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SOME CONVERGENCE RESULTS FOR THE JUNGCK-MANN AND THE JUNGCK-ISHIKAWA ITERATION PROCESSES IN THE CLASS OF GENERALIZED ZAMFIRESCU OPERATORS

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Abstract. In this paper, we shall establish some strong convergence results for the recently introduced Jungck-Mann iteration process of Singh et al. [**18**] and the newly introduced Jungck-Ishikawa iteration process in the class of non-selfmappings in an arbitrary Banach space. Our results are generalizations and extensions of those of Berinde [**4**], Rhoades [**13, 14**] as well as some other analogous ones in the literature.

1. Introduction

Let $(E, \|\cdot\|)$ be a Banach space and $T : E \to E$ a self-map of E. Suppose that $F_T = \{\, p \in E \mid Tp = p \,\}$ is the set of fixed points of T.

There are several iteration processes for which the fixed points of operators have been approximated over the years by various authors. In the Banach space setting we shall state some of these iteration processes as follows:

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n = 0, 1, \dots, \tag{1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, is called the Mann iteration process (see Mann [**12**]).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned}\right\} \qquad n = 0, 1, \dots, \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$, is called the Ishikawa iteration process (see Ishikawa [**7**]).

The following process is the iteration one introduced by Singh et al [**18**] to establish some stability results: Let $(X, \|.\|)$ be a normed linear space and $S,\ T : Y \to X$ such that $T(Y) \subseteq S(Y)$. Then, for $x_0 \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n T x_n, \qquad n = 0, 1, \dots, \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ is called the *Jungck-Mann* iteration process.

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While the iteration processes (1.2) and (1.3) extend (1.1), both (1.2) and (1.3) are independent.

Berinde [**4**] obtained some strong convergence results in an arbitrary Banach space for the Ishikawa iteration process by employing the following contractive definition:

For a mapping $T : E \to E$, there exist real numbers $\alpha,\ \beta, \gamma$ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$, $0 \leq \gamma < \frac{1}{2}$ respectively such that for each $x, y \in E$, at least one of the following is true:

$$
\left.\begin{array}{ll}
(z_1) & d(Tx, Ty) \leq \alpha d(x, y) \\
(z_2) & d(Tx, Ty) \leq \beta\left[d(x, Tx) + d(y, Ty)\right] \\
(z_3) & d(Tx, Ty) \leq \gamma\left[d(x, Ty) + d(y, Tx)\right].
\end{array}\right\} \tag{1.4}
$$

(1.4) is called the *Zamfirescu contraction condition*. It was employed by Zamfirescu [**19**] to prove some fixed point results. The condition (z_1) of (1.4) is the well known *contraction condition* or *Banach's contraction condition* introduced by Banach [**1**], see also Zeidler [**20**] and several other references. Any mapping satisfying the condition (z_2) of (1.4) is called a *Kannan mapping,* while the mapping satisfying the condition (z_3) is called *Chatterjea operator.* See Chatterjea [**6**] for the detail on the Chatterjea operator. The condition (1.4) was used by Berinde [**4**] to obtain some convergence results.

In the next section, we shall employ both Jungck-Mann and Jungck-Ishikawa iteration processes to extend the results of Berinde [**4**] for non-selfmappings in an arbitrary Banach space. In establishing our results, a more general contractive condition than (1.4) will be considered.

2. Preliminaries

We shall introduce the following iteration processes in establishing our results:

Let $(E, \|\cdot\|)$ be a Banach space and Y an arbitrary set. Let $S,\ T : Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y),\ S(Y)$ is a complete subspace of E and S is injective. Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ iteratively by

$$
\left.\begin{aligned}
Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Ty_n \\
Sy_n &= (1-\beta_n)Sx_n + \beta_n Tx_n
\end{aligned}\right\}, \qquad n = 0, 1, \ldots, \tag{2.1}
$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\mathrm{n}}\}_{\mathrm{n}=0}^{\infty}$ are sequences in $[0, 1]$.

The iteration process (2.1) will be called the *Jungck-Ishikawa* iteration process. If in (2.1), S is the identity operator, $Y = E,\ \beta_n = 0$, then we obtain the Mann iteration process defined in (1.1).
Furthermore, with S being injective, if $\beta_n = 0$, then we get the Jungck-Mann iteration process defined in (1.3).

Indeed, (2.1) reduces to some other interesting iteration processes such as Picard and Jungck iterations amongst others. See [**1, 8, 18, 20**] and some other references for Picard, Jungck and Jungck-type iteration processes.

In addition to the Jungck-Mann and Jungck-Ishikawa iteration processes defined in (1.3) and (2.1) respectively, we shall employ the following contractive definition:

Definition 2.1. For two non-selfmappings $S,\ T : Y \to E$ with $T(Y) \subseteq S(Y)$, there exist real numbers $\alpha,\ \beta, \gamma$ satisfying $0 \leq \alpha < 1,\ 0 \leq \beta < \frac{1}{2},\ 0 \leq \gamma < \frac{1}{2}$ respectively such that for each $x,\ y \in Y$, at least one of the following is true:

$$\left.\begin{array}{ll} (gz_1) & d(Tx, Ty) \leq \alpha d(Sx, Sy) \\ (gz_2) & d(Tx, Ty) \leq \beta\left[d(Sx, Tx) + d(Sy, Ty)\right] \\ (gz_3) & d(Tx, Ty) \leq \gamma\left[d(Sx, Ty) + d(Sy, Tx)\right]. \end{array}\right\} \tag{2.2}$$

The ondition (2.2) will be called the *generalized Zamfirescu contraction* for the pair (S, T).

Moreover, the condition (gz_2) will be called the *generalized Kannan condition* for the pair $(S,\ T)$, while the condition (gz_3) will be called the *generalized chatterjea condition* for the pair $(S,\ T)$. However, the condition (gz_1) is contained in [**8, 18**].

The contractive condition (2.2) is more general than the Zamfirescu contraction defined in (1.4) in the sense that if in (2.2), S is the identity operator and $Y = E$, then we obtain (1.4).

In this paper we shall introduce both the Jungck-Mann and the Jungck-Ishikawa iteration processes defined in (1.3) and (2.1) to establish some strong convergence results for non-selfmappings in an arbitrary Banach space by employing the contractive condition (2.2). Our results are generalizations and extensions of some of the results of Kannan [**9, 10**], Rhoades [**13, 14**] and Berinde [**3, 4**]. We shall employ the concept of coincidence points of two non-selfmappings.

Definition 2.2. Let X and Y be two nonempty sets and $S,\ T : X \to Y$ two mappings. Then an element $x^* \in X$ is a *coincidence point* of S and T if and only if $Sx^* = Tx^*$.

We denote the set of the coincidence points of S and T by $C(S,\ T)$. There are several papers and monographs on the coincidence point theory. However, we refer our readers to Rus [**16**] and Rus et al [**17**] for the Definition 2.2 and some coincidence point results.

3. Main Results

We shall establish the following theorems as our main results.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be an arbitrary Banach space and Y an arbitrary set. Suppose that $S,\ T : Y \to E$ are non-selfoperators such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E, and S an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy the condition* (2.2). *For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration process defined by* (2.1), *where $\{\alpha_n\}_{n=0}^{\infty}$* and *$\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.*

Proof. We shall first establish that the condition (2.2) implies

$$(\star) \qquad \|Tx - Ty\| \leq 2\delta\|Sx - Tx\| + \delta\|Sx - Sy\|, \qquad \forall\, x, y \in Y.$$

Denoting

$$(\star\star) \qquad \delta = \max\left\{\alpha,\ \frac{\beta}{1-\beta},\ \frac{\gamma}{1-\gamma}\right\},$$

then we have $0 \leq \delta < 1$.

By the triangle inequality and the condition (gz_2) of (2.2), we have

$$\|Tx - Ty\| \leq \beta\left[\|Sx - Tx\| + \|Sy - Sx\| + \|Sx - Tx\| + \|Tx - Ty\|\right],$$

from which it follows that

$$(3.1) \qquad \|Tx - Ty\| \leq \frac{2\beta}{1-\beta}\|Sx - Tx\| + \frac{\beta}{1-\beta}\|Sx - Sy\|.$$

Again, using the triangle inequality and the condition (gz_3) of (2.2), we obtain

$$\|Tx - Ty\| \leq \gamma\left[\|Sx - Tx\| + \|Tx - Ty\| + \|Sy - Sx\| + \|Sx - Tx\|\right],$$

from which it follows that

$$(3.2) \qquad \|Tx - Ty\| \leq \frac{2\gamma}{1-\gamma}\|Sx - Tx\| + \frac{\gamma}{1-\gamma}\|Sx - Sy\|.$$

Now by condition $0 \leq \delta < 1$ given in $(\star\star)$ and also with (gz_1), (3.1) and (3.2) we obtain $(\star)$. Hence, the condition (2.2) implies $(\star)$.

Indeed, we shall use the condition $(\star)$ in the rest of the proof.

Let $C(S,\ T)$ be the set of the coincidence points of S and T. We shall now use the condition (3.4) to establish that S and T have a unique coincidence point z (i.e. $Sz = Tz = p$): Injectivity of S is sufficient. Suppose that there exist $z_1,\ z_2 \in C(S,T)$ such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$.

If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$.

If $p_1 \neq p_2$, then we have by the contractiveness condition (2.2) for S and T that

$$\begin{aligned} 0 < \|p_1 - p_2\| = \|Tz_1 - Tz_2\| &\leq 2\delta\|Sz_1 - Tz_1\|) + \delta\|Sz_1 - Sz_2\| \\ &= \delta\|p_1 - p_2\|, \end{aligned}$$

which leads to

$$(1-\delta)\|p_1 - p_2\| \leq 0,$$

from which it follows that $1 - \delta > 0$ since $\delta \in [0,1)$, but $\|p_1 - p_2\| \leq 0$, which is a contradiction since the norm is nonnegative. Therefore, we have $\|p_1 - p_2\| = 0$, that is, $p_1 = p_2 = p$.

Since $p_1 = p_2$, then we have $p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2$, leading to $Sz_1 = Sz_2 \ \Rightarrow\ z_1 = z_2 = z$ (since S is injective).

Hence, $z \in C(S,T)$, that is, z is a unique coincidence point of S and T.

We now prove that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p (where $Sz = Tz = p$) using again, the condition $(\star)$. Therefore, we have

$$\begin{aligned}\|Sx_{n+1} - p\| &= \|(1-\alpha_n)(Sx_n - p) + \alpha_n(Ty_n - p)\| \\ &\leq (1-\alpha_n)\|Sx_n - p\| + \alpha_n\|Tz - Ty_n\| \\ &\leq (1-\alpha_n)\|Sx_n - p\| + \delta\alpha_n\|p - Sy_n\|. \qquad (3.3)\end{aligned}$$

Now, we have that

$$\begin{aligned}\|p - Sy_n\| &\leq (1-\beta_n)\|Sx_n - p\| + \beta_n\|p - Tx_n\| \\ &= (1-\beta_n)\|Sx_n - p\| + \beta_n\|Tz - Tx_n\| \\ &\leq (1-\beta_n+\delta\beta_n)\|Sx_n - p\|. \qquad (3.4)\end{aligned}$$

Using (3.4) in (3.3) yields

$$\begin{aligned}\|Sx_{n+1} - p\| &\leq [1-(1-\delta)\alpha_n - \delta(1-\delta)\alpha_n\beta_n]\|Sx_n - p\| \\ &\leq [1-(1-\delta)\alpha_n]\|Sx_n - p\| \\ &\leq \Pi_{k=0}^{n}[1-(1-\delta)\alpha_k]\|Sx_0 - p\| \\ &\leq \mathrm{e}^{-[(1-\delta)\sum_{k=0}^{n}\alpha_k]}\|Sx_0 - p\| \to 0 \quad \text{as} \quad n \to \infty, \qquad (3.5)\end{aligned}$$

since $\sum_{k=0}^{\infty}\alpha_k = \infty$ and $\delta \in [0,1)$. Hence, from (3.5) we obtain $\|Sx_n - p\| \to 0$ as $n \to \infty$, that is, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. □

Remark 3.2. Theorem 3.1 is a generalization and extension of a multitude of results. In particular, Theorem 3.1 is a generalization and extension of both Theorem 1 and Theorem 2 of Berinde [**4**], Theorem 2 and Theorem 3 of Kannan [**10**], Theorem 3 of Kannan [**11**], Theorem 4 of Rhoades [**13**] as well as Theorem 8 of Rhoades [**14**]. Also, both Theorem 4 of Rhoades [**13**] and Theorem 8 of Rhoades [**14**] are Theorem 4.10 and Theorem 5.6 of Berinde [**3**] respectively.

Theorem 3.3. *Let $(E, \|.\|)$ be an arbitrary Banach space and Y an arbitrary set. Suppose that S, $T : Y \to E$ are non-selfoperators such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E, and S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy the condition (2.2). For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Mann iteration process defined by (1.3), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.*

Proof. The proof of this theorem follows a similar argument as in that of Theorem 3.1. □

Remark 3.4. Theorem 3.3 is a generalization and extension of Theorem 1 of Berinde [**4**], Theorem 2 and Theorem 3 of Kannan [**10**], Theorem 3 of Kannan [**11**] as well as Theorem 4 of Rhoades [**13**].

Remark 3.5. If $S = I$ (identity operator) and $Y = E$ in Theorem 3.1, then the coincidence point z becomes a fixed point of T. If in addition T satisfies the condition (1.4), we have that the Ishikawa iteration process defined in (1.2) converges strongly to the fixed point z. It is also true that if $S = I$ (identity operator), $Y = E$ and that T satisfies condition (1.4) in Theorem 3.3, then the Mann iteration process obtained from (1.1) converges strongly to the fixed point z.

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