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ON PSEUDO-SEQUENCE-COVERING π-IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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Abstract. In this paper, we characterize pseudo-sequence-covering π-images of locally separable metric spaces by means of fcs-covers and point-star networks. We also investigate pseudo-sequence-covering π-s-images of locally separable metric spaces.

1. Introduction

Determining what spaces the images of "nice" spaces under "nice" mappings are is one of the central questions of general topology [**3**]. In the past, some noteworthy results on images of metric spaces have been obtained [**9, 15**]. Recently, π-images of metric spaces have attracted attention again [**4, 5, 7, 11, 16**]. It is known that a space is a pseudo-sequence-covering π-image of a metric space (resp. separable metric space) if and only if it has a point-star network of fcs-covers (resp. countable fcs-covers) [**4, 5**]. This leads us to investigate pseudo-sequence-covering π-images of locally separable metric spaces. That is, we have the following question.

Question 1.1. *How are pseudo-sequence-covering π-images of locally sparable metric spaces characterized?*

On the other hand, pseudo-sequence-covering π-s-images of metric spaces have been characterized by means of point-star networks of point-countable fcs-covers (see [**11**], for example). This leads us to consider the following question.

Question 1.2. *How are pseudo-sequence-covering π-s-images of locally sparable metric spaces characterized?*

Taking these questions into account, we characterize pseudo-sequence-covering π-images of locally separable metric spaces by means of fcs-covers and point-star networks. Then we give a complete answer to Question 1.1. As the application

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of this result, we get a characterization of pseudo-sequence-covering π-s-images of locally separable metric spaces to answer Question 1.2.

Throughout this paper, all spaces are assumed to be Hausdorff, all mappings are assumed continuous and onto, a convergent sequence includes its limit point, $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and let $\mathcal{P}$ be a collection of subsets of X, we denote $\mathrm{st}(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$ and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\}$ converging to x is *eventually* (resp. *frequently*) in A if $\{x_n : n \geq n_0\} \cup \{x\} \subset A$ for some $n_0 \in \mathbb{N}$ (resp. $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset A$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$). Note that some notions are different in different references, and some different notions in different references are coincident. Please, terms which are not defined here, see [**2, 15**].

2. Main results

Let $\mathcal{P}$ be a collection of subsets of a space X and let K be a subset of X.

$\mathcal{P}$ is *point-countable* [**15**] if every point of X meets only countably many members of $\mathcal{P}$.

For each $x \in X$, $\mathcal{P}$ is a *network at x* [**8**] if $x \in P$ for every $P \in \mathcal{P}$, and if $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

$\mathcal{P}$ is a *k-cover for K in X*, if for each compact subset H of K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup \mathcal{F}$. When $K = X$, a k-cover for K in X is a *k-cover for X*.

$\mathcal{P}$ is a *cfp-cover for K in X* if for each compact subset H of K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$ where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. Note that such $\mathcal{F}$ is a *full cover* in the sense of [**1**], and if K is closed, $\mathcal{F}$ is a *cfp-cover* for K in the sense of [**8**]. When $K = X$, a cfp-cover for K in X is a *cfp-cover for X* [**16**].

$\mathcal{P}$ is an *fcs-cover for K in X* if for each convergent sequence S converging to x in K, there exists a finite subfamily $\mathcal{F}$ of $(\mathcal{P})_x$ such that S is eventually in $\bigcup \mathcal{F}$. When $K = X$, an fcs-cover for K in X is an *fcs-cover of X* [**4**], or an *sfp-cover for X* [**11**], or a *wcs-cover* [**5**].

$\mathcal{P}$ is a *cs^*-cover for K in X*, if for each convergent sequence S in K, S is frequently in some $P \in \mathcal{P}$. When $K = X$, a cs^*-cover for K in X is a *cs^*-cover* for X [**16**].

A k-cover (resp. cfp-cover, fcs-cover, cs^*-cover) for K in X is also called a *k-cover* (resp. *cfp-cover*, *fcs-cover*, *cs^*-cover*) *in X for K*, and a k-cover (resp. cfp-cover, fcs-cover, cs^*-cover) for X is abbreviated to a k-cover (resp. cfp-cover, fcs-cover, cs^*-cover).

It is clear that if $\mathcal{P}$ is a k-cover (resp. cfp-cover, fcs-cover, cs^*-cover), then $\mathcal{P}$ is a k-cover (resp. cfp-cover, fcs-cover, cs^*-cover) for K in X.

Remark. The following statements hold.

1. closed k-cover for K in $X \Longrightarrow$ cfp-cover for K in $X \Longrightarrow$ k-cover for K in X,

2. cfp-cover for K in $X \Longrightarrow fcs$-cover for K in $X \Longrightarrow cs^*$-cover for K in X.

For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a cover for X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *refinement sequence* for X, if $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for each $n \in \mathbb{N}$. A refinement sequence for X is a *refinement* of X in the sense of [**3**].

Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is be refinement sequence for X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *point-star network* for X, if $\{\mathrm{st}\,(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x for each $x \in X$. A point--star network for X is a *σ-strong network* for X in the sense of [**16**], and, without the assumption of a refinement sequence, a *point-star network* in the sense of [**12**]. It is easy to see that if each $\mathcal{P}_n$ is countable, every members of $\mathcal{P}_n$ can be chosen closed in X.

Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-star network for a space X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and A_n is endowed with discrete topology. Put

$$M = \Big\{a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} A_n$, is a metric space with a metric d described as follows.

Let $a = (\alpha_n), b = (\beta_n) \in M$. If $a = b$, then $d(a, b) = 0$. If $a \neq b$, then $d(a, b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Define $f : M \longrightarrow X$ by choosing $f(a) = x_a$, then f is a mapping, and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev's system* [**16**], and without the assumption of a refinement sequence in the notion of point-star networks, $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev's system* in the sense of [**12**].

Let $f : X \longrightarrow Y$ be a mapping; Then,

f is a *π-mapping* [**4**] if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

f is an *s-mapping* [**11**], if for each $y \in Y$, $f^{-1}(y)$ is a separable subset of X.

f is a *π-s-mapping* [**11**], if f is both π-mapping and s-mapping.

f is a *pseudo-sequence-covering mapping* [**3**], if every convergent sequence of Y is the image of some compact subset of X.

f is a *subsequence-covering mapping* [**3**], if for every convergent sequence S of Y, there is a compact subset K of X such that $f(K)$ is a subsequence of S.

f is a *sequentially-quotient mapping* [**3**], if for every convergent sequence S of Y, there is a convergent sequence L of X such that $f(L)$ is a subsequence of S.

f is a *quotient mapping* [**14**], if U is open in Y whenever $f^{-1}(U)$ is open in X.

f is a *pseudo-open mapping* [**9**], if $y \in \mathrm{int} f(U)$ whenever $f^{-1}(y) \subset U$ with U open in X. A pseudo-open mapping is a *hereditarily quotient mapping* in the sense of [**2**].

Let X be a space and let A be a subset of X. A is *sequential open* [**16**], if for each $x \in A$ and each convergent sequence S converging to x, S is eventually in A. X is a *sequential space* [**16**], if every sequential open subset of X is open in X. X is a *Fréchet space*, if for each $x \in \overline{A}$, there exists a sequence in A converging to x.

For a mapping $f : X \longrightarrow Y$, f is a pseudo-sequence-covering or sequentially-quotient $\Longrightarrow$ a f is subsequence-covering. Also, a f is quotient if and only if a f is subsequence-covering such that Y is sequential [**17**].

Lemma 2.1. *Let $\mathcal{P}$ be a countable cover for a convergent sequence S in a space X. Then the following propositions are equivalent.*

1. *$\mathcal{P}$ is a cfp-cover for S in X,*
2. *$\mathcal{P}$ is an fcs-cover for S in X,*
3. *$\mathcal{P}$ is a cs^*-cover for S in X.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Obviously.

(3) $\Longrightarrow$ (1). Let H be a compact subset of S. We can assume that H is a subsequence of S. Since $\mathcal{P}$ is countable, put $(\mathcal{P})_x = \{P_n : n \in \mathbb{N}\}$ where x is the limit point of S. Then H is eventually in $\bigcup_{n\le k} P_n$ for some $k \in \mathbb{N}$. If not, then for any $k \in \mathbb{N}$, H is not eventually in $\bigcup_{n\le k} P_n$. So, for every $k \in \mathbb{N}$, there exists $x_{n_k} \in S - \bigcup_{n\le k} P_n$. We may assume $n_1 < n_2 < \ldots < n_{k-1} < n_k < n_{k+1} < \ldots$. Put $H' = \{x_{n_k} : k \in \mathbb{N}\} \cup \{x\}$, then H' is a subsequence of S. Since $\mathcal{P}$ is a cs^*-cover for S in X, there exists $m \in \mathbb{N}$ such that H' is frequently in P_m. This contradicts the construction of H'. So H is eventually in $\bigcup_{n\le k} P_n$ for some $k \in \mathbb{N}$. It implies that $\mathcal{P}$ is a cfp-cover for S in X. □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a mapping.*

1. *If $\mathcal{P}$ is a k-cover in X for a compact set K, then $f(\mathcal{P})$ is a k-cover for $f(K)$ in Y.*
2. *If $\mathcal{P}$ is a cfp-cover in X for a compact set K, then $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.*

Proof. (1). Let H be a compact subset of $f(K)$. Then $G = f^{-1}(H) \cap K$ is a compact subset of K and $f(G) = H$. Since $\mathcal{P}$ is a k-cover for K in X, there is a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $G \subset \bigcup \mathcal{F}$. Hence $f(\mathcal{F})$ is a finite subfamily of $f(\mathcal{P})$ such that $H \subset \bigcup f(\mathcal{F})$. It implies that $f(\mathcal{P})$ is a k-cover for $f(K)$ in Y.

(2). Let H be a compact subset of $f(K)$. Then $L = f^{-1}(H) \cap K$ is a compact subset of K satisfying $f(L) = H$. Since $\mathcal{P}$ is a cfp-cover for K in X, there is a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $L \subset \bigcup\{C_F : F \in \mathcal{F}\}$ where $C_F \subset F$, and C_F is closed for every $F \in \mathcal{F}$. Because L is compact, every C_F can be chosen compact. It implies that every $f(C_F)$ is closed (in fact, every $f(C_F)$ is compact), and $f(C_F) \subset f(F)$. We get that $H = f(L) \subset \bigcup\{f(C_F) : F \in \mathcal{F}\}$, and $f(\mathcal{F})$ is a finite subfamily of $\mathcal{P}$. Then $\mathcal{P}$ is a cfp-cover for $f(K)$ in Y. □

Theorem 2.3. *The following propositions are equivalent for a space X*

1. *X is a pseudo-sequence-covering π-image of a locally separable metric space,*
2. *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ of countable covers for X_λ satisfying the following conditions:*
   (a) *For each $x \in U$ with U open in X, there is $n \in \mathbb{N}$ such that*

$$\bigcup\{\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \textit{ with } x \in X_\lambda\} \subset U,$$

(b) *For each convergent sequence S of X, there is a finite subset Λ_S of Λ such that S has a finite compact cover $\{S_\lambda : \lambda \in \Lambda_S\}$, and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is an fcs-cover for S_λ in X_λ.*

Proof. (1) $\Longrightarrow$ (2). Let $f : M \longrightarrow X$ be a pseudo-sequence-covering π-mapping from a locally separable metric space M with a metric d onto X. Since M is a locally separable metric space, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ where each M_λ is a separable metric space by [**2**, 4.4.F]. For each $\lambda \in \Lambda$, let D_λ be a countable dense subset of M_λ, and put $f_\lambda = f|_{M_\lambda}$ and $X_\lambda = f_\lambda(M_\lambda)$. For each $a \in M_\lambda$ and $n \in \mathbb{N}$, put $B(a, 1/n) = \{b \in M_\lambda : d(a, b) < 1/n\}$, $\mathcal{B}_{\lambda,n} = \{B(a, 1/n) : a \in D_\lambda\}$, and $\mathcal{P}_{\lambda,n} = f_\lambda(\mathcal{B}_{\lambda,n})$. It is clear that $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a cover sequence of countable covers for X_λ and $\mathcal{P}_{\lambda,n+1}$ is a refinement of $\mathcal{P}_{\lambda,n}$ for every $n \in \mathbb{N}$. We only need to prove that conditions (a) and (b) are satisfied.

Condition (a): For each $x \in U$ with U open in X. Since f is a π-mapping, $\mathrm{d}(f^{-1}(x), M - f^{-1}(U)) > 2/(n-1)$ for some $n \in \mathbb{N}$. Then, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get

$$\mathrm{d}(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) > 2/(n-1)$$

where $U_\lambda = U \cap X_\lambda$. Let $a \in D_\lambda$ and $x \in f_\lambda(B(a, 1/n)) \in \mathcal{P}_{\lambda,n}$. We shall prove that $B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda)$. In fact, if $B(a, 1/n) \not\subset f_\lambda^{-1}(U_\lambda)$, then pick $b \in B(a, 1/n) - f_\lambda^{-1}(U_\lambda)$. Note that $f_\lambda^{-1}(x) \cap B(a, 1/n) \neq \emptyset$, pick $c \in f_\lambda^{-1}(x) \cap B(a, 1/n)$, then

$$\mathrm{d}(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \le \mathrm{d}(c, b) \le \mathrm{d}(c, a) + \mathrm{d}(a, b) < 2/n < 2/(n-1).$$

It is a contradiction. So $B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda)$, thus $f_\lambda(B(a, 1/n)) \subset U_\lambda$. Then $\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. It implies that

$$\bigcup\{\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U.$$

Condition (b): For each convergent sequence S of X, since a f is pseudo-sequence-covering, $S = f(K)$ for some compact subset K of M. By compactness of K, $K_\lambda = K \cap M_\lambda$ is compact and $\Lambda_S = \{\lambda \in \Lambda : K_\lambda \neq \emptyset\}$ is finite. For each $\lambda \in \Lambda_S$, put $S_\lambda = f(K_\lambda)$, then $\{S_\lambda : \lambda \in \Lambda_S\}$ is a finite compact cover for S. For each $n \in \mathbb{N}$, since $\mathcal{B}_{\lambda,n}$ is a cfp-cover for K_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for S_λ in X_λ by Lemma 2.2. It follows from Lemma 2.1 that $\mathcal{P}_{\lambda,n}$ is an fcs-cover for S_λ in X_λ

(2) $\Longrightarrow$ (1). For each $\lambda \in \Lambda$, let $x \in U_\lambda$ with U_λ open in X_λ. We get that $U_\lambda = U \cap X_\lambda$ with some U open in X. Since $\bigcup\{\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U$ for some $n \in \mathbb{N}$, $\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. It implies $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ. Then the Ponomarev's system $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ exists. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space with a metric d_λ described as follows.

Let $a = (\alpha_n), b = (\beta_n) \in M_\lambda$. If $a = b$, then $\mathrm{d}_\lambda(a, b) = 0$. If $a \neq b$, then $\mathrm{d}_\lambda(a, b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Put $M = \oplus_{\lambda\in\Lambda} M_\lambda$ and define $f : M \longrightarrow X$ by choosing $f(a) = f_\lambda(a)$ for every $a \in M_\lambda$ with some $\lambda \in \Lambda$. Then f is a mapping and M is a locally separable metric space with a metric d as follows.

Let $a, b \in M$. If $a, b \in M_\lambda$ for some $\lambda \in \Lambda$, then $\mathrm{d}(a,b) = \mathrm{d}_\lambda(a,b)$. Otherwise, $\mathrm{d}(a,b) = 1$. We only need to prove that f is a pseudo-sequence-covering π-mapping.

(a) *f is a π-mapping.* Let $x \in U$ with U open in X, then

$$\bigcup\{\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U$$

for some $n \in \mathbb{N}$. So, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get

$$\mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$$

where $U_\lambda = U \cap X_\lambda$. It implies that

$$\mathrm{d}_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n.$$

In fact, if $a = (\alpha_k) \in M_\lambda$ such that $\mathrm{d}_\lambda(f_\lambda^{-1}(x), a) < 1/n$, then there is $b = (\beta_k) \in f_\lambda^{-1}(x)$ such that $\mathrm{d}_\lambda(a,b) < 1/n$. So $\alpha_k = \beta_k$ if $k \leq n$. Note that $x \in P_{\beta_n} \subset \mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Then

$$f_\lambda(a) \in P_{\alpha_n} = P_{\beta_n} \subset \mathrm{st}\,(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda.$$

Hence $a \in f_\lambda^{-1}(U_\lambda)$. It implies that $\mathrm{d}_\lambda(f_\lambda^{-1}(x), a) \geq 1/n$ if $a \in M_\lambda - f_\lambda^{-1}(U_\lambda)$. So

$$\mathrm{d}_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n.$$

Therefore

$$\begin{aligned} &\mathrm{d}(f^{-1}(x), M - f^{-1}(U)) = \inf\{d(a,b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\ &\quad = \min\big\{1, \inf\{d_\lambda(a,b) : a \in f_\lambda^{-1}(x), b \in M_\lambda - f^{-1}(U_\lambda), \lambda \in \Lambda\}\big\} \geq 1/n > 0. \end{aligned}$$

It implies that f is a π-mapping.

(b) *f is pseudo-sequence-covering.* For each convergent sequence S of X, there is a finite subset Λ_S of Λ such that S has a finite compact cover $\{S_\lambda : \lambda \in \Lambda_S\}$ and for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is an fcs-cover for S_λ in X_λ. By Lemma 2.1 $\mathcal{P}_{\lambda,n}$ is a cfp-cover for S_λ in X_λ. It follows from Lemma 13 in [**12**] that $S_\lambda = f_\lambda(K_\lambda)$ with some compact subset K_λ of M_λ. Put $K = \bigcup\{K_\lambda : \lambda \in \Lambda_S\}$, then K is a compact subset of M and $f(K) = S$. It implies that f is a pseudo-sequence-covering. □

Remark. 1. For each $\lambda \in \Lambda$, $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

2. Since each $\mathcal{P}_{\lambda,n}$ is countable, every member of $\mathcal{P}_{\lambda,n}$ can be chosen closed in X_λ. Hence, it is possible to replace the prefix "*fcs-*" in (b) of Theorem 2.3.(2) by "*k-*", "*cfp-*", or "*cs^*-*"

By [**2**, 2.4.F, 2.4.G], [**3**, Proposition 2.1], and Theorem 2.3, we get a characterization of pseudo-sequence-covering quotient (resp. pseudo-open) π-images of locally separable metric spaces as follows.

Corollary 2.4. *The following propositions are equivalent:*

1. *a space X is a pseudo-sequence-covering quotient (resp. pseudo-open) π-image of a locally separable metric space,*

2. *a space X is a sequential (resp. Fréchet) space having a cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ of countable covers for X_λ satisfying conditions (a) and (b) in Theorem 2.3.(2).*

In the next, we investigate pseudo-sequence-covering π-s-images of locally separable metric spaces.

Corollary 2.5. *The following propositions are equivalent:*

1. *a space X is a pseudo-sequence-covering π-s-image of a locally separable metric space,*
2. *a space X has a point-countable cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ of countable covers for X_λ satisfying conditions (a) and (b) in Theorem 2.3.(2).*

Proof. (1) $\Longrightarrow$ (2). By using notations and arguments in proof (1) $\Longrightarrow$ (2) of Theorem 2.3 again, X has a cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ of countable covers for X_λ satisfying conditions (a) and (b) in Theorem 2.3.(2). It suffices to prove that $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable. For each $x \in X$, since f is an s-mapping, $f^{-1}(x)$ is separable in M. Then $f^{-1}(x)$ meets only countably many M_λ's. It implies that x meets only coutably many X_λ's, i.e., $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable.

(2) $\Longrightarrow$ (1). By using notations and arguments in proof (2) $\Longrightarrow$ (1) of Theorem 2.3 again, X is a pseudo-sequence-covering π-image of a locally separable metric space under the mapping f. We shall prove that f is also an s-mapping. For each $x \in X$, since $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable, $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$ is countable. Note that each M_λ is separable metric, $f_\lambda^{-1}(x)$ is separable. It implies that $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda_x\}$ is separable, i.e., f is an s-mapping. □

Similar to Corollary 2.4, we get the following.

Corollary 2.6. *The following propositions are equivalent:*

1. *a space X is a pseudo-sequence-covering quotient (resp. pseudo-open) π-s-image of a locally separable metric space,*
2. *a space X is a sequential (resp. Fréchet) space having a point-countable cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ of countable covers for X_λ satisfying conditions (a) and (b) in Theorem 2.3.(2).*

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