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TRANSVERSALS OF RECTANGULAR ARRAYS

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Abstract. The paper deals with m by n rectangular arrays whose mn cells are filled with symbols. A section of the array consists of m cells, one from each row and no two from the same column. The paper focuses on the existence of sections that do contain symbols with high multiplicity.

1. Introduction

An n by n array of cells filled with symbols $1, 2, \ldots, n$ such that each symbol appears in each row and each column exactly once is called a Latin square. A section is a set of n cells, one from each row such that no two cells are in the same column. A section is called a transversal if each of its symbols is distinct. H. J. Ryser [**5**] conjectured that every n by n Latin square has a transversal for odd n. P. W. Shor [**6**] proved that an n by n Latin square has a section with

$$n - 5.53(\ln n)^2$$

distinct symbols. S. K. Stein [**7**] showed that if an n by n array is filled with symbols $1, 2, \ldots, n$ such that each symbol appears exactly n times then there is a section with $0.63n$ distinct symbols. P. Erdös and J. H. Spencer [**4**] proved that if an n by n array is filled with symbols such that each symbol appears at most $(n-1)/16$ times, then the array has a transversal. In this paper we will use the Erdös-Spencer technique to show that m by n arrays have sections in which no symbol appears with high multiplicity.

2. The graph G

Consider an m by n table filled with symbols $1, 2, \ldots$ such that each symbol appears at most k times. In order to avoid trivial cases we assume that $2 \leq m \leq n$. For a given value of m and n there is a large number of such tables. We will work with a fixed table. The symbol in the a-th row and the b-th column is denoted by $f(a, b)$. The s cells

$$[x_1, y_1], \ldots, [x_s, y_s]$$

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in the table is called an s-clique if

(1) $x_1, \dots, x_s$ are distinct numbers,
(2) $y_1, \dots, y_s$ are distinct numbers,
(3) $f(x_1, y_1) = \dots = f(x_s, y_s)$.

Again to avoid non-desired cases we assume that $2 \leq s \leq m \leq n$. Let T be the set of all s-cliques in the table. We define a graph G in the following way. Let the elements of T be the vertices of G. Two distinct vertices

$$\{[x_1, y_1], \dots, [x_s, y_s]\} \text{ and } \{[x'_1, y'_1], \dots, [x'_s, y'_s]\}$$

are connected if

$$\{x_1, \dots, x_s\} \cap \{x'_1, \dots, x'_s\} \neq \emptyset$$

or

$$\{y_1, \dots, y_s\} \cap \{y'_1, \dots, y'_s\} \neq \emptyset.$$

Note that the degree of a vertex of G is at most

$$[s(m-s) + s(n-s) + s^2]\binom{k-1}{s-1}.$$

The reason is the following. Choose an s-clique C. Then consider the s rows and s columns of the table that contain a cell from C. These s rows and s columns occupy $s(m-s) + s(n-s) + s^2$ cells of the table. Let us call this the shaded area of the table. Another s-clique C' is connected to C if and only if C' has a cell from the shaded area. There are at most $s(m-s) + s(n-s) + s^2$ choices for such a cell. The common cell contains a symbol. This symbol appears at most k times in the table. So there are at most $\binom{k-1}{s-1}$ choices for the remaining $s-1$ cells of the clique C'.

3. The probability space Ω

Let ω be an injective map from $\{1, \dots, m\}$ to $\{1, \dots, n\}$. The set of cells

$$[i, \omega(i)],\ 1 \leq i \leq m$$

is called a section of the table. Intuitively a section consists of m cells of the table such that no two cells are in the same row and no two cells are in the same column.

Let Ω be the probability space consisting of all sections of the table. Clearly,

$$|\Omega| = n(n-1)\cdots(n-m+1).$$

We assign the same probability to each element of Ω. For an element $\{[x_1, y_1], \dots, \dots, [x_s, y_s]\}$ of T we define $A([x_1, y_1], \dots, [x_s, y_s])$ to be the subset of Ω which contains all ω with $\omega(x_1) = y_1, \dots, \omega(x_s) = y_s$. Intuitively, $A([x_1, y_1], \dots, [x_s, y_s])$ is the set of all sections that contain the cells $[x_1, y_1], \dots, [x_s, y_s]$. For notational convenience we number the elements of T by $1, 2, \dots, \mu$ and identify the elements of T by their numbers. If the vertex $\{[x_1, y_1], \dots, [x_s, y_s]\}$ is numbered by i, then $A([x_1, y_1], \cdots, [x_s, y_s])$ will be denoted by A_i. As an example suppose that

$\{[1,1],\ldots,[s,s]\}$ is a vertex of G and is numbered by 1. The event A_1 consists of all the ω for which

$$\omega(1)=1,\ \omega(2)=2,\ldots,\omega(s)=s.$$

$$\begin{aligned}\Pr[A_1] &= \frac{[n-s][n-s-1]\cdots[n-s-(m-s)+1]}{n(n-1)\cdots(n-m+1)}\\ &= \frac{1}{n(n-1)\cdots(n-s+1)}\\ &= p.\end{aligned}$$

In general $\Pr[A_i]=p$ for all i, $1\le i\le\mu$.

4. The conditional probabilities

The content of this section is the following lemma.

Lemma 1. *Suppose that the vertex* 1 *is not adjacent to any of the vertices* $2,\ldots,t$ *in the graph* G *and that* $\Pr[\overline{A}_2\cdots\overline{A}_t]>0$. *Then* $\Pr[A_1|\overline{A}_2\cdots\overline{A}_t]\le p$.

Proof. By definition

$$\Pr[A_1|\overline{A}_2\cdots\overline{A}_t]=\frac{\Pr[A_1\overline{A}_2\cdots\overline{A}_t]}{\Pr[\overline{A}_2\cdots\overline{A}_t]}.$$

The event $A_1\overline{A}_2\cdots\overline{A}_t$ is the set of all ω for which

$$\omega\in A_1,\ \omega\notin A_2,\ldots,\omega\notin A_t.$$

Intuitively $A_1\overline{A}_2\cdots\overline{A}_t$ is the set of all sections that contain the clique $\{[1,1],\ldots,$ $\ldots,[s,s]\}$ associated with A_1 and do not contain any of the cliques associated with the events $A_2,\ldots,A_t$. Let $S(y_1,\ldots,y_s)$ be the set of all ω with

$$\omega(1)=y_1,\ldots,\omega(s)=y_s,\ \omega\notin A_2,\ldots,\omega\notin A_t.$$

Intuitively $S(y_1,\ldots,y_s)$ is the set of all sections that contain the clique

$$\{[1,y_1],\ldots,[s,y_s]\}$$

and do not contain any of the cliques associated with $A_2,\ldots,A_t$. Clearly, $S(1,\ldots,$ $\ldots,s)=A_1\overline{A}_2\cdots\overline{A}_t$ and the sets $S(y_1,\ldots,y_s)$ form a partition of the set $\overline{A}_2\cdots\overline{A}_t$ as $y_1,\ldots,y_s$ vary over the possible $n(n-1)\cdots(n-s+1)$ values. Next we try to establish that $|S(1,\ldots,s)|\le|S(y_1,\ldots,y_s)|$. If $S(1,\ldots,s)=\emptyset$, then $|S(1,\ldots,s)|\le|S(y_1,\ldots,y_s)|$ holds. So we may assume that $S(1,\ldots,s)\ne\emptyset$. Choose an ω from $S(1,\ldots,s)$. Consider the cells $[1,y_1],\ldots,[s,y_s]$. Then define the sets A, B, C in the following way. Let

$$\begin{aligned}A &= \{y_1,\ldots,y_s\},\\ B &= \{a:\ a\in A,\ a\le s\},\\ C &= \{a:\ a\in A,\ a>s,\ a\in \text{ range of } \omega\}.\end{aligned}$$

Table 1. An illustration in the $s = 8$, $u = 3$, $v = 4$ case.

	i_1		i_2				i_3	i_4			j_1	j_2	j_3	
		y_1		y_3	y_4	y_7			y_2		y_6	y_8	y_5	
	$*$										$\times$			x_1
			$*$									$\times$		x_2
										$\times$				
							$*$						$\times$	x_3
8								$\times$				$\bullet$		
7						$\bullet$	$\times$							
6						$\times$					$\bullet$			
5					$\times$								$\bullet$	
4				$\times$	$\bullet$									
3			$\times$	$\bullet$										
2		$\times$							$\bullet$					
1	$\times$	$\bullet$												
	1	2	3	4	5	6	7	8						

Suppose that C has u elements, say $j_1, \ldots, j_u$. Then $\{1, \ldots, s\} \setminus B$ has at least u elements, say $i_1, \ldots, i_v$. There are $x_1, \ldots, x_u$ such that $\omega(x_1) = j_1, \ldots, \omega(x_u) = j_u$. Clearly, $x_1, \ldots, x_u \geq s + 1$. Define ω^* by

$$\begin{array}{lclclcl} \omega^*(1) & = & y_1 & , \ldots , & \omega^*(s) & = & y_s, \\ \omega^*(x_1) & = & i_1 & , \ldots , & \omega^*(x_u) & = & i_u \end{array}$$

and $\omega^*(x) = \omega(x)$ for all x, $s + 1 \leq x \leq m$, $x \notin \{x_1, \ldots, x_u\}$. Note that $\omega^* \in S(y_1, \ldots, y_s)$. From a given ω^* we can reconstruct ω without any ambiguity. Namely setting

$$\begin{array}{lclclcl} \omega(1) & = & 1 & , \ldots , & \omega(s) & = & s, \\ \omega(x_1) & = & j_1 & , \ldots , & \omega(x_u) & = & j_u \end{array}$$

and $\omega(x) = \omega^*(x)$ for all x, $s + 1 \leq x \leq m$, $x \notin \{x_1, \ldots, x_u\}$. Thus the map $* : S(1, \ldots, s) \to S(y_1, \ldots, y_s)$ defined by $\omega \to \omega^*$ is injective. This gives that $|S(1, \ldots, s)| \leq |S(y_1, \ldots, y_s)|$. Table 1 illustrates our consideration in the $s = 8$, $u = 3$, $v = 4$ special case. The cells $[1, \omega(1)], \ldots, [m, \omega(m)]$ are marked with "$\times$" and the cells $[1, y_1], \ldots, [s, y_s]$ are marked with "$\bullet$".

Now turn back to the probability estimations.

$$\Pr[A_1 \overline{A}_2 \cdots \overline{A}_t] = \frac{|S(1, \ldots, s)|}{|\Omega|}.$$

If $|S(1, \ldots, s)| = 0$, then $\Pr[A_1 | \overline{A}_2 \cdots \overline{A}_t] = 0 \leq p$ and we are done. So we may assume that $|S(1, \ldots, s)| \neq 0$.

$$\begin{aligned} \Pr[\overline{A}_2 \cdots \overline{A}_t] &= \frac{\sum |S(y_1, \ldots, y_s)|}{|\Omega|} \\ &\geq \frac{1}{|\Omega|} [n(n-1) \cdots (n-s+1)] |S(1, \ldots, s)|. \end{aligned}$$

Thus

$$\Pr[A_1|\overline{A}_2\cdots\overline{A}_t] \le \frac{1}{n(n-1)\cdots(n-s+1)} = p.$$

□

5. Applications

We quote a version of the Lovász local lemma. For more details see [**1**].

Lemma 2. *Let $A_1,\dots,A_\mu$ be events in a probability space Ω such that $\Pr[A_1] = \cdots = \Pr[A_\mu] = p$. Let G be a graph on $\{1,\dots,\mu\}$ such that each vertex in G has degree at most d. Suppose that $\Pr[A_i|\overline{A}_{j(1)}\cdots\overline{A}_{j(t)}] \le p$ whenever i is not adjacent to any of the vertices $j(1),\dots,j(t)$. Then $4dp \le 1$ implies $\Pr[\overline{A}_1\cdots\overline{A}_\mu] > 0$.*

Let us turn to the applications.

(a) In the $s = 2$ case $d = 2(m+n-2)(k-1)$, $p = 1/[n(n-1)]$. If $k-1 \le [n(n-1)]/[8(m+n-2)]$, then the $4dp \le 1$ condition holds and the Lovász local lemma guarantees the existence of a transversal. When $m = n$, this reduces to a result similar to that of Erdös and Spencer.

In the remaining part we consider only n by n arrays, that is, we will assume that $m = n$.

(b) In the $s = 3$ case $d = (6n-9)(k-1)(k-2)/2$, $p = 1/[n(n-1)(n-2)]$. If

$$\frac{n(n-1)(n-2)}{2(6n-9)(k-1)(k-2)} \ge 1$$

then the condition $4dp \le 1$ holds and by the Lovász local lemma there is a section in which each symbol appears at most twice. We can say that for large n if each symbol appears at most $0.28n$ times in the table, then there is a section in which no symbol appears more than twice.

We would like to point out that P. J. Cameron and I. M. Wanless [**2**] show that every Latin square of order n contains a section in which no symbol occurs more than twice.

We single out one more special case. In this case each symbol appears at most n times in an n by n table. So the conditions are similar to the conditions of Stein's result described in the introduction.

(c) In the $s = 6$ case $d = (12n-36)(k-1)\cdots(k-5)/120$, $p = 1/[n(n-1)\cdots(n-5)]$. If $k = n$, then the condition $4dp \le 1$ holds and by the Lovász local lemma there is a section in which each symbol appears at most 5 times.

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