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SOME COVERING SPACES AND TYPES OF COMPACTNESS

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Abstract. In this paper we shall study covering spaces such as fully normal spaces, absolutely countably compact, minimal Hausdorff, $\aleph$-space, realcompact, locally paracompact, $w-$compact, maximal compact. Moreover, we give refinements of some theorems rasied in [**1**], also we shall give partial solutions of some open problems raised in [**2**], and [**3**].

In 1996 M. L. Puertas suggested the following question: If every proper subspace (A, τ_A) of the space (X, τ) has a property P, should the original space (X, τ) have the property P? Nowadays, such kind of topological properties are known as *properly hereditary properties*. More precisely, a topological property is called a *properly hereditary property* if every proper subspace has the property, then the whole space has the property. Moreover, if every proper closed (open, F_σ, G_δ, etc.) subspace has the property, then the whole space has the property, we call such a property *properly closed (open,* F_σ, G_δ, etc.) *hereditary*.

F. Arenas in [**3**] studied Puertas's problem and proved that topological properties like *separation axioms* (T_0, T_1, T_2, T_3), *separability, countability axioms*, and *metrizability* are properly hereditary properties. At the end of his paper Arenas [**3**] suggested some open problems. Some of these problems were solved by Al-Bsoul in [**1**] and [**2**]. Also, in [**1**] and [**2**], Al-Bsoul proved that many topological properties are properly hereditary properties. Moreover, Al-Bsoul suggested new open problems concerning this concept.

In this paper some open problems raised in [**2**] and [**3**] will be solved. Moreover, we proved that the following topological properties are properly hereditary properties: *fully normality, absolutely countably compactness*, *locally paracompactness, minimal Hausdorff, realcompactness*, *maximal compactness,* $\aleph$*-space* and ω*-space*. Also, we managed to improve some results in [**1**].

1. Some Types of Compactness

Arenas [**3**] proved that compactness, local compactness are properly hereditary properties. In this section we shall study more types of compactness according to this property. For the next result we need the following definition.

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Definition 1 ([**8**])**.** A space X is called absolutely countably compact if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a finite subset $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$.

Theorem 1. *Being an absolutely countably compact is a properly hereditary property.*

Proof. Suppose that every proper subspace of a space (X,τ) is absolutely countably compact. Let $\mathcal{A} = \{A_s : s \in S\}$ be an open cover of X and D be a dense subset of X. Evidently, if $D = X$ or D is degenerate, then X is absolutely countably compact. Thus, we may assume that $D \neq X$ and D is non-degenerate. Thus, we have two cases:

(i) $X = D \cup \{x_1\}$ for some $x_1 \in X \backslash D$, then D is a dense subset of $Z = X \setminus \{x_1\}$ and hence D has a finite subset K_1 such that $\mathrm{St}\,(K_1,\mathcal{U}) = Z$ where $\mathcal{U} = \{A_s \setminus \{x_1\} : s \in S\}$. Now, if $x_1 \in \mathrm{St}\,(K_1,\mathcal{A})$ then it is done, but if $x_1 \notin \mathrm{St}\,(K_1,\mathcal{A})$, choose $x_2 \in D$ such that $\{x_1, x_2\}$ is not open in X. Now, $D \setminus \{x_2\}$ is dense in the subspace $X \setminus \{x_2\}$. So, $D \setminus \{x_2\}$ has a finite subset K_2 such that $\mathrm{St}\,(K_2,\mathcal{W}) = X \setminus \{x_2\}$ where $\mathcal{W} = \{A_s \setminus \{x_2\} : s \in S\}$. Take $K = K_1 \cup K_2$, then $\mathrm{St}\,(K,\mathcal{A}) = X$.

(ii) $X \neq D \cup \{x\}$ for any $x \in X$. Let x_3 and x_4 be two distinct points in $X \setminus D$, then D is a dense subset to both subspaces $X \setminus \{x_3\}$ and $X \setminus \{x_4\}$, so, D has a finite subset K_3 such that $\mathrm{St}\,(K_3,\mathcal{C}) = X \setminus \{x_3\}$ where $\mathcal{C} = \{A_s \setminus \{x_3\} : s \in S\}$, and a finite subset K_4 such that $\mathrm{St}\,(K_4,\mathcal{L}) = X \setminus \{x_4\}$ where $\mathcal{L} = \{A_s \setminus \{x_4\} : s \in S\}$. Take $E = K_3 \cup K_4$, then $\mathrm{St}\,(E,\mathcal{A}) = X$. □

A subset A of a topological space X is called a zero-set if $A = f^{-1}(0)$ for a continuous function $f : X \to I$, the complement of A is called a cozero-set. Let A be a subset of X, τ-closure of A, $\mathrm{cl}_\tau A$ is the set of all points $x \in X$ such that any cozero-set neighborhood of x intersects A. Now, we define w-compact as follows:

Definition 2 ([**5**])**.** A topological space (X,τ) is called w-compact if any open cover $\{U_\alpha : \alpha \in \Delta\}$ of X contains a finite subfamily $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $X = \mathrm{cl}_\tau(U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n})$.

Theorem 2. *w-compactness is a properly closed hereditary property.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of the topological space (X,τ) and $Y = X \setminus U_{\alpha_0}$ for some $\alpha_0 \in \Delta$, so $\{U_\alpha \cap Y : \alpha \in \Delta\}$ is an open cover of Y and hence it has a finite subfamily $\{U_{\alpha_1} \cap Y,\ U_{\alpha_2} \cap Y,\ \ldots,\ U_{\alpha_n} \cap Y\}$ such that

$$Y = \mathrm{cl}_{\tau'}((U_{\alpha_1} \cap Y) \cup (U_{\alpha_2} \cap Y) \cup \ldots \cup (U_{\alpha_n} \cap Y))$$

where τ' is the topology on Y.

Hence, $\{U_{\alpha_0}, U_{\alpha_1},\ U_{\alpha_2},\ \ldots,\ U_{\alpha_n}\}$ is the required subfamily of $\mathcal{U}$. □

Recall that a compact space X is maximal compact iff every compact subset is closed. The maximal compactness is not a properly open (closed) hereditary property. To see this, consider the following examples:

Example 1. Let $X = \mathbb{R}$ and topologize X as $\tau = \{\mathbb{R}, \phi, \{0\}\}$. Then, every proper open subspace of X is maximal compact, but X is not maximal compact.

Example 2. Every proper closed subspace of the topological space $(\mathbb{R}, \tau_{cof.})$ is maximal compact, but $(\mathbb{R}, \tau_{cof.})$ is not maximal compact.

In the next result, we shall show that maximal compactness is a properly hereditary property for $\mathrm{Card}(X) \geqslant 3$. The condition $\mathrm{Card}(X) \geqslant 3$ is necessary as we shall see in the next example.

Example 3. Let $X = \{1, 2\}$ and $\tau = \{X, \phi, \{1\}\}$. Then every proper subspace of X is maximal compact, but X itself is not maximal compact.

Theorem 3. *If every proper subspace of a topological space X is maximal compact with* $\mathrm{Card}(X) \geqslant 3$*, then X is finite and has the discrete topology.*

Proof. Suppose that every proper subspace of a topological space X is maximal compact. Let $x_1 \in X$, and let $x_2 \in X \setminus \{x_1\}$. So, the subspace $X \setminus \{x_1, x_2\}$ is a compact subset of $X \setminus \{x_2\}$, hence it is closed in $X \setminus \{x_2\}$, let $x_3 \in X \setminus \{x_1, x_2\}$, then the subspace $X \setminus \{x_1, x_3\}$ is a compact subset of $X \setminus \{x_3\}$, thus it is closed in $X \setminus \{x_3\}$ and hence $(X \setminus \overline{X \setminus \{x_1, x_2\}}^X) \cap (X \setminus \overline{X \setminus \{x_1, x_3\}}^X) = \{x_1\}$. Therefore, $\{x_1\}$ is open. $\square$

2. Covering Spaces

Let us study some types of covering properties that are properly hereditary properties. Recall that a topological space (X, τ) is called a $T_{\frac{1}{2}}$-space iff every singleton in X is either open or closed.

Definition 3 ([**4**])**.** A topological space (X, τ) is called fully normal iff every open cover of X has an open star refinement.

Lemma 1 ([**8**])**.** *A barycentric refinement of a barycentric refinement of a cover $\mathcal{U}$ is a star refinement of $\mathcal{U}$.*

Theorem 4. *If every proper subspace of (X, τ) is fully normal $T_{\frac{1}{2}}$-space, then (X, τ) is fully normal.*

Proof. If X has the discrete topology, then it is fully normal. Assume that τ is not the discrete topology. Let $\mathcal{A} = \{A_s : s \in S\}$ be any open cover of X and let $x_1 \in X$ be such that $\{x_1\}$ is closed in X.

Let $Y = X \setminus \{x_1\}$, so $\{A_s \setminus \{x_1\} : s \in S\}$ is an open cover of Y, hence it has an open barycentric refinement, say, $\mathcal{B} = \{B_t : t \in T\}$. Let $x_2 \in X$ be such that $\{x_2\}$ is closed in X, if there is no such x_2, we are done.

Let $Z = X \setminus \{x_2\}$, so $\{A_s \setminus \{x_2\} : s \in S\}$ is an open cover of Z, hence it has an open barycentric refinement $\mathcal{D} = \{D_\alpha : \alpha \in \triangle\}$. Now, for all $x \in X \setminus \{x_1, x_2\}$, there exist $B_{t_x} \in \mathcal{B}$ and $D_{\alpha_x} \in \mathcal{D}$ such that x belongs to both B_{t_x} and D_{α_x}. Also, there exist $B_{t_{x_2}} \in \mathcal{B}$, and $D_{\alpha_{x_1}} \in \mathcal{D}$ that contain x_2, x_1, respectively. Let $F = B_{t_{x_2}} \cup D_{\alpha_{x_1}}$, so $\mathcal{F} = \left\{B_{t_{x_2}},\ D_{\alpha_{x_1}}\right\}$ is an open cover of F, hence it has an open barycentric refinement, say $\mathcal{G}$, and $\mathrm{St}\,(x_1, \mathcal{G}) \cap \mathrm{St}\,(x_2, \mathcal{G}) = \phi$.

It is easy to see that the family

$$\mathcal{C} = \{B_{t_x} \cap D_{\alpha_x} : x \in X \setminus \{x_1, x_2\}\} \cup \{\mathrm{St}\,(x_1, \mathcal{G}),\ \mathrm{St}\,(x_2, \mathcal{G})\}$$

is an open barycentric refinement of $\mathcal{A}$.

Now, since $\mathcal{C}$ is an open cover of X, so it has an open barycentric refinement $\mathcal{W}$. Therefore, $\mathcal{W}$ is an open star refinement of $\mathcal{A}$. □

Now, [**2**, Theorem 3.1] becomes an easy consequence.

Corollary 1. *If every proper subspace of (X, τ) is paracompact, then (X, τ) is paracompact.*

Definition 4 ([**5**])**.** A cover $\mathcal{A}$ of a space X is called a k-network for X, if for any open set U in X and any compact subset K of U, there exists a finite subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $K \subseteq \cup\mathcal{B} \subseteq U$.

Definition 5 ([**5**])**.** A regular space X is called $\aleph$-space if it has a σ-locally finite k-network.

Theorem 5. *$\aleph$-space is a properly hereditary property.*

Proof. Suppose that every proper subspace of a space X is an $\aleph$-space. Let $\mathcal{A}$ be an open cover of X. If X has the indiscrete topology, then X is an $\aleph$-space.

If X does not have the indiscrete topology, then there exist two nonempty disjoint open subsets U and V of X. Let $Y = X \setminus U$, $Z = X \setminus V$, so there exist two σ-locally finite k-networks $\mathcal{C} = \cup_{i=1}^{\infty}\mathcal{C}_{2i-1}$and $\mathcal{D} = \cup_{j=1}^{\infty}\mathcal{D}_{2j}$ of Y and Z, respectively.

It is not hard to see that $\mathcal{B} = \cup_{n=1}^{\infty}\mathcal{B}_n$ is a σ-locally finite k-network of X where

$$\mathcal{B}_n = \begin{cases} \mathcal{C}_n & \text{if } n \text{ is odd} \\ \mathcal{D}_n & \text{if } n \text{ is even} \end{cases}$$

□

Definition 6 ([**7**])**.** A topological space (X, τ) is called locally paracompact iff for every $x \in X$ there exists an open neighborhood U_x of x such that $\overline{U_x}$ is paracompact.

Theorem 6. *If every proper subspace of a topological space X is locally paracompact, then X is locally paracompact.*

Proof. If X is finite, then X is paracompact. So, we may assume that X is infinite. Let $x_1 \in X$ and let x_2 be such that $x_1 \neq x_2$. So, there exists an open neighborhood V_{x_1} of x_1 in the subspace $Y = X \setminus \{x_2\}$ such that $\overline{V_{x_1}}^Y$ is paracompact, we have two cases:

(I) V_{x_1} is open in X. If $x_2 \notin \overline{V_{x_1}}^X$, then $\overline{V_{x_1}}^X$ is paracompact. So, we assume that $x_2 \in \overline{V_{x_1}}^X$, hence we have two subcases to be considered:

(1) $\overline{V_{x_1}}^X \neq X$, let $x_3 \in X \setminus \overline{V_{x_1}}^X$, so x_1 has an open neighborhood U_{x_1} in the subspace $Z = X \setminus \{x_3\}$ such that $\overline{U_{x_1}}^Z$ is paracompact, so $V_{x_1} \cap U_{x_1}$ is an open neighborhood of x_1 in X. Now, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of the subspace $\overline{V_{x_1} \cap U_{x_1}}^X$, so for all $\alpha \in \Delta$ there exists an open set F_α in $\overline{U_{x_1}}^Z$ such that $U_\alpha = F_\alpha \cap \overline{V_{x_1} \cap U_{x_1}}^X$. Thus, the family $\{F_\alpha : \alpha \in \Delta\} \cup \left\{\overline{U_{x_1}}^Z \setminus \overline{V_{x_1} \cap U_{x_1}}^X\right\}$ is an open cover of the subspace $\overline{U_{x_1}}^Z$ and hence it has a locally finite open refinement, say $\{V_\gamma : \gamma \in \Gamma\}$. Now, the family $\left\{V_\gamma \cap (\overline{V_{x_1} \cap U_{x_1}}^X) : \gamma \in \Gamma\right\}$ is a locally finite open refinement of $\mathcal{U}$.

(2) $\overline{V_{x_1}}^X = X$. We have two cases:

(a) If $V_{x_1} = X \setminus \{x_2\}$, then X is paracompact.

(b) $V_{x_1} \neq X \setminus \{x_2\}$. If there exists $x_4 \in X \setminus (V_{x_1} \cup \{x_2\})$ such that x_1 has a neighborhood, say U_{x_1}, in $X \setminus \{x_4\}$ with $\overline{U_{x_1}}^{X\setminus\{x_4\}}$ is paracompact and $\overline{U_{x_1}}^{X\setminus\{x_4\}} \neq X \setminus \{x_4\}$, so $V_{x_1} \cap U_{x_1}$ is open in X and $\overline{V_{x_1} \cap U_{x_1}}^X \neq X$. Let $p \in X \setminus \overline{V_{x_1} \cap U_{x_1}}^X$, then x_1 has a neighborhood, say F_{x_1}, in $X \setminus \{p\}$ with $\overline{F_{x_1}}^{X\setminus\{p\}}$ is paracompact. So, by a similar way as in case (I.1), $\overline{(V_{x_1} \cap U_{x_1}) \cap F_{x_1}}^X$ is paracompact. Otherwise, $X \setminus \{x_4\}$ is paracompact and since $X \setminus \{x_2\}$ is paracompact, then $\{x\}$ is closed for all $x \in X \setminus \{x_2, x_4\}$. Let $\{x_5, x_6\} \subseteq X \setminus \{x_1, x_2, x_4\}$ and let $\mathcal{A} = \{A_s : s \in S\}$ be an open cover of X, then $\{A_s \setminus \{x_5\} : s \in S\}$ (respectively, $\{A_s \setminus \{x_6\} : s \in S\}$) is an open cover of $X \setminus \{x_5\}$ (respectively, $X \setminus \{x_6\}$) and hence it has a locally finite open refinement, say $\{J_l : l \in L\}$ (respectively, $\{I_k : k \in K\}$). Hence the family

$$\{J_{l_x} \cap I_{k_x} : x \in X \setminus \{x_5, x_6\}\} \cup \left\{J_{l_{x_6}}, I_{k_{x_5}}\right\}$$

is a locally finite open refinement of $\mathcal{A}$.

(II) V_{x_1} is not open in X. If there exists $x_0 \in X \setminus \{x_1\}$ such that x_1 has a neighborhood in $X \setminus \{x_0\}$, say W_{x_1}, such that $\overline{W_{x_1}}^{X\setminus\{x_0\}}$ is paracompact and it is open in X, then we have Case (I). Otherwise, for all $x \in X \setminus \{x_1\}$, we have $\{x\}$ is not closed and hence $X \setminus \{x\}$ is not paracompact. Now, let $y \in X \setminus \{x_1\}$, so x_1 has an open neighborhood in $X \setminus \{y\}$, say B_{x_1}, such that $\overline{B_{x_1}}^{X\setminus\{y\}}$ is paracompact, let $z \in X \setminus (\overline{B_{x_1}}^{X\setminus\{y\}} \cup \{y\})$, so x_1 has an open neighborhood in $X \setminus \{z\}$, say D_{x_1}, such that $\overline{D_{x_1}}^{X\setminus\{z\}}$ is paracompact. Thus $D_{x_1} \cap (B_{x_1} \cup \{y\})$ is open in X and $\overline{D_{x_1} \cap (B_{x_1} \cup \{y\})}^X \neq X$. Let $q \in X \setminus \overline{D_{x_1} \cap (B_{x_1} \cup \{y\})}^X$, then x_1 has a neighborhood, say H_{x_1}, in $X \setminus \{q\}$ with $\overline{H_{x_1}}^{X\setminus\{q\}}$ which is paracompact. So, by a similar way as in Case (I.1), $\overline{(D_{x_1} \cap (B_{x_1} \cup \{y\})) \cap H_{x_1}}^X$ is paracompact.

□

3. More Topological Properties

In this section, we shall study the topological properties: minimal Hausdorff, realcompact, extremely disconnected and improve a result about δ-*normal* spaces. Recall that a Hausdorff space X is called minimal Hausdorff if every one-to-one continuous map of X to a Hausdorff space Y is a homeomorphism.

Theorem 7. *Being a minimal Hausdorff is a properly hereditary property.*

Proof. Suppose that every proper subspace of X is minimal Hausdorff. Since every proper subspace of X is Hausdorff, then X is a Hausdorff space. Let f be a one to one continuous map of X to a Hausdorff space Y and let $x_1 \in X$. So, $g : X \setminus \{x_1\} \to Y \setminus \{f(x_1)\}$ is one to one and continuous where $g = f$ on $X \setminus \{x_1\}$ and hence g is a homeomorphism. Now, since $f(X) = f(\{x_1\}) \cup f(X \setminus \{x_1\}) = f(\{x_1\}) \cup g(X \setminus \{x_1\}) = f(\{x_1\}) \cup Y \setminus \{f(x_1)\} = Y$, so f is onto.

Let U be a proper open subset of X. If $x_1 \notin U$, then $f(U) = g(U)$, and since g is a homeomorphism, we have $g(U)$ open in $Y \setminus \{f(x_1)\}$, so it is open in Y. If $x_1 \in U$, let $x_2 \in X \setminus U$, so $h : X \setminus \{x_2\} \to Y \setminus \{f(x_2)\}$ is one-to-one and continuous where $h = f$ on $X \setminus \{x_2\}$ and hence h is a homeomorphism that implies $f(U) = h(U)$ is open in $Y \setminus h(\{x_2\})$ and so in Y. □

To study the next result we need the following definition.

Definition 7 ([4])**.** A Tychonoff space X is called realcompact if there is no Tychonoff space Y which satisfies the following conditions:

(1) There exists a homeomorphism embedding $r : X \to Y$ such that $r(X) \neq \mathrm{Cl}(r(X))$ and $\mathrm{Cl}(r(X)) = Y$.
(2) For every continuous real valued function $f : X \to \mathbb{R}$, there exists a continuous function $g : Y \to \mathbb{R}$ such that $g \circ r = f$.

Theorem 8. *If every proper subspace of a disconnected space X is realcompact, then so is X.*

Proof. Since X is a disconnected space, so there exist two nonempty disjoint open subsets of X, say U and V, such that $U \cup V = X$. Let F be a closed subset of X and $x_1 \notin F$, so x_1 belongs to U or V, say U.

If $U \cap F = \phi$, then the function $h : X \to I$ where

$$h(x) = \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{if } x \in U \end{cases}$$

is continuous with $h(x_1) = 0$ and $h(F) = 1$.

If $U \cap F \neq \phi$, then there exists a continuous function $h_U : U \to I$ such that $h_U(x_1) = 0$, $h_U(U \cap F) = 1$ and hence the function $h : X \to I$ where

$$h(x) = \begin{cases} 1 & \text{if } x \in V \\ h_U(x) & \text{if } x \in U \end{cases}$$

is continuous with $h(x_1) = 0$ and $h(F) = 1$. Thus, X is Tychonoff.

Now, suppose that there exists a Tychonoff space Y satisfying conditions (1) and (2) in Definition 7, so there exists a homeomorphic embedding $r : X \to Y$ such that $r(X) \neq \overline{r(X)}^Y = Y$ and hence $r(U) \neq \overline{r(U)}^Y$ or $r(V) \neq \overline{r(V)}^Y$, say $r(U) \neq \overline{r(U)}^Y$, let $W = \overline{r(U)}^Y$, so the function $r\,|_U\colon U \to W$ is a homeomorphic embedding such that $r(U) \neq \overline{r(U)}^W = W$.

Now, let $h : U \to \mathbb{R}$ be a continuous real valued function, thus the function

$$f(x) = \begin{cases} h(x) & \text{if } x \in U \\ 0 & \text{if } x \in V \end{cases}$$

is a continuous real valued function from X to $\mathbb{R}$, so there exists a continuous function $g : Y \to \mathbb{R}$ such that $g \circ r = f$, so $g\,|_W\colon W \to \mathbb{R}$ is continuous and $(g\,|_W) \circ (r\,|_U) = h$, which is a contradiction. $\square$

In the next result we managed to improve [**1**, Theorem 3.5] by removing the condition T_1. For this we need the following definition

Definition 8 ([**8**])**.** A topological space X is called a δ-normal space if whenever A and B are disjoint closed subsets of X, there exist two disjoint G_δ-sets H and K such that $A \subseteq H$ and $B \subseteq K$.

Theorem 9. *If every proper subspace of X is δ-normal, then X is δ-normal provided that* $\mathrm{Card}(X) \neq 3$.

Proof. It is clear that X is δ-normal whenever $\mathrm{Card}(X) \leq 2$. For $\mathrm{Card}(X) \geq 4$, let A and B be two disjoint nonempty closed subsets of X. If $X = A \cup B$, the proof is completed. So, we assume that $X \neq A \cup B$. Let $x_1 \in X \setminus (A \cup B)$, so there exist two disjoint G_δ-sets $H_1 = \cap_{i=1}^{\infty} H_i^1$ and $K_1 = \cap_{i=1}^{\infty} K_i^1$ in $Y = X \setminus \{x_1\}$ such that $A \subseteq H_1$ and $B \subseteq K_1$, where H_i^1 and K_i^1 are open in Y for all $i \in \mathbb{N}$. Hence, there exist open sets U_i^1 and V_i^1 in X such that $H_i^1 = U_i^1 \cap Y$ and $K_i^1 = V_i^1 \cap Y$ for all $i \in \mathbb{N}$. Thus, $U^1 = \cap_{i=1}^{\infty} U_i^1$ and $V^1 = \cap_{i=1}^{\infty} V_i^1$ are G_δ-sets in X. Now, if $U^1 \cap V^1 = \phi$, the proof is completed. If $U^1 \cap V^1 \neq \phi$, then $U^1 \cap V^1 = \{x_1\}$, we have two cases:

(a) There exists $x_2 \in X \setminus (A \cup B \cup \{x_1\})$. So, there exist two disjoint $G_\delta-$sets $H_2 = \cap_{i=1}^{\infty} H_i^2$ and $K_2 = \cap_{i=1}^{\infty} K_i^2$ in $Z = X \setminus \{x_2\}$ such that $A \subseteq H_2$ and $B \subseteq K_2$, where H_i^2 and K_i^2 are open in Z for all $i \in \mathbb{N}$. Hence, there exist open sets U_i^2 and V_i^2 in X such that $H_i^2 = U_i^2 \cap Z$ and $K_i^2 = V_i^2 \cap Z$ for all $i \in \mathbb{N}$. Thus, $U^2 = \cap_{i=1}^{\infty} U_i^2$ and $V^2 = \cap_{i=1}^{\infty} V_i^2$ are G_δ-sets in X. Then $U = \cap_{i=1}^{\infty} U_i$ and $V = \cap_{i=1}^{\infty} V_i$ are two disjoint G_δ-sets in X such that $A \subseteq U$ and $B \subseteq V$, where $U_i = U_i^1 \cap U_i^2$ and $V_i = V_i^1 \cap V_i^2$.

(b) $X = A \cup B \cup \{x_1\}$. So $\mathrm{Card}(A)$ or $\mathrm{Card}(B)$ is not equal to 1, say $\mathrm{Card}(A) \neq 1$. If $x \in A$ there exist two disjoint G_δ-subsets $H_x = \cap_{i=1}^{\infty} H_i^x$ and $K_x = \cap_{i=1}^{\infty} K_i^x$ in $W_x = X \setminus \{x\}$ such that $A \setminus \{x\} \subseteq H_x$ and $B \subseteq K_x$, where H_i^x and K_i^x are open in W_x for all $i \in \mathbb{N}$. Hence, there exist open sets U_i^x and V_i^x in X such that $H_i^x = U_i^x \cap W_x$ and $K_i^x = V_i^x \cap W_x$ for all $i \in \mathbb{N}$. Thus, $U^x = \cap_{i=1}^{\infty} U_i^x$ and $V^x = \cap_{i=1}^{\infty} V_i^x$ are G_δ-sets in X. If there exists $x_3 \in A$ such that $x_1 \in U^{x_3}$, then $\cap_{i=1}^{\infty}(V_i^{x_3} \cap V_i^1)$ and U^1 are two disjoint

G_δ-sets of X such that $A \subseteq U^1$ and $B \subseteq \cap_{i=1}^{\infty}(V_i^{x_3} \cap V_i^1)$. Otherwise, there exist $U_{i_0}^{x_4}$ and $U_{i_1}^{x_5}$ such that $x_1 \notin \left\{U_{i_0}^{x_4} \cup U_{i_1}^{x_5}\right\}$ for some $\{x_4, x_5\} \subseteq A$ and $\{i_0, i_1\} \subseteq \mathbb{N}$ and hence V^1 and $\cap_{i=1}^{\infty}(U_i^1 \cap \left\{U_{i_0}^{x_4} \cup U_{i_1}^{x_5}\right\})$ are two disjoint $G_\delta-$sets of X such that $A \subseteq \cap_{i=1}^{\infty}(U_i^1 \cap \left\{U_{i_0}^{x_4} \cup U_{i_1}^{x_5}\right\})$ and $B \subseteq V^1$.

□

The following example shows that Theorem 9 is not true if the cardinality of X is 3.

Example 4. Let $X = \{x, y, z\}$ be topologized as follows:

$$\tau = \{\phi, \{x\}, \{x, y\}, \{x, z\}, X\},$$

so every proper subspace of X is δ-normal, but X is not δ-normal.

We improve [**1**, Theorem 3.8] for any space with cardinality greater than 3 as follows.

Theorem 10. *If every proper subspace of a topological space X is an extremely disconnected space, then X is extremely disconnected provided that* $\mathrm{Card}(X) \geqslant 4$.

Proof. Let U be a nonempty proper open set in X, we have two cases:

(I) If $\mathrm{Card}(X \setminus U) = 1$, then $\overline{U}^X$ is either U or X.

(II) If $\mathrm{Card}(X \setminus U) \neq 1$, so let x_1, x_2 be two distinct points in $X \setminus U$ and let $Y = X \setminus \{x_1\}$, $Z = X \setminus \{x_2\}$. So, $\overline{U}^Y = V_1$ (respectively, $\overline{U}^Z = V_2$) is open in Y (respectively, Z), then $\overline{U}^X = V_1 \cup V_2$. Now, it is easy to prove that $V_1 \cup V_2$ is open in X. Therefore, X is extremely disconnected.

□

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