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OSCILLATION OF FIXED POINTS OF SOLUTIONS OF SOME LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the relationship between solutions and their derivatives of the differential equation $f^{(k)} + A(z)f = 0$, $k \geq 2$, where $A(z)$ is a transcendental meromorphic function with $\rho_p(A) = \rho > 0$ and meromorphic functions of finite iterated $p-$order.

1. Introduction and statement of result

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [**3**], [**8**]. For the definition of the iterated order of a merormorphic function, we use the same definition as in [**4**], [**2**, p. 317], [**5**, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1 ([**4**], [**5**])**.** Let f be a meromorphic function. Then the iterated $p-$order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log r} \qquad (p \geq 1 \text{ is an integer}), \tag{1.1}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [**3**], [**8**]). For $p = 1$, this notation is called order and for $p = 2$ hyper-order.

Definition 1.2 ([**4**], [**5**])**.** The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational},\\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{for } f \text{ transcendental} \\ & \text{for which some } j \in \mathbb{N} \\ & \text{with } \rho_j(f) < +\infty \text{ exists},\\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty \\ & \text{for all } j \in \mathbb{N}. \end{cases} \tag{1.2}$$

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Definition 1.3 ([**4**])**.** Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \varlimsup_{r\to+\infty} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log r} \qquad (p \geq 1 \ \text{ is an integer}), \tag{1.3}$$

where $\overline{N}(r, \frac{1}{f})$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$. For $p = 1$, this notation is called the exponent of convergence of the sequence of distinct zeros and for $p = 2$ the hyper-exponent of convergence of the sequence of distinct zeros.

Definition 1.4 ([**6**])**.** Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f) = \overline{\lambda}_p(f - z) = \varlimsup_{r\to+\infty} \frac{\log_p \overline{N}(r, \frac{1}{f-z})}{\log r} \qquad (p \geq 1 \ \text{ is an integer}). \tag{1.4}$$

For $p = 1$, this notation is called the exponent of convergence of the sequence of distinct fixed points and for $p = 2$ the hyper-exponent of convergence of the sequence of distinct fixed points [**7**]. Thus $\overline{\tau}_p(f) = \overline{\lambda}_p(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + A(z)f = 0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades [**10**]. However, there are a few studies on the fixed points of solutions of differential equations. In [**9**], Wang and Lü have investigated the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives and have obtained the following result:

Theorem A ([**9**])**.** *Suppose that $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \varliminf_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation*

$$f^{''} + A(z)f = 0, \tag{1.6}$$

satisfies that f and $f^{'}$, $f^{''}$ all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f^{'}) = \overline{\tau}(f^{''}) = \rho(f) = +\infty, \tag{1.7}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f^{'}) = \overline{\tau}_2(f^{''}) = \rho_2(f) = \rho. \tag{1.8}$$

Recently, Theorem A has been generalized to higher order differential equations by Liu Ming-Sheng and Zhang Xiao-Mei as follows (see [**7**]):

Theorem B ([**7**]). *Suppose that $k \geq 2$ and $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \varliminf_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. Then every meromorphic solution $f(z) \not\equiv 0$ of (1.5), satisfies that f and $f^{'}, f^{''}, \dots, f^{(k)}$ all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f^{'}) = \overline{\tau}(f^{''}) = \dots = \overline{\tau}(f^{(k)}) = \rho(f) = +\infty, \tag{1.9}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f^{'}) = \overline{\tau}_2(f^{''}) = \dots = \overline{\tau}_2(f^{(k)}) = \rho_2(f) = \rho. \tag{1.10}$$

The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1.5) and meromorphic functions of finite iterated p-order. We obtain an extension of Theorem B. In fact, we prove the following result:

Theorem 1.1. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \varliminf_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

If $\varphi(z) \not\equiv 0$ is a meromorphic function with finite p-iterated order $\rho_p(\varphi) < +\infty$, then every meromorphic solution $f(z) \not\equiv 0$ of (1.5), satisfies

$$\overline{\lambda}_p(f-\varphi) = \overline{\lambda}_p(f^{'}-\varphi) = \dots = \overline{\lambda}_p(f^{(k)}-\varphi) = \rho_p(f) = +\infty, \tag{1.11}$$

$$\overline{\lambda}_{p+1}(f-\varphi) = \overline{\lambda}_{p+1}(f^{'}-\varphi) = \dots = \overline{\lambda}_{p+1}(f^{(k)}-\varphi) = \rho_{p+1}(f) = \rho. \tag{1.12}$$

Setting $p = 1$ and $\varphi(z) = z$ in Theorem 1.1, we obtain the following corollary:

Corollary 1.1. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ of (1.5) satisfies that f and $f^{'}, f^{''}, \dots$ $\dots, f^{(k)}$ all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f^{'}) = \overline{\tau}(f^{''}) = \dots = \overline{\tau}(f^{(k)}) = \rho(f) = +\infty, \tag{1.13}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f^{'}) = \overline{\tau}_2(f^{''}) = \dots = \overline{\tau}_2(f^{(k)}) = \rho_2(f) = \rho. \tag{1.14}$$

2. Auxiliary Lemmas

We need the following lemmas in the proofs of our theorem.

Lemma 2.1 ([**4**]). *Let f be a meromorphic function for which $i(f) = p \geq 1$ and $\rho_p(f) = \sigma$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m(r, \frac{f^{(k)}}{f}) = O(\exp_{p-2}\left\{r^{\sigma+\varepsilon}\right\}), \tag{2.1}$$

outside of a possible exceptional set E of finite linear measure.

To avoid some problems caused by the exceptional set we recall the following Lemma.

Lemma 2.2. ([**1**, p. 68]) *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.3. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(1) *all poles of f are of uniformly bounded multiplicity or that*
(2) *$\delta(\infty, f) > 0$.*

Then every meromorphic solution $f(z) \not\equiv 0$ of (1.5) satisfies $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$.

Proof. First, we prove that $\rho_p(f) = +\infty$. We suppose that $\rho_p(f) = \beta < +\infty$ and then we obtain a contradiction. Rewrite (1.5) as

$$A = -\frac{f^{(k)}}{f}. \tag{2.2}$$

By Lemma 2.1, there exist a set E with finite linear measure such that

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{r^{\beta+\varepsilon}\right\}\right), \qquad \rho_p(f) = \beta < +\infty, \tag{2.3}$$

for $r \notin E$.

It follows from the definition of deficiency $\delta(\infty, A)$ that for sufficiently large r, we have

$$m(r, A) \geq \frac{\delta}{2} T(r, A). \tag{2.4}$$

So when $r \notin E$ is sufficiently large, we have by (2.2)–(2.4)

$$T(r, A) \leq \frac{2}{\delta} m(r, A) = \frac{2}{\delta} m\left(r, \frac{f^{(k)}}{f}\right) = O(\exp_{p-2}\left\{r^{\beta+\varepsilon}\right\}). \tag{2.5}$$

By Lemma 2.2, we have for any $\alpha > 1$

$$T(r, A) \leq O(\exp_{p-2}\left\{\alpha r^{\beta+\varepsilon}\right\}) \tag{2.6}$$

for a sufficiently large r. Therefore, by the definition of iterated order, we obtain that $i(A) \leq p-1$

$$\rho_{p-1}(A) \leq \beta + \varepsilon < +\infty \tag{2.7}$$

and this contradicts $\rho_p(A) = \rho > 0$. Hence $\rho_p(f) = +\infty$.

By using the same proof as in the proof of Theorem 2.1 [**6**], we obtain that $\rho_{p+1}(f) = \rho_p(A) = \rho$. □

Lemma 2.4. *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite p-iterated order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = F, \tag{2.8}$$

then $\overline{\lambda}_p(f) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f) = \rho_{p+1}(f) = \rho$.

Proof. By (2.8), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \ldots + A_1\frac{f'}{f} + A_0\right). \tag{2.9}$$

If f has a zero at z_0 of order $\alpha (> k)$ and if $A_0, A_1, \ldots, A_{k-1}$ are all analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$N\left(r, \frac{1}{f}\right) \leq k\, \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r, A_j). \tag{2.10}$$

By (2.9), we have

$$m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k-1} m(r, A_j) + m\left(r, \frac{1}{F}\right) + O(1). \tag{2.11}$$

Applying the Lemma 2.1, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O(\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}) \qquad \text{for } j = 1, \ldots, k \tag{2.12}$$

where $\rho_{p+1}(f) = \rho < +\infty$, holds for all r outside a set $E \subset (0, +\infty)$ with a linear measure $m(E) = \mu < +\infty$. By (2.10)–(2.12), we get

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq k\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=0}^{k-1} T(r, A_j) + T(r, F) + O(\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}) \\ &\quad (|z| = r \notin E). \end{aligned} \tag{2.13}$$

Set

$$\sigma = \max\left\{\rho_p(F), \rho_p(A_j) : j = 0, \ldots, k-1\right\}.$$

Then for a sufficiently large r, we have

$$T(r,A_0)+...+T(r,A_{k-1})+T(r,F)\leq (k+1)\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}. \tag{2.14}$$

Thus, by (2.13), (2.14) we have

$$\begin{aligned}&T(r,f)\leq k\overline{N}\left(r,\frac{1}{f}\right)+(k+1)\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}+O(\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\})\\&\qquad (|z|=r\notin E).\end{aligned} \tag{2.15}$$

Hence for any f with $\rho_p(f)=+\infty$ and $\rho_{p+1}(f)=\rho$, by (2.15) and Lemma 2.2, we have

$$\overline{\lambda}_p(f)\geq\rho_p(f)=+\infty$$

and $\overline{\lambda}_{p+1}(f)\geq\rho_{p+1}(f)$. Since $\overline{\lambda}_{p+1}(f)\leq\rho_{p+1}(f)$ we have $\overline{\lambda}_{p+1}(f)=\rho_{p+1}(f)=\rho$. □

3. Proof of Theorem 1.1

Suppose that $f(z)\not\equiv 0$ is a meromorphic solution of the equation (1.5). Then by Lemma 2.3 we have $\rho_p(f)=+\infty$ and $\rho_{p+1}(f)=\rho_p(A)$. Set $w_j=f^{(j)}-\varphi$ $(j=0,1,\ldots,k)$, then $\rho_p(w_j)=\rho_p(f)=+\infty$, $\rho_{p+1}(w_j)=\rho_{p+1}(f)=\rho_p(A)$, $(j=0,1,\ldots,k)$, $\overline{\lambda}_p(w_j)=\overline{\lambda}_p(f^{(j)}-\varphi)$, $(j=0,1,\ldots,k)$. Differentiating both sides of $w_j=f^{(j)}-\varphi$ and replacing $f^{(k)}$ with $f^{(k)}=-Af$, we obtain

$$w_j^{(k-j)}=-Af-\varphi^{(k-j)}\qquad (j=0,1,\ldots,k). \tag{3.1}$$

Then we have

$$f=-\frac{w_j^{(k-j)}+\varphi^{(k-j)}}{A}. \tag{3.2}$$

Substituting (3.2) into the equation (1.5), we get

$$\left(\frac{w_j^{(k-j)}}{A}\right)^{(k)}+w_j^{(k-j)}=-\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)}+\varphi^{(k-j)}\right). \tag{3.3}$$

By (3.3) we can write

$$\begin{aligned}&w_j^{(2k-j)}+\Phi_{2k-j-1}w_j^{(2k-j-1)}+...+\Phi_{k-j}w_j^{(k-j)}\\&=-A((\frac{\varphi^{(k-j)}}{A})^{(k)}+A(\frac{\varphi^{(k-j)}}{A})),\end{aligned} \tag{3.4}$$

where $\Phi_{k-j-1}(z),\ldots,\Phi_{2k-j}(z)$, $(j=0,1,\ldots,k)$ are meromorphic functions with $\rho_p(\Phi_{k-j})\leq\rho,\ldots,\rho_p(\Phi_{2k-j-1})\leq\rho$, $(j=0,1,\ldots,k)$. By $A\not\equiv 0$ and $\rho_p(\frac{\varphi^{(k-j)}}{A})<+\infty$, then by Lemma 2.3, we have

$$-A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)}+A\left(\frac{\varphi^{(k-j)}}{A}\right)\right)\not\equiv 0. \tag{3.5}$$

Hence, by Lemma 2.4, we have $\overline{\lambda}_p(w_j) = \rho_p(w_j) = +\infty$ and $\overline{\lambda}_{p+1}(w_j) = \rho_{p+1}(w_j) = \rho_p(A)$. Thus

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \rho_p(f) = +\infty \qquad (j = 0, 1, \dots, k),$$
$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho \quad (j = 0, 1, \dots, k).$$

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