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A NOTE ON NEIGHBORHOODS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. The purpose of the present paper is to make use of the familiar concept of neighborhoods of analytic functions. Several inclusion relations associated with the (n, δ) neighborhoods of various subclasses defined by Sălăgean operator are proved. Special cases of these results are shown to yield known results in the literature.

1. INTRODUCTION

Let $\mathcal{T}(j)$ be the class of functions in the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; \qquad j \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Let Ω be the class of functions $\omega(z)$ analytic in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$.

For $f(z)$ and $g(z)$ in $\mathcal{T}(j)$, $f(z)$ is said to be subordinate to $g(z) \in \mathcal{U}$ if there exists an analytic function $\omega(z) \in \Omega$ such that $f(z) = g(\omega(z))$. This subordination [**6**] is denoted by

$$f(z) \prec g(z).$$

Following [**1, 7, 9**] we define the (j, δ)-neighborhood of a function $f(z) \in \mathcal{A}(j)$ by

$$\mathbf{N}_{j,\delta}(f) = \{g \in \mathcal{T}(j);\ g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,\ \sum_{k=j+1}^{\infty} k|a_k - b_k| \leq \delta\}. \tag{1.2}$$

In particular, for the identity function $e(z) = z$, we have

$$\mathbf{N}_{j,\delta}(f) = \{g \in \mathcal{T}(j);\ g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,\ \sum_{k=j+1}^{\infty} k|b_k| \leq \delta\}. \tag{1.3}$$

The purpose of this paper is to investigate the (j, δ)-neighborhoods of the certain subclasses of the class $\mathcal{T}(j)$ of normalized analytic functions in $\mathcal{U}$ with negative coefficients.

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For a function $f(z) \in \mathcal{A}(j)$, we define

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^n f(z) &= D(D^{n-1} f(z)), \qquad (n \in \mathbb{N}) \end{aligned} \tag{1.4}$$

where D^n is the differential operator introduced by Sălăgean [**10**]. Using the differential operator D^n, we define the class $\mathcal{T}_j(n,m,A,B)$ as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}(j)$ is in the class $\mathcal{T}_j(n,m,A,B)$ if and only if

$$\frac{D^{n+m} f(z)}{D^n f(z)} \prec \frac{1+Az}{1+Bz}, \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ m \in \mathbb{N}) \tag{1.5}$$

for $-1 \le B < A \le 1$ and for all $z \in \mathcal{U}$.

The operator D^{n+m} was studied by Sekine [**11**], Aouf et al. [**2**], Aouf et al. [**3**] and Hossen et al.[**8**]. We note that $\mathcal{T}_j(n,m,1-2\alpha,-1) = \mathcal{T}_j(n,m,\alpha)$[**4**], $\mathcal{T}_j(0,1,\alpha) = \mathcal{S}_j^*(\alpha)$, the class of starlike functions of order α and $\mathcal{T}_j(1,1,\alpha) = \mathcal{C}_j(\alpha)$, the class of convex functions of order α (Chatterjea [**5**] and Srivastava et al.[**12**]).

2. Neighborhood for the class $\mathcal{T}_j(n,m,A,B)$

For the class $\mathcal{T}_j(n,m,A,B)$, we prove the following lemma.

Lemma 2.1. *A function $f(z) \in \mathcal{T}(j)$ is in the class $\mathcal{T}_j(n,m,A,B)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n[(1-B)k^m - (1-A)]a_k \le A - B \tag{2.1}$$

for $n \in \mathbb{N}_0,\ m \in \mathbb{N}$ and $-1 \le B < A \le 1$.

Proof. Suppose $f(z) \in \mathcal{T}_j(n,m,A,B)$, then

$$\frac{D^{n+m} f(z)}{D^n f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}.$$

Therefore

$$\omega(z) = \frac{D^n f(z) - D^{n+m} f(z)}{BD^{n+m} f(z) - AD^n f(z)}$$

hence

$$\begin{aligned} |\omega(z)| &= \left| \frac{D^{n+m} f(z) - D^n f(z)}{BD^{n+m} f(z) - AD^n f(z)} \right| \\ &= \left| \frac{\displaystyle\sum_{k=j+1}^{\infty} k^n(k^m-1)a_k z^k}{(A-B)z + \displaystyle\sum_{k=j+1}^{\infty} k^n(Bk^m - A)a_k z^k} \right| < 1. \end{aligned}$$

Thus

$$\Re\left\{\frac{\sum\limits_{k=j+1}^{\infty} k^n(k^m-1)a_k z^k}{(A-B)z+\sum\limits_{k=j+1}^{\infty} k^n(Bk^m-A)a_k z^k}\right\}<1. \tag{2.2}$$

Take $z=r$ with $0<r<1$. Then for sufficiently small r, the denominator of (2.2) is positive and so it is positive for all r with $0<r<1$, since $\omega(z)$ is analytic for $|z|<1$. Then (2.2) gives

$$\sum_{k=j+1}^{\infty} k^n(1-k^m)a_k r^k<(B-A)r-B\sum_{k=j+1}^{\infty} k^{n+m}a_k r^k+A\sum_{k=j+1}^{\infty} k^n a_k r^k$$

i.e.,

$$\sum_{k=j+1}^{\infty} k^n[(1-B)k^m-(1-A)]a_k r^k<(A-B)r$$

and (2.1) follows on letting $r\to 1$.

Conversely, for $|z|=r,\ 0<r<1,$ we have $r^k<r,$ i.e.,

$$\sum_{k=j+1}^{\infty} k^n[(1-B)k^m-(1-A)]a_k r^k<\sum_{k=j+1}^{\infty} k^n[k^m(1-B)-(1-A)]a_k r<(A-B)r,$$

by (2.1), so we have,

$$\left|\sum_{k=j+1}^{\infty} k^n(k^m-1)a_k z^k\right|\le\sum_{k=j+1}^{\infty} k^n(k^m-1)a_k r^k$$

i.e.,

$$\left|\sum_{k=j+1}^{\infty} k^n(k^m-1)a_k z^k\right|<(A-B)r+\sum_{k=j+1}^{\infty}(Bk^m-A)k^n a_k r^k$$

i.e.,

$$\left|\sum_{k=j+1}^{\infty} k^n(k^m-1)a_k z^k\right|\le\left|(A-B)z+\sum_{k=j+1}^{\infty}(Bk^m-A)k^n a_k z^k\right|.$$

This proves that $\dfrac{D^{n+m}f(z)}{D^n f(z)}$ is of the form $\dfrac{1+A\omega(z)}{1+B\omega(z)}$ and hence $f(z)\in\mathcal{T}_j(n,m,A,B)$ and the proof is complete. □

Applying the above lemma, we prove the following.

Theorem 2.2. $\mathcal{T}_j(n,m,A,B) \subset \mathcal{N}_{j,\delta}(e)$, *where*

$$\delta = \frac{A-B}{(j+1)^{n-1}[(1-B)(j+1)^m-(1-A)]} \tag{2.3}$$

Proof. It follows from (2.1) that if $f(z) \in \mathcal{T}_j(n,m,A,B)$, then

$$(j+1)^{n-1}[(1-B)(j+1)^m-(1-A)] \sum_{k=j+1}^{\infty} k a_k \le A-B \tag{2.4}$$

which implies

$$\sum_{k=j+1}^{\infty} k a_k \le \frac{A-B}{(j+1)^{n-1}[(1-B)(j+1)^m-(1-A)]} = \delta. \tag{2.5}$$

Using (1.3), we get the result. $\square$

Putting $j=1$ in Theorem 2.2, we have the following.

Corollary 2.3. $\mathcal{T}_1(n,m,A,B) \subset \mathcal{N}_{1,\delta}(e)$, *where*

$$\delta = \frac{A-B}{2^{n-1}[(1-B)2^m-(1-A)]}.$$

3. Neighborhoods for the classes $\mathcal{R}_j(n,A,B)$ and $\mathcal{P}_j(n,A,B)$

We define the following classes.

Definition 3.1. A function $f(z) \in \mathcal{T}(j)$ is said to be in the class $f(z) \in \mathcal{R}_j(n,A,B)$ if it satisfies

$$(D^n f(z))' \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{3.1}$$

for $-1 \le B < A \le 1$ and $n \in \mathbb{N}_0$.

Definition 3.2. A function $f(z) \in \mathcal{T}(j)$ is said to be a member of the class $\mathcal{P}_j(n,A,B)$ if it satisfies

$$\frac{D^n f(z)}{z} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{3.2}$$

for $-1 \le B < A \le 1$ and $n \in \mathbb{N}_0$.

So, we have the following results.

Lemma 3.3. *A function $f(z) \in \mathcal{T}(j)$ is in the class $\mathcal{R}_j(n,A,B)$ if and only if*

$$\sum_{k=j+1}^{\infty} (1-B)k^{n+1} a_k \le A-B. \tag{3.3}$$

Lemma 3.4. *A function $f(z) \in \mathcal{T}(j)$ is in the class $\mathcal{P}_j(n,A,B)$ if and only if*

$$\sum_{k=j+1}^{\infty} (1-B)k^{n} a_k \le A-B \tag{3.4}$$

From the above Lemmas, we see that $\mathcal{R}_j(n,A,B) \subset \mathcal{P}_j(n,A,B)$

Theorem 3.5. $\mathcal{R}_j(n,A,B) \subset \mathcal{N}_{j,\delta}(e)$ *where*

$$\delta = \frac{A-B}{(j+1)^n(1-B)}. \tag{3.5}$$

Proof. If $f(z) \in \mathcal{R}_j(n,A,B)$, we have

$$(j+1)^n \sum_{k=j+1}^{\infty} (1-B)ka_k \leq A-B \tag{3.6}$$

which implies

$$\sum_{k=j+1}^{\infty} ka_k \leq \frac{A-B}{(1-B)(j+1)^n} = \delta. \tag{3.7}$$

□

Corollary 3.6. $\mathcal{R}_1(n,A,B) \subset \mathcal{N}_{1,\delta}(e)$ *where* $\delta = \dfrac{A-B}{2^n(1-B)}$

Theorem 3.7. $\mathcal{P}_j(n,A,B) \subset \mathcal{N}_{j,\delta}(e)$ *where*

$$\delta = \frac{A-B}{(j+1)^{n-1}(1-B)}. \tag{3.8}$$

Proof. If $f(z) \in \mathcal{P}_j(n,A,B)$ we have

$$(j+1)^{n-1} \sum_{k=j+1}^{\infty} (1-B)ka_k \leq A-B$$

which gives

$$\sum_{k=j+1}^{\infty} ka_k \leq \frac{A-B}{(1-B)(j+1)^{n-1}} = \delta \tag{3.9}$$

that, in view of definition (1.3) proves Theorem 3.7. □

Putting $j=1$ in Theorem 3.7, we have the following.

Corollary 3.8.

$$\mathcal{P}_1(n,A,B) \subset \mathcal{N}_{1,\delta}(e)$$

where

$$\delta = \frac{A-B}{2^{n-1}(1-B)}$$

4. Neighborhood of the class $\mathcal{K}_j(n,m,A,B,C,D)$

Definition 4.1. A function $f(z) \in \mathcal{T}(j)$ is said to be in the class $\mathcal{K}_j(n,m,A,B,C,D)$ if it satisfies

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{A-B}{1-B} \qquad (z \in \mathcal{U}) \tag{4.1}$$

for $-1 \le B < A \le 1, -1 \le D < C \le 1$ and $g(z) \in \mathcal{T}_j(n,m,C,D)$.

Theorem 4.2. *$\mathcal{N}_{j,\delta}(g) \subset \mathcal{K}_j(n,m,A,B,C,D)$ where $g(z) \in \mathcal{T}_j(n,m,C,D)$ and*

$$\frac{1-A}{1-B} = 1 - \frac{(j+1)^m[(1-D)(j+1)^m - (1-C)]\delta}{(j+1)^n[(1-D)(j+1)^m - (1-C)] - (C-D)} \tag{4.2}$$

where

$$\delta \le (1-D)(j+1) - (C-D)(j+1)^{1-n}[(1-D)(j+1)^m - (1-C)]^{-1}.$$

Proof. Let $f(z)$ be in $\mathcal{N}_{j,\delta}(g)$ for $g(z) \in \mathcal{T}_j(n,m,C,D)$ then

$$\begin{aligned}\sum_{k=j+1}^{\infty} k|a_k - b_k| &\le \delta \sum_{k=j+1}^{\infty} b_k \\ &\le \frac{C-D}{(j+1)^n[(1-D)(j+1)^m - (1-C)]}.\end{aligned} \tag{4.3}$$

Consider,

$$\begin{aligned}&\left|\frac{f(z)}{g(z)} - 1\right| \\ &\le \frac{\sum_{k=j+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=j+1}^{\infty} b_k} \\ &\le \frac{\delta}{j+1}.\frac{(j+1)^n[(j+1)^m(1-D) - (1-C)]}{(j+1)^n[(j+1)^m(1-D) - (1-C)] - (C-D)} \\ &= \frac{(j+1)^{n-1}[(j+1)^m(1-D) - (1-C)]}{(j+1)^n[(j+1)^m(1-D) - (1-C)] - (C-D)} \\ &= \frac{A-B}{1-B}.\end{aligned}$$

This implies that $f(z) \in \mathcal{K}_j(n,m,A,B,C,D)$. □

Putting $j = 1$ in Theorem 4.2, we have the following.

Corollary 4.3. *$\mathcal{N}_{1,\delta}(g) \subset \mathcal{K}_1(n,m,A,B,C,D)$ where $g(z) \in \mathcal{T}_1(n,m,C,D)$ and*

$$\alpha = 1 - \frac{2^{n-1}[2^m(1-D) - (1-B)]\delta}{2^n[2^m(1-D) - (1-B)] - (C-D)}.$$

Remark 4.4. For $A = 1-2\alpha$ $B = -1, C = 1-2\beta, D = -1$ we get the results obtained by Aouf [**4**].

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