X, \Im)$ is closed-homogeneous, then by Corollary 3.2, $(X, \Im)$ is homogeneous, hence by Lemma 1.8, it follows that $(X, \Im_a)$ is homogeneous. Since $|\Im_a| < \infty$, then by Corollary 3.4 it follows that $(X, \Im_a)$ is a closed-homogeneous. □

Corollary 3.6. *Let $(X, \Im)$ be a closed-homogeneous fuzzy topological space for which $|X| < \infty$ or $|\Im| < \infty$. Then for every $a \in [0, 1)$, $(X, \Im_a)$ is closed--homogeneous.*

Question 3.5. Let $(X, \Im)$ be a closed-homogeneous fuzzy topological space with both X and $\Im$ are infinite sets. Is it true that $(X, \Im_a)$ is closed-homogeneous for all $a \in [0, 1)$?

One may raise the following question.

Suppose $(X, \Im)$ is a fuzzy topological space such that for every $a \in [0, 1)$, $(X, \Im_a)$ is closed-homogeneous. Is it true that $(X, \Im)$ is closed-homogeneous?

The following is a counter example for the above question.

Example 3.1. Let $X = \{a, b\}$ and let $\Im = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, 0.5\}$ where

$$\begin{array}{ll} \lambda_1(a) = 0.5, & \lambda_1(b) = 1, \\ \lambda_2(a) = 1, & \lambda_2(b) = 0.5, \\ \lambda_3(a) = 0.7, & \lambda_3(b) = 0.5, \\ \lambda_4(a) = 0.7, & \lambda_4(b) = 1, \end{array}$$

then $\Im$ is a fuzzy topology on X, $\Im_a = \{\emptyset, X\}$ if $0 \leq a < 0.5$ and $\Im_a = \tau_{\text{disc}}$ if $a \geq 0.5$. Therefore, for each $a \in [0, 1)$, $(X, \Im_a)$ is a closed-homogeneous. Now if $(X, \Im)$ is closed-homogeneous, then there exists $h \in FH(X, \Im)$ such that $h(a) = b$, but $(h(\lambda_3))(a) = 0.5$ and $(h(\lambda_3))(b) = 0.7$ which show that $h(\lambda_3) \notin \Im$.

A fuzzy topological space $(X, \Im)$ is called T_2 if for any two distinct fuzzy points p, q in X, there exist $\lambda_1, \lambda_2 \in \Im$ such that $p \in \lambda_1, q \in \lambda_2$ and $\lambda_1 \cap \lambda_2 = 0$.

A fuzzy topological space $(X, \Im)$ is called zero-dimensional if $\Im$ has a base consisting of clopen fuzzy sets.

Definition 3.2. A fuzzy topological space $(X, \Im)$ is called a crisp zero-dimensional if it is zero-dimensional and the support of each clopen fuzzy set in X is an open subspace of $(X, \Im)$.

For Question 3.3, we give the following theorem.

Theorem 3.5. *Every crisp zero-dimensional homogeneous T_2 fuzzy topological space is closed-homogeneous.*

Proof. Let $(X, \Im)$ be a crisp zero-dimensional homogeneous T_2 fuzzy topological space. Let β be a base for $\Im$ which consists of clopen fuzzy sets. Let $0 \neq \lambda \in \Im$ and let p, q be any two fuzzy points in X such that $p, q \in \lambda$ and $p(x_p) = q(x_q)$. If p, q are not distinct, then $p = q$ and so the identity function on X is the required fuzzy homeomorphism. If p, q are distinct, then since $(X, \Im)$ is T_2, there exist $\mu_1, \mu_2 \in \beta$ such that $p \in \mu_1$, $q \in \mu_2$ and $\mu_1 \cap \mu_2 = 0$. Choose $\upsilon_1, \upsilon_2 \in \beta$ such that $p \in \upsilon_1 \subseteq \lambda$ and $q \in \upsilon_2 \subseteq \lambda$. Let $\upsilon = \upsilon_1 \cup \upsilon_2$, $\gamma_1 = \mu_1 \cap \upsilon$, and $\gamma_2 = \mu_2 \cap \upsilon$. Since $(X, \Im)$ is homogeneous, there exists $f \in FH(X, \Im)$ such that $f(x_p) = x_q$ and hence, $f(p) = q$. Let $\lambda_1 = \gamma_1 \cap f^{-1}(\gamma_2)$ and $\lambda_2 = f(\lambda_1)$. Define $h : (X, \Im) \to (X, \Im)$ by

$$h(t) = \begin{cases} f(t) & \text{if } t \in S(\lambda_1) \\ f^{-1}(t) & \text{if } t \in S(\lambda_2) \\ t & \text{if } t \in X - (S(\lambda_1) \cup S(\lambda_2)) \end{cases}$$

Let $\mu \in \Im$. Then $\mu = \mu \cap (\chi_{S(\lambda_1)} \cup \chi_{S(\lambda_2)} \cup \chi_{(X-S(\lambda_1)\cup S(\lambda_2))})$, and so

$$\begin{aligned} h^{-1}(\mu) &= h^{-1}(\mu \cap \chi_{S(\lambda_1)}) \cup h^{-1}(\mu \cap \chi_{S(\lambda_2)}) \cup h^{-1}(\mu \cap \chi_{\chi(X-S(\lambda_1)\cup S(\lambda_2))}) \\ &= f(\mu \cap \chi_{S(\lambda_1)}) \cup f^{-1}(\mu \cap \chi_{S(\lambda_2)}) \cup (\mu \cap \chi_{\chi(X-S(\lambda_1)\cup S(\lambda_2))}). \end{aligned}$$

Therefore, $h^{-1}(\mu) \in \Im$ and hence h is fuzzy continuous. Similarly we can show that h^{-1} is a fuzzy continuous. It is not difficult to see that h is a bijection, hence $h \in FH(X, \Im)$. Now since $x_p \in S(\lambda_1)$ and $f(p) = q$, hence $h(x_p) = x_q$. Since $p(x_p) = q(x_q)$, it follows that $h(p) = q$. Since $\lambda_1 \cup \lambda_2 \subseteq \lambda$, then $S(\lambda_1 \cup \lambda_2) \subseteq S(\lambda)$ and so for every $t \in X - S(\lambda)$, $h(t) = t$. Thus, by Theorem 3.1(iv) it follows that $(X, \Im)$ is a closed-homogeneous. □

Question 3.6. Is it true that every T_2 zero-dimensional homogeneous fuzzy topological space is closed-homogeneous?

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