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ON THE UNIQUENESS RESULT FOR THE DIRICHLET PROBLEM AND INVEXITY

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Abstract. We provide an existence and uniqueness theorem for the Dirichlet problem

$$\operatorname{div} H_z(y, \nabla x(y)) = \nabla_x F(y, x(y)).$$

The assumption that both H and F are invex with respect to the second variable is imposed and the direct variational method is applied. The application is also shown.

1. Introduction

We show a generalization of results presented in [**2**] to the case of partial differential equations of real-valued functions. We assume that both functions H and F are invex [**1**], instead of one being convex and the other one invex. Thus our uniqueness result applies to much more nonlinear problems since the class of invex functions is broader than the class of convex ones.

Let Ω be a bounded, convex subset of $\mathbb{R}^n$ with regular boundary $\delta\Omega$. We shall consider the existence of solutions to the Dirichlet problem

$$\operatorname{div} H_z(y, \nabla x(y)) = \nabla_x F(y, x(y)), \qquad x_{|\delta\Omega} = 0, \tag{1.1}$$

where $H\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$. We seek for solutions to (1.1) on

$$W_0^{1,p}(\Omega) := \left\{x\colon \Omega \to \mathbb{R}: \quad x \in L^p(\Omega),\ \nabla x \in L^p(\Omega),\ x_{|\delta\Omega} = 0\right\},$$

normed by

$$\|x\|_{W^{1,p}} = \left(\|x\|_p^p + \|\nabla x\|_p^p\right)^{\frac{1}{p}}.$$

The *Sobolev inequality* for $W^{m,p}(\Omega)$ [**3**] states that for fixed $1 \le p \le q$, $k \le m$, $f \in W^{m,p}(\Omega)$ there exists a constant C, independent of Ω and f such that

S1. if $kp < n$, then $\|f\|_{W^{m-k,q}} \le C\|f\|_{W^{m,p}}$, for $p \le q \le \frac{np}{n-kp}$,

S2. if $kp = n$, then $\|f\|_{W^{m-k,q}} \le C\|f\|_{W^{m,p}}$, for $p \le q < +\infty$.

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If $k = m = 1$, $q = p$, then $W^{0,p} = L^p$. In this case the above inequalities are equivalent to the inequality

$$\|f\|_p \leq C\|f\|_{W^{1,p}}, \quad \text{for } p \leq n. \tag{1.2}$$

We assume that

(A1) H is a Caratheodory function, i.e. it is measurable with respect to the first variable and continuous in the second one. It is also Gâteaux differentiable in the second variable. There exist constants $c_i > 0$, functions $d_i \in L^1(\Omega)$, $i = 1, \dots, 4$ such that for all $w \in \mathbb{R}^n$ and almost all $y \in \Omega$

$$c_2\|w\|^p + d_2(y) \leq H(y, w) \leq c_1\|w\|^p + d_1(y), \tag{1.3}$$

$$c_4\|w\|^p + d_4(y) \leq \|H_z(y, w)\|_{W^{1,p}} \leq c_3\|w\|^p + d_3(y). \tag{1.4}$$

(A2) F is a Caratheodory function, Gâteaux differentiable in the second variable. There exist a constant $a < c_2 \left(C^{-p} - 1\right)$ and functions $b_{p-j} \in L^1(\Omega)$, $j = 1, \dots, p$ such that

$$F(y, x) \geq -a\|x\|^p - \sum_{j=1}^{p} b_{p-j}(y)\|x\|^{p-j}, \tag{1.5}$$

for all $x \in \mathbb{R}$ and almost all $y \in \Omega$, where C is the best Sobolev constant in inequality (1.2) [**3**]. Moreover, for any $r > 0$ there exists a function $g_r \in L^1(\Omega)$ such that for all $x \in \mathbb{R}$, $\|x\| \leq r$ and $y \in \Omega$ *a.e.*

$$F(y, x) \leq g_r(y), \tag{1.6}$$

$$\|F_z(y, x)\|_{W^{1,p}} \leq g_r(y). \tag{1.7}$$

(A3) Either H is invex with respect to the second variable for *a.e.* y and F is strictly invex with respect to the second variable for *a.e.* y or H is strictly invex in the second variable for *a.e.* y and F is invex with respect to the second variable for *a.e.* y.

(A4) $\liminf_{n\to\infty} H(y, w_n) \geq H(y, \bar{w})$ for every sequence $w_n \rightharpoonup \bar{w}$ weakly in $W^{1,p}(\Omega)$.

Let us recall that f defined on $\mathbb{R}^n$ is said to be (strictly) invex if there exists an operator $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for all $y_1, y_2 \in \mathbb{R}^n$ the following inequality holds

$$f(y_1) - f(y_2) \geq \eta(y_1, y_2)^T \cdot \nabla f(y_2),$$

(with the strict inequality above). It is well known that every stationary point of (strictly) invex functional f minimizes it on $\mathbb{R}^n$ (uniquely) [**1**].

2. Existence and Uniqueness

We shall prove that (1.1) is a Euler-Lagrange equation for some Gâteaux differentiable, coercive and weakly lower semicontinuous functional J. Therefore as in [**6**], the existence of solutions (1.1) will be guaranteed. Put $J\colon W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$J(x) = \int_\Omega \big[H\left(y, \nabla x(y)\right) + F(y, x(y))\big]\,\mathrm{d}y. \tag{2.1}$$

Lemma 2.1. *The functional J is Gâteaux differentiable and has at each $x \in W_0^{1,p}(\Omega)$ and in each direction $h \in W^{1,\infty}(\Omega)$ the Gâteaux variation*

$$\delta J(x;h) = \int_\Omega \big[H_z(y, \nabla x(y))\nabla h(y) + F_z(y, x(y))h(y)\big]\,\mathrm{d}y.$$

Proof. Let $h \in W^{1,\infty}(\Omega)$. By assumptions **(A1)** and **(A2)** we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}J(x+\varepsilon h)_{|\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega\big[H(y, \nabla x(y)+\varepsilon\nabla h(y)) + F(y, x(y)+\varepsilon h(y))\big]\,\mathrm{d}y_{|\varepsilon=0}\\
&= \int_\Omega\big[H_z(y, \nabla x(y)+\varepsilon h(y))\nabla h(y) + F_z(y, x(y)+\varepsilon h(y))h(y)\big]\,\mathrm{d}y_{|\varepsilon=0}\\
&= \int_\Omega\big[H_z(y, \nabla x(y))\nabla h(y) + F_z(y, x(y))h(y)\big]\,\mathrm{d}y.
\end{aligned}$$

It is clear that the function $h \longmapsto \int_\Omega\big[H_z(y, \nabla x(y))\nabla h(y) + F_z(y, x(y))h(y)\big]\,\mathrm{d}y$ is linear. Applying the Hölder inequality, (1.4) and (1.7) we obtain its continuity. $\square$

By the above Lemma it follows that (1.1) is Euler-Lagrange equation for the functional J given by (2.1).

Lemma 2.2. *The functional J is coercive on $W_0^{1,p}(\Omega)$.*

Proof. By the assumptions **(A1)–(A2)**, inequalities (1.3), (1.5) and (1.2) we have

$$\begin{aligned}
J(x) &\geq c_2\int_\Omega \|\nabla x(y)\|^p\mathrm{d}y + \int_\Omega d_2(y)\mathrm{d}y - a\int_\Omega\|x\|^p\mathrm{d}y - \sum_{j=1}^{p}\int_\Omega b_{p-j}(y)\|x\|^{p-j}\mathrm{d}y\\
&\geq \left[c_2\left(\frac{1}{C^p}-1\right)-a\right]\int_\Omega\|x\|^p\mathrm{d}y + \int_\Omega d_2(y)\mathrm{d}y - \sum_{j=1}^{p}\int_\Omega b_{p-j}(y)\|x\|^{p-j}\mathrm{d}y\\
&= \left[c_2\left(\frac{1}{C^p}-1\right)-a\right]\|x\|_p^p + \int_\Omega d_2(y)\mathrm{d}y - \sum_{j=1}^{p}\int_\Omega b_{p-j}(y)\|x\|^{p-j}\mathrm{d}y.
\end{aligned}$$

Passing $\|x\|_p \to \infty$ we obtain the assertion of the lemma. $\square$

Lemma 2.3. *The functional J is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$.*

Proof. We shall prove the weak lower semicontinuity of the functionals

$$J_1(x) = \int_\Omega H(y, \nabla x(y))\mathrm{d}y \quad\text{and}\quad J_2(x) = \int_\Omega F(y, x(y))\mathrm{d}y.$$

Let (x_n) be weakly convergent in $W_0^{1,p}(\Omega)$ to a certain $\bar{x}$. By the Fàtou Lemma and **(A4)** we have that for almost all $y \in \Omega$

$$\liminf_{n\to\infty} \int_\Omega H(y, \nabla x_n(y))\mathrm{d}y \geq \int_\Omega \liminf_{n\to\infty} H(y, \nabla x_n(y))\mathrm{d}y \geq \int_\Omega H(y, \nabla \bar{x}(y))\mathrm{d}y.$$

To prove weak lower semicontinuity of J_2 we observe that Rellich-Kondrashov Theorem [**3**] provides strong convergence of a sequence (x_n) to $\bar{x}$ on every open and bounded subset of $L^p(\Omega)$. It implies that (x_n) is bounded and there exists a subsequence, still denoted by (x_n), such that $\lim_{n\to\infty} x_n(y) = \bar{x}(y)$ for $y \in \Omega$ *a.e.* Hence by continuity of F in the second variable (see [**4**]), (1.6) and Lebesgue Dominated Convergence Theorem [**3**] we obtain

$$\liminf_{n\to\infty} \int_\Omega F(y, x_n(y))\mathrm{d}y = \liminf_{n\to\infty} \int_\Omega F(y, x_n(y))\mathrm{d}y = \int_\Omega F(y, \bar{x}(y))\mathrm{d}y.$$

Finally, we get

$$\liminf_{n\to\infty} J(x_n) = \liminf_{n\to\infty} J_1(x_n) + \lim_{n\to\infty} J_2(x_n) \geq J_1(\bar{x}) + J_2(\bar{x}) = J(\bar{x}).$$

□

Lemma 2.4. *Let $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$. Assume that both $f(y, x(y))$ and $g(y, z(y))$ are invex in the second variable. If f and/or g is strictly invex with respect to the second variable, then $J = \int_\Omega [f(y, x(y)) + g(y, z(y))]\,\mathrm{d}y$ is strictly invex.*

Proof. Let

$$J_1(x) = \int_\Omega f(y, x(y))\mathrm{d}y \quad \text{and} \quad J_2(z) = \int_\Omega g(y, z(y))\mathrm{d}y.$$

Assume that f is invex and g is strictly invex. We have the following inequalities, for all $x, \bar{x} \in \mathbb{R}$ and $z, \bar{z} \in \mathbb{R}^n$

$$\begin{aligned} J_1(x) - J_1(\bar{x}) &\geq \int_\Omega \eta(x, \bar{x}) \cdot \nabla_x f(y, \bar{x})\mathrm{d}y, \\ J_2(z) - J_2(\bar{z}) &> \int_\Omega \rho(z, \bar{z})^T \cdot \nabla_z g(y, \bar{z})\mathrm{d}y, \end{aligned}$$

for some $\eta\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\rho\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$. It follows that

$$\begin{aligned} J_1(x) + J_2(z) - (J_1(\bar{x}) + J_2(\bar{z})) &> \int_\Omega \left[\eta(x, \bar{x}) \cdot \nabla_x f(y, \bar{x}) + \rho(z, \bar{z})^T \cdot \nabla_z g(y, \bar{z})\right] \mathrm{d}y \\ &= \int_\Omega \langle (\eta(x, \bar{x}), \rho(z, \bar{z})), (\nabla_x f(y, \bar{x}), \nabla_z g(y, \bar{z})\rangle \mathrm{d}y, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}\times\mathbb{R}^n$. The remaining two cases follow in the same manner. □

Now we formulate the main result of the paper.

Theorem 2.5 (Existence)**.** *Assume* **(A1)**, **(A2)** *and* **(A4)**. *Then there exists a solution to* (1.1).

Proof. Since the functional J is weakly lower semicontinuous and coercive on $W_0^{1,p}(\Omega)$ it follows by [**6**, Proposition 1.2] that every $\bar{x}$ for which $J'(\bar{x}) = 0$ minimizes J on $W_0^{1,p}(\Omega)$ and satisfies (1.1). □

Theorem 2.6 (Uniqueness)**.** *Assume* **(A1)–(A4)** *Then there exists a unique solution to* (1.1)*.*

Proof. The existence is given by Theorem 2.5. Uniqueness follows by Lemma 2.4. □

3. Application

Let $1 < r < |\frac{2}{5}|$. Consider the following two-dimensional Dirichlet problem on $W^{1,2}(\Omega)$

$$\begin{aligned}(1 + 2r \sin \|\nabla x\|^2)\triangle x + 4r \cos \|\nabla x\|^2 (x_u^2 x_{uu} + x_v^2 x_{vv}) \\ = 2x + \sin x + x \cos x, \qquad x_{|\delta\Omega} = 0,\end{aligned} \tag{3.1}$$

where $\Omega = \{y = (u, v) \in \mathbb{R}^2 \colon \|y\| \le 1\}$. Here

$$H(y, z) = \frac{1}{2}|z|^2 - r \cos |z|^2 \quad \text{and} \quad F(y, x) = x^2 + x \sin(x).$$

Both $H(y, z)$ and $F(y, x)$ are not convex with respect to z and x, respectively. They are actually strictly invex since each of them has only one stationary point which is the global minimizer [**7**].

We see that the set

$$\{z \in \mathbb{R}^2 \colon H(y, z) \le \alpha\}$$

is convex for any $y \in \Omega$ and $\alpha > -r$. Indeed, H is a radial function with respect to the second variable. Moreover, $f(u) = \frac{1}{2}|u|^2 - r \cos |u|^2$ is even and its derivative is positive on the positive half-line and negative for $u < 0$. Thus f is strictly increasing for $u > 0$, strictly decreasing for $u < 0$ and $f(0) = -r$. We conclude that $\{z \in \mathbb{R}^2 \colon H(y, z) \le \alpha\}$ is convex in $\mathbb{R}^2$.

Assumptions **(A1)–(A4)** are clearly satisfied. The unique solution to (3.1) is thus guaranteed by Theorem 2.6.

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