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COINCIDENCE POINTS FOR CONTRACTION TYPE MAPS

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Abstract. Some new results on the existence of coincidence points for multivalued f-contractive maps have been obtained. Our results improve and extend various known results existing in the literature.

1. Introduction and Preliminaries

Using the concept of Hausdorff metric, many authors have proved fixed point and coincidence point results in the setting of metric spaces. For example, using the Hausdorff metric, Nadler [**12**] has introduced a notion of multivalued contraction maps and proved a multivalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space has a fixed point. Since then various fixed point results concerning multivalued contractions have appeared. For example, see [**2, 3, 4, 7, 11, 13, 15, 18**]. On the other hand, using the concept of Hausdorff metric, Kaneko [**8**] has introduced a notion of multivalued f-contraction maps and proved coincidence point result for such maps with commutativity condition, extending the corresponding results of Jungck [**6**], Nadler [**12**] and others. This result has been generalized in different directions. For example, see [**9, 10, 14, 16, 17**].

Without using the concept of Hausdorff metric, most recently Feng and Liu [**5**] introduced a notion of multivalued contractive maps and proved a fixed point result for such maps, extending Nadler's fixed point result concerning multivalued contractions. The aim of this paper is to obtain some results on the existence of coincidence points for multivalued f-contractive maps with and without commutativity condition. Our results unify, improve and extend a number of fixed point and coincidence point results including the corresponding results of Feng and Liu [**5**], Kaneko [**8**], Nadler [**12**].

Throughout this paper, (X, d) is a metric space; $S(X)$ is the family of nonempty subsets of X; $C(X)$ is the family of nonempty closed subsets of X; $CB(X)$ is the family of nonempty closed bounded subsets of X; for any $A, B \in S(X)$,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

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where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from the point a to the subset B. It is well known that H is a metric on $CB(X)$ and is known as the Hausdorff metric on $CB(X)$. H on $C(X)$ is also a metric except that it takes also the value $+\infty$ if (X,d) is unbounded.

We also use the following notions.

Let $T : X \to S(X)$ be a multivalued map and $f : X \to X$ a single-valued injective map such that $T(X) \subset f(X)$. Then

i) T is called *contraction* if there exists a constant $h \in (0,1)$ such that
$$H(T(x),T(y)) \leq hd(x,y), \quad x,y \in X.$$
ii) T is called f-*contraction* if there exists a constant $h \in (0,1)$ such that
$$H(T(x),T(y)) \leq hd(f(x),f(y)), \quad x,y \in X.$$
iii) For a positive number $b < 1$, Feng and Liu [**5**] defined a subset of X as
$$I_b^x = \{y \in T(x) : bd(x,y) \leq d(x,T(x))\}$$
and here we define a subset J_b^x of X as
$$J_b^x = \{w \in T(x) : bd(f(x),w) \leq d(f(x),T(x))\}.$$
iv) We say T is *contractive* if there exists a constant $h \in (0,1)$ and for any $x \in X$ there is $y \in I_b^x$ such that
$$d(y,T(y)) \leq hd(x,y),$$
where $h < b$.

v) We say T is f-*contractive* if there exists a constant $h \in (0,1)$ and for all $x \in X$ there is $w \in J_b^x$ such that
$$d(w,Tf^{-1}(w)) \leq hd(f(x),w),$$
where $h < b$ and $f^{-1}(w)$ denotes the counterimage of w.

vi) we say f and T weakly commute if $f(T(x)) \subset T(f(x))$, for all $x \in X$.

vii) we say a sequence $\{x_n\} \subset X$ is f-*iterative* sequence of T at $x_0 \in X$ if for each $n \geq 1$, $f(x_n) \in T(x_{n-1})$.

It has been observed [**5**] that each multivalued contraction map $T : X \to CB(X)$ is a multivalued contractive map. Also, we note that each multivalued f-contraction map is multivalued f-contractive, because for any $x \in X$, $w \in T(x)$,
$$d(w,Tf^{-1}(w)) \leq H(T(x),Tf^{-1}(w)) \leq h\,d(f(x),w)).$$
Moreover, in particular, if $f = I$, the identity map on X then each multivalued f-contraction map is a contraction, each multivalued f-contractive map is contractive and $J_b^x = I_b^x$.

A point $x \in X$ is called a fixed point of T if $x \in T(x)$ and the set of fixed points of T is denoted by $\mathrm{Fix}(T)$. A point $x \in X$ is called a coincidence point of f and T if $f(x) \in T(x)$. We denote by $C(f \cap T)$ the set of coincidence points of f and T.

A real valued function φ on X is called lower semi-continuous if, for any sequence $\{x_n\} \subset X$, $x_n \to x \in X$ implies that $\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n)$. A multivalued map $T : X \to S(X)$ is called upper semi-continuous if, for each open set $A \subset X$, the set $T^{-1}(A) = \{x \in X : T(x) \subset A\}$ is open. We say T is closed if its graph $G(T) = \{(x, y) \in X \times X : y \in T(x)\}$ is closed. Recall that the graph of an upper semi-continuous multivalued map $T : X \to C(X)$ is closed. See, for instance [**1**].

To make the paper self-contained, we recall the following fixed point and coincidence point results.

Theorem 1.1 ([**5**]). *Let X be a complete metric space and let $T : X \to C(X)$ be a multivalued contractive map such that the real-valued function g on X, $g(x) = d(x, T(x))$ is lower semi-continuous. Then* $\mathrm{Fix}(T) \neq \emptyset$.

Theorem 1.2 ([**8**]). *Let X be a complete metric space and let $T : X \to CB(X)$ be a multivalued f-contraction map which commutes with continuous map f. Then $C(f \cap T) \neq \emptyset$.*

2. Main Results

First, we prove our basic lemma on the existence of f-iterative sequences, which is useful for proving coincidence point theorems.

Lemma 2.1. *Let $T : X \to C(X)$ be an f-contractive map. Then, there exists an f-iterative sequence $\{x_n\}$ of T at $x_0 \in X$ such that $\{f(x_n)\}$ is a Cauchy sequence.*

Proof. Since, for any $x \in X$, $T(x)$ is closed, the set J_b^x is nonempty for any $b \in (0, 1)$. Now, let $x_0 \in X$, then by the definition of T , there exists $w_1 \in J_b^{x_0}$ such that

$$d(w_1, Tf^{-1}(w_1)) \leq h\, d(f(x_0), w_1).$$

Since $w_1 \in J_b^{x_0} \subset T(x_0) \subset f(X)$ and f is injective, there exists $x_1 \in X$ such that $f(x_1) = w_1$. Also, since the counterimage of w_1 is x_1, denoted by $f^{-1}(w_1) = x_1$, we get

$$d(f(x_1), T(x_1)) \leq h\, d(f(x_0), f(x_1)),$$

that is,

$$d(f(x_1), Tf^{-1}(f(x_1))) \leq h\, d(f(x_0), f(x_1)).$$

Also, since $x_1 \in X$, there exists $w_2 \in J_b^{x_1}$ such that

$$d(w_2, Tf^{-1}(w_2)) \leq h\, d(f(x_1), w_2).$$

Since $w_2 \in J_b^{x_1} \subset T(x_1) \subset f(X)$, there exists $x_2 \in X$ such that $f(x_2) = w_2$, thus

$$d(f(x_2), T(x_2)) \leq h\, d(f(x_1), f(x_2)).$$

By continuing this process we get $w_n \in J_b^{x_{n-1}}$ such that for all $n \geq 1$

$$d(w_n, Tf^{-1}(w_n)) \leq h\, d(f(x_{n-1}), w_n).$$

Since $w_n \in J_b^{x_{n-1}} \subset T(x_{n-1}) \subset f(X)$ there exists $x_n \in X$ such that $f(x_n) = w_n$. Thus $f(x_n) \in T(x_{n-1})$, that is, $\{x_n\}$ is f-iterative sequence of T at $x_0 \in X$ such that for all $n \geq 1$

$$d(f(x_n), T(x_n)) \leq h\, d(f(x_{n-1}), f(x_n))$$

and

$$b\, d(f(x_n), f(x_{n-1})) \leq d(f(x_{n-1}), T(x_{n-1})),$$

and hence for all $n \geq 1$,

$$\begin{aligned} d(f(x_{n+1}), f(x_n)) &\leq \frac{h}{b}\, d(f(x_{n-1}), f(x_n)) \\ &\leq \frac{h^2}{b^2}\, d(f(x_{n-1}), f(x_{n-2})) \\ &\vdots \\ &\leq \frac{h^n}{b^n}\, d(f(x_1), f(x_0)). \end{aligned}$$

Take $a^n = \frac{h^n}{b^n}$ for all $n \geq 1$, then we have $d(f(x_{n+1}), f(x_n)) \leq a^n d(f(x_1), f(x_0))$. Now, for $n, m \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned} d(f(x_m),&f(x_n)) \\ &\leq d(f(x_m), f(x_{m-1})) + d(f(x_{m-1}), f(x_{m-2})) + \cdots + d(f(x_{n+1}), f(x_n)) \\ &\leq a^{m-1} d(f(x_1), f(x_0)) + a^{m-2} d(f(x_1), f(x_0)) + \cdots + a^n d(f(x_1), f(x_0)) \\ &\leq \frac{a^n}{1-a}\, d(f(x_1), f(x_0)). \end{aligned}$$

since $h < b$, so $a^n \to 0$ as $n \to \infty$ and thus $\{f(x_n)\}$ is a Cauchy sequence in X. □

Now, we prove the following result on the existence of coincidence points.

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to C(X)$ be an f-contractive upper semi-continuous map which commutes weakly with continuous map f then $C(f \cap T) \neq \emptyset$.*

Proof. By Lemma 2.1, we get an f-iterative sequence $\{x_n\}$ of T at $x_0 \in X$ such that $\{f(x_n)\}$ is a Cauchy sequence. Since X is complete, so let $f(x_n) \to p \in X$. Since f is continuous, $f(f(x_n)) \to f(p)$. Note that for each $n \geq 1$, $f(x_n) \in T(x_{n-1})$, and thus by the weak commutativity of f and T we get

$$f(f(x_n)) \in f(T(x_{n-1})) \subseteq T(f(x_{n-1})).$$

Since, T is upper semi-continuous, so its graph $G(T)$ is closed and hence $f(p) \in T(p)$. □

Remark 2.3. Theorem 2.2 is an extension of Theorem 1.2. Indeed, if a multivalued map $T : X \to CB(X)$ satisfies the conditions of Theorem 1.2 then, as we observed, T is f-contractive. Also we note that f and T commute weakly. Thus, T satisfies all the conditions of Theorem 2.2, and hence $C(f \cap T) \neq \emptyset$.

Since $J_b^x \subset T(x)$, we obtain another extension of Theorem 1.2 as follows:

Corollary 2.4. *Let X be a complete metric space and let $T : X \to C(X)$ be a multivalued upper semi-continuous map which commutes weakly with a single-valued injective continuous map f on X such that $T(X) \subset f(X)$. If there exists a constant $h \in (0, 1)$ such that for any $x \in X$, $w \in T(x)$, $d(w, Tf^{-1}(w)) \leq h\, d(f(x), w)$, then $C(f \cap T) \neq \emptyset$.*

Without using the concept of Hausdorff metric and without continuity and weakly commute conditions of f and T we prove the following coincidence point result. Here, we consider a real valued function g on X defined by $g(w) = d(w, Tf^{-1}(w))$.

Theorem 2.5. *Let (X, d) be a complete metric space and let $T : X \to C(X)$ be an f-contractive map. Further, if the function g is lower semi-continuous, then $C(f \cap T) \neq \emptyset$.*

Proof. By Lemma 2.1, we obtain an f-iterative sequence $\{x_n\}$ of T at $x_0 \in X$ such that $\{f(x_n)\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $p \in X$ such that $\lim_{n\to\infty} f(x_n) = p$. Now, as in the proof of Lemma 2.1, for all $n \geq 1$

$$b\, d(f(x_n), f(x_{n-1})) \leq d(f(x_{n-1}), T(x_{n-1})),$$

and

$$d(f(x_n), T(x_n)) \leq h\, d(f(x_{n-1}), f(x_n)).$$

thus for all $n \geq 1$,

$$d(f(x_n), T(x_n)) \leq \frac{h}{b}\, d(f(x_{n-1}), T(x_{n-1})),$$

Since $h < b$, we have for all $n \geq 1$

$$d(f(x_n), T(x_n)) \leq d(f(x_{n-1}), T(x_{n-1})),$$

and

$$\begin{aligned} d(f(x_n), T(x_n)) &\leq \frac{h}{b}\, d(f(x_{n-1}), T(x_{n-1})) \\ &\leq \frac{h^2}{b^2}\, d(f(x_{n-2}), T(x_{n-2})) \\ &\vdots \\ &\leq \frac{h^n}{b^n}\, d(f(x_0), T(x_0)). \end{aligned}$$

Put $f(x_n) = w_n \in X$, then we note that $\lim_{n\to\infty} w_n = p$ and the sequence $\{g(w_n)\} = \{d(w_n, Tf^{-1}(w_n))\}$ is decreasing and converges to 0. Since g is lower semi-continuous we have

$$0 \leq g(p) \leq \liminf_{n\to\infty} g(w_n) = 0.$$

Hence, $g(p) = d(p, Tf^{-1}(p)) = 0$. Let $f^{-1}(p) = y \in X$, then $d(f(y), T(y)) = 0$, since $T(y)$ is closed, we get $f(y) \in T(y)$. □

Remark 2.4. If we take $f = I$, the identity map in Theorem 2.5, then we get Theorem 1.2, due to Feng and Liu [**5**] which is an extension of the Nadler's fixed point result.

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