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SOME FAMILIES OF p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We introduce two subclasses $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$ $(0 \le \alpha < p-j+1,\ 1 \le j \le p,\ p \in N = \{1,2,\dots\})$ of p-valent starlike and p-valent convex functions with negative coefficents. In this paper we obtain coefficient inequalities, distortion theorems, extreme points and integral operators for functions belonging to the classes $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$. We also determine the radii of close-to-convexity and convexity for the functions belonging to the class $T^*(p,\alpha,j)$. Also we obtain several results for the modified Hadamard products of functions belonging to the classes $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in N = \{1,2,\dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p-valent starlike of order α $(0 \le \alpha < p)$ if $f(z)$ satisfies the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U) \tag{1.2}$$

and

$$\int_0^{2\Pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \mathrm{d}\theta = 2p\pi \qquad (z \in U). \tag{1.3}$$

We denote by $S(p,\alpha)$ the class of p-valent starlike functions of α. Also a function $f(z) \in A(p)$ is called p-valent convex of order α $(0 \le \alpha < p)$ if $f(z)$ satisfies the following conditions

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U) \tag{1.4}$$

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and

$$\int_0^{2\Pi} \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\mathrm{d}\theta = 2p\pi \qquad (z\in U). \tag{1.5}$$

We denote by $K(p,\alpha)$ the class of p-valent convex functions of order α. We note that

$$f(z)\in K(p,\alpha) \qquad \text{if and only if} \qquad \frac{zf'(z)}{p}\in S(p,\alpha) \qquad (0\le\alpha<p). \tag{1.6}$$

The class $S(p,\alpha)$ was introduced by Patil and Thakare [**3**] and the class $K(p,\alpha)$ was introduced by Owa [**1**].

For $0\le\alpha<p-j+1$, $1\le j\le p$ and $p\in N$, we say $f(z)\in A(p)$ is in the class $S(p,\alpha,j)$ if it satisfies the following inequality:

$$\mathrm{Re}\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\}>\alpha \qquad (z\in U). \tag{1.7}$$

Also for $0\le\alpha<p-j+1$, $1\le j\le p$ and $p\in N$, we say $f(z)\in A(p)$ is in the class $K(p,\alpha,j)$ if it satisfies the following inequality:

$$\mathrm{Re}\left\{1+\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\}>\alpha \qquad (z\in U). \tag{1.8}$$

It follows from (1.7) and (1.8) that:

$$f(z)\in K(p,\alpha,j) \qquad \text{if and only if} \qquad \frac{zf^{(j)}(z)}{p-j+1}\in S(p,\alpha,j). \tag{1.9}$$

The classes $S(p,\alpha,j)$ and $K(p,\alpha,j)$ were studied by Srivastava et al. [**6**] (see also Nunokawa [**2**]). We note that $S(p,\alpha,1)=S(p,\alpha)$ and $K(p,\alpha,1)=K(p,\alpha)$.

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form:

$$f(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k \qquad (a_k\ge 0;\ p\in N). \tag{1.10}$$

We denote by $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$ the classes obtained by taking intersections, respectively, of the classes $S(p,\alpha,j)$ and $K(p,\alpha,j)$ with $T(p)$, that is

$$T^*(p,\alpha,j)=S(p,\alpha,j)\cap T(p)$$

and

$$C(p,\alpha,j)=K(p,\alpha,j)\cap T(p).$$

We note that:

(i) $T^*(p,\alpha,1)=T^*(p,\alpha)$ and $C(p,\alpha,1)=C(p,\alpha)$ (Owa [**1**]);
(ii) $T^*(1,\alpha,1)=T^*(\alpha)$ and $C(1,\alpha,1)=C(\alpha)$ (Silverman [**5**]).

In this paper we obtain coefficient inequalities, distortion theorems, extreme points and integral operators for functions belonging to the classes $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$. We also determine the radii of close-to-convexity and convexity for the functions belonging to the class $T^*(p,\alpha,j)$. Also we obtain several results for the modified Hadamard products of functions belonging to the classes $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$.

2. Coefficient Estimates

Theorem 1. *Let the function* $f(z)$ *be defined by* (1.10). *Then* $f(z)\in T^*(p,\alpha,j)$ *if and only if*

$$\sum_{k=p+1}^{\infty}\frac{\delta(k,j-1)}{\delta(p,j-1)}(k-j+1-\alpha)a_k\leq(p-j+1-\alpha), \tag{2.1}$$

where

$$\delta(p,j)=\frac{p!}{(p-j)!}=\begin{cases} p(p-1)\ldots\ldots(p-j+1) & (j\neq 0),\\ 1 & (j=0).\end{cases} \tag{2.2}$$

Proof. Assume that the inequality (2.1) holds true. Then we obtain

$$\begin{aligned}\left|\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-(p-j+1)\right| &= \left|\frac{\sum_{k=p+1}^{\infty}\frac{k!(k-p)}{(k-j+1)!}a_kz^{k-p}}{\frac{p!}{(p-j+1)!}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-j+1)!}a_kz^{k-p}}\right|\\ &\leq\frac{\sum_{k=p+1}^{\infty}\frac{k!(k-p)}{(k-j+1)!}a_k}{\frac{p!}{(p-j+1)!}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-j+1)!}a_k}\\ &\leq p-j+1-\alpha.\end{aligned} \tag{2.3}$$

This shows that the values of $\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}$ lie in a circle which is centered at $w=(p-j+1)$ and whose radius is $p-j+1-\alpha$. Hence $f(z)$ satisfies the condition (1.7).

Conversely, assume that the function $f(z)$ defined by (1.10) is in the class $T^*(p,\alpha,j)$. Then we have

$$\begin{aligned}&\operatorname{Re}\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\}\\ &=\operatorname{Re}\left\{\frac{\frac{p!}{(p-j)!}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-j)!}a_kz^{k-p}}{\frac{p!}{(p-j+1)!}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-j+1)!}a_kz^{k-p}}\right\}>\alpha\end{aligned} \tag{2.4}$$

for $0\leq\alpha<p-j+1$, $1\leq j\leq p$, $p\in N$ and $z\in U$. Choose values of z on the real axis so that $\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}$ is real. Upon clearing the denominator in (2.4) and letting

$z \to 1^-$ through real values, we can see that

$$\frac{p!}{(p-j)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j)!} a_k \geq \alpha \left(\frac{p!}{(p-j+1)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j+1)!} a_k \right). \tag{2.5}$$

Thus we have the required inequality (2.1). □

Corollary 1. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p,\alpha,j)$. Then we have*

$$a_k \leq \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} \qquad (k \geq p+1;\ p \in N). \tag{2.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} z^k \qquad (k \geq p+1\,;\ p \in N). \tag{2.7}$$

Theorem 2. *Let the function $f(z)$ be defined by (1.10). Then $f(z) \in C(p,\alpha,j)$ if and only if*

$$\sum_{k=p+1}^{\infty} \frac{\delta(k,j)}{\delta(p,j)} (k-j+1-\alpha) a_k \leq (p-j+1-\alpha). \tag{2.8}$$

Proof. Since $f(z) \in C(p,\alpha,j)$ if and only if $\frac{zf^{(j)}(z)}{p-j+1} \in T^*\ (p,\alpha,j)$, we have the theorem by replacing a_k with $\left(\frac{k-j+1}{p-j+1}\right) a_k \quad (k \geq p+1)$ in Theorem 1. □

Corollary 2. *.Let the function $f(z)$ defined by (1.10) be in the class $C(p,\alpha,j)$. Then we have*

$$a_k \leq \frac{\delta(p,j)(p-j+1-\alpha)}{\delta(k,j)(k-j+1-\alpha)} \qquad (k \geq p+1;\ p \in N). \tag{2.9}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\delta(p,j)(p-j+1-\alpha)}{\delta(k,j)(k-j+1-\alpha)} z^k \qquad (k \geq p+1;\ p \in N). \tag{2.10}$$

3. Extreme points

From Theorem 1 and Theorem 2, we see that both $T^*(p,\alpha,j)$ and $C(p,\alpha,j)$ are closed under convex linear combinations, which enables us to determine the extreme points for these classes.

Theorem 3. *Let*

$$f_p(z) = z^p \tag{3.1}$$

and

$$f_k(z) = z^p - \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} z^k \qquad (k \geq p+1;\ p \in N). \tag{3.2}$$

Then $f(z) \in T^*(p,\alpha,j)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z), \tag{3.3}$$

where $\lambda_k \geq 0$ $(k \geq p)$ *and* $\sum_{k=p}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} \lambda_k z^k. \tag{3.4}$$

Then it follows that

$$\begin{aligned} &\sum_{k=p+1}^{\infty} \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)} \cdot \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} \lambda_k \\ &= \sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_p \leq 1. \end{aligned} \tag{3.5}$$

Therefore, by Theorem 1, $f(z) \in T^*(p,\alpha,j)$.

Conversely, assume that the function $f(z)$ defined by (1.10) belongs to the class $T^*(p,\alpha,j)$. Then

$$a_k \leq \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} \qquad (k \geq p+1; \quad \in N). \tag{3.6}$$

Setting

$$\lambda_k = \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)} a_k \qquad (k \geq p+1; \quad p \in N) \tag{3.7}$$

and

$$\lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k, \tag{3.8}$$

we see that $f(z)$ can be expressed in the form (3.3). This completes the proof of Theorem 3. □

Corollary 3. *The extreme points of the class* $T^*(p,\alpha,j)$ *are the functions* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{\delta(p,j-1)(p-j+1-\alpha)}{\delta(k,j-1)(k-j+1-\alpha)} z^k \qquad (k \geq p+1; p \in N).$$

Similarly, we have

Theorem 4. *Let*

$$f_p(z) = z^p \tag{3.9}$$

and

$$f_k(z) = z^p - \frac{\delta(p,j)(p-j+1-\alpha)}{\delta(k,j)(k-j+1-\alpha)} z^k \qquad (k \geq p+1; \quad p \in N). \tag{3.10}$$

Then $f(z) \in C(p,\alpha,j)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z), \tag{3.11}$$

where $\lambda_k \geq 0\,(k \geq p)$ and $\sum\limits_{k=p}^{\infty} \lambda_k = 1$.

Corollary 4. *The extreme points of the class $C(p,\alpha,j)$ are the functions $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{\delta(p,j)(p-j+1-\alpha)}{\delta(k,j)(k-j+1-\alpha)} z^k \qquad (k \geq p+1; \quad p \in N).$$

4. Distortion theorems

Theorem 5. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p,\alpha,j)$. Then, for $|z| = r < 1$,*

$$\begin{aligned} r^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} r^{p+1} &\leq |f(z)| \\ &\leq r^p + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} r^{p+1} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} pr^{p-1} - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)} r^p &\leq \left| f^{'}(z) \right| \\ &\leq pr^{p-1} + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)} r^p. \end{aligned} \tag{4.2}$$

The equalities in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} z^{p+1} \qquad (z = \pm r). \tag{4.3}$$

Proof. Since $f(z) \in T^*(p,\alpha,j)$, in view of Theorem 1, are have

$$\begin{aligned} \frac{\delta(p+1,j-1)(p-j+2-\alpha)}{\delta(p,j-1)} \sum_{k=p+1}^{\infty} a_k &\leq \sum_{k=p+1}^{\infty} \frac{\delta(k,j-1)}{\delta(p,j-1)} (k-j+1-\alpha) a_k \\ &\leq (p-j+1-\alpha), \end{aligned}$$

which evidently yields

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}. \tag{4.4}$$

Consequently, for $|z| = r < 1$, we obtain

$$|f(z)| \leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k \leq r^p + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} r^{p+1}$$

and

$$|f(z)| \geq r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k \geq r^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} r^{p+1},$$

which prove the assertion (4.1) of Theorem 5.

Also from Theorem 1, it follows that

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)}. \tag{4.5}$$

Consequently, for $|z| = r < 1$, we have

$$\begin{aligned}\left|f^{'}(z)\right| &\leq p r^{p-1} + \sum_{k=p+1}^{\infty} k a_k r^{k-1} \leq p r^{p-1} + r^p \sum_{k=p+1}^{\infty} k a_k \\ &\leq p r^{p-1} + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)} r^p\end{aligned}$$

and

$$\begin{aligned}\left|f^{'}(z)\right| &\geq p\, r^{p-1} - \sum_{k=p+1}^{\infty} k a_k\, r^{k-1} \geq p\, r^{p-1} - r^p \sum_{k=p+1}^{\infty} k a_k \\ &\geq p\, r^{p-1} - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)} r^p,\end{aligned}$$

which prove the assertion (4.2) of Theorem 5. □

Finally, it is easy to see that the bounds in (4.1) and (4.2) are attained for the function $f(z)$ given already by (4.3).

Corollary 5. *Let the function $f(z)$ defined by* (1.10) *be in the class $T^*(p, \alpha, j)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(p-j+2-\alpha)(p+1) - (p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}. \tag{4.6}$$

The result is sharp, with the extremal function $f(z)$ given by (4.3).

Theorem 6. *Let the function $f(z)$ defined by* (1.10) *be in the class $C(p, \alpha, j)$. Then, for $|z| = r < 1$,*

$$\begin{aligned}r^p - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)} r^{p+1} &\leq |f(z)| \\ &\leq r^p + \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)} r^{p+1}\end{aligned} \tag{4.7}$$

and

$$pr^{p-1} - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)} r^p \leq \left| f^{'}(z) \right|$$
$$\leq pr^{p-1} + \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)} r^p. \tag{4.8}$$

The results are sharp.

Proof. The proof of Theorem 6 is obtained by using the same technique as in the proof of Theorem 5 with the aid of Theorem 2. Further we can show that the bounds of Theorem 6 are sharp for the function $f(z)$ defined by

$$f(z) = z^p - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)} z^{p+1}. \tag{4.9}$$

□

Corollary 6. *Let the function $f(z)$ defined by* (1.10) *be in the class $C(p, \alpha, j)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(p-j+2-\alpha)(p+1) - (p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)}. \tag{4.10}$$

The result is sharp, with the extremal function $f(z)$ given by (4.9).

5. Integral operators

Theorem 7. *Let the function $f(z)$ defined by* (1.10) *be in the class $T^*(p, \alpha, j)$, and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) \mathrm{d}t \tag{5.1}$$

also belongs to the class $T^*(p, \alpha, j)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=p+1}^{\infty} b_k \, z^k, \tag{5.2}$$

where

$$b_k = \left(\frac{c+p}{c+k} \right) a_k.$$

Therefore

$$\begin{aligned}\sum_{k=p+1}^{\infty} & \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)} b_k \\ &= \sum_{k=p+1}^{\infty} \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)} \left(\frac{c+p}{c+k}\right) a_k \\ &\leq \sum_{k=p+1}^{\infty} \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)} a_k \leq (p-j+1-\alpha),\end{aligned}$$

since $f(z) \in T^*(p,\alpha,j)$. Hence, by Theorem 1, $f(z) \in T^*(p,\alpha,j)$. □

Corollary 7. *Under the same conditions as Theorem 7, a similar proof shows that the function $F(z)$ defined by* (5.1) *is in the class $C(p,\alpha,j)$, whenever $f(z)$ is in the class $C(p,\alpha,j)$.*

6. Radii of close-to-convexity and convexity for the class $\mathrm{T}^*(p,\alpha,j)$

Theorem 8. *Let the function $f(z)$ defined by* (1.10) *be in the class $T^*(p,\alpha,j)$, then $f(z)$ is p-valently close-to-convex of order ϕ $(0 \leq \phi < p)$ in $|z| < r_1$, where*

$$(6.1)\quad r_1 = \inf_k \left\{ \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)} \left(\frac{p-\phi}{k}\right)\right\}^{\frac{1}{k-p}} \qquad (k \geq p+1).$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

Proof. We must show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \phi$ for $|z| < r_1$. We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+1}^{\infty} k\, a_k \, |z|^{k-p}.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \phi$ if

$$(6.2)\qquad \sum_{k=p+1}^{\infty} \left(\frac{k}{p-\phi}\right) a_k \, |z|^{k-p} \leq 1.$$

Hence, by Theorem 1, (6.2) will be true if

$$\left(\frac{k}{p-\phi}\right) |z|^{k-p} \leq \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)}$$

or if

$$(6.3)\quad |z| \leq \left\{ \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)} \left(\frac{p-\phi}{k}\right)\right\}^{\frac{1}{k-p}} \qquad (k \geq p+1).$$

The theorem follows easily from (6.3). □

Theorem 9. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p,\alpha,j)$ then $f(z)$ is p-valently convex of order ϕ $(0 \le \phi < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_k \left\{ \frac{\delta(p,j-1)(k-j+1-\alpha)}{\delta(k,j-1)(p-j+1-\alpha)} \cdot \left(\frac{p(p-\phi)}{k(k-\phi)} \right) \right\}^{\frac{1}{k-p}} \quad (k \ge p+1). \tag{6.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

Proof. It is sufficient to show that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \le p - \phi \qquad \text{for} \qquad |z| < r_2.$$

We have

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \le \frac{\sum\limits_{k=p+1}^{\infty} k(k-p)a_k\, |z|^{k-p}}{p - \sum\limits_{k=p+1}^{\infty} k\, a_k\, |z|^{k-p}}.$$

Thus $\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \le p - \phi$ if

$$\sum_{k=p+1}^{\infty} \frac{k(k-\phi)}{p(p-\phi)} a_k\, |z|^{k-p} \le 1 \tag{6.5}$$

Hence, by Theorem 1, (6.5) will be true if

$$\frac{k(k-\phi)}{p(p-\phi)} |z|^{k-p} \le \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)}.$$

or if

$$|z| \le \left\{ \frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)} \cdot \left(\frac{p(p-\phi)}{k(k-\phi)} \right) \right\}^{\frac{1}{k-p}} \quad (k \ge p+1). \tag{6.6}$$

The theorem follows easily from (6.6). □

7. Modified Hadamard products

Let the functions $f_v(z)$ $(v = 1,2)$ be defined by

$$f_v(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,v} z^k \qquad (a_{k,v} \ge 0; \quad v = 1,2). \tag{7.1}$$

Then the modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{7.2}$$

Theorem 10. *Let the functions $f_v(z)$ $(v = 1, 2)$ defined by* (7.1) *be in the class $T^*(p, \alpha, j)$. Then $(f_1 * f_2)(z) \in T^*(p, \gamma, j)$, where*

$$\begin{aligned}\gamma = {} & (p-j+1) \\ & - \frac{(p-j+1-\alpha)^2(p-j+2)}{(p-j+2-\alpha)^2(p+1)-(p-j+1-\alpha)^2(p-j+2)}.\end{aligned} \tag{7.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silveman [**4**], we need to find the largest γ such that

$$\sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\gamma)}{\delta(p, j-1)(p-j+1-\gamma)} a_{k,1} a_{k,2} \leq 1. \tag{7.4}$$

Since

$$\sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} a_{k,1} \leq 1 \tag{7.5}$$

and

$$\sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} a_{k,2} \leq 1, \tag{7.6}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\frac{(k-j+1-\gamma)}{(p-j+1-\gamma)} a_{k,1} a_{k,2} \leq \frac{(k-j+1-\alpha)}{(p-j+1-\alpha)} \sqrt{a_{k,1} a_{k,2}} \qquad (k \geq p+1), \tag{7.8}$$

that is

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(k-j+1-\alpha)(p-j+1-\gamma)}{(k-j+1-\gamma)(p-j+1-\alpha)}. \tag{7.9}$$

Note that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \qquad (k \geq p+1). \tag{7.10}$$

Consequently, we need only to prove that

$$\frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \leq \frac{(k-j+1-\alpha)(p-j+1-\gamma)}{(k-j+1-\gamma)(p-j+1-\alpha)} \qquad (k \geq p+1) \tag{7.11}$$

or, equivalently, that

$$\gamma \le (p-j+1) - \frac{\delta(p,j-1)(p-j+1-\alpha)^2(k-p)}{\delta(k,j-1)(k-j+1-\alpha)^2-\delta(p,j-1)(p-j+1-\alpha)^2} \quad (k \ge p+1). \tag{7.12}$$

Since

$$D(k) = (p-j+1) - \frac{\delta(p,j-1)(p-j+1-\alpha)^2(k-p)}{\delta(k,j-1)(k-j+1-\alpha)^2-\delta(p,j-1)(p-j+1-\alpha)^2} \tag{7.13}$$

is an increasing function of k $(k \ge p+1)$, letting $k = p+1$ in (7.13) we obtain

$$\gamma \le D(p+1) = (p-j+1) - \frac{(p-j+1-\alpha)^2(p-j+2)}{(p-j+2-\alpha)^2(p+1)-(p-j+1-\alpha)^2(p-j+2)}, \tag{7.14}$$

which completes the proof Theorem 10. □

Finally, by taking the functions

$$f_v(z) = z^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} z^{p+1} \qquad (v = 1, 2;\ \ p \in N) \tag{7.15}$$

we can see that the result is sharp.

Corollary 8. *Let the functions $f_v(z)$ $(v = 1, 2)$ be the same as in Theorem 10, we have*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} \sqrt{a_{k,1}a_{k,2}}\, z^k \tag{7.16}$$

belongs to the class $T^*(p, \alpha, j)$.

The result follows from the inequality (7.7). It is sharp for the same functions as in Theorem 10.

Corollary 9. *Let the functions $f_\nu(z)(\nu = 1, 2)$ defined by (7.1) be in the class $C(p, \alpha, j)$. Then $(f_1 * f_2)(z) \in C(p, \lambda, j)$ where*

$$\lambda = (p-j+1) - \frac{(p-j+1-\alpha)^2(p-j+1)}{(p-j+2-\alpha)^2(p+1)-(p-j+1-\alpha)^2(p-j+1)}. \tag{7.17}$$

The result is sharp for the functions

$$f_\nu(z) = z^p - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)} z^{p+1} \qquad (\nu = 1, 2 \quad p \in N). \tag{7.18}$$

Using arguments similar to those in the proof of Theorem 10, we obtain the following result.

Theorem 11. *Let the function $f_1(z)$ defined by (7.1) be in the class $T^*(p,\alpha,j)$ and the function $f_2(z)$defined by (7.1) be in the class $T^*(p,\tau,j)$, then $(f_1*f_2)(z)\in T^*(p,\zeta,j)$, where*

$$\zeta=(p-j+1) - \frac{(p-j+1-\alpha)(p-j+1-\tau)(p-j+2)}{(p-j+2-\alpha)(p-j+2-\tau)(p+1)-(p-j+1-\alpha)(p-j+1-\tau)(p-j+2)}. \tag{7.19}$$

The result is the best possible for the functions

$$f_1(z)=z^p-\frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}z^{p+1} \qquad (p\in N) \tag{7.20}$$

and

$$f_2(z)=z^p-\frac{(p-j+1-\tau)(p-j+2)}{(p-j+2-\tau)(p+1)}z^{p+1}\ (p\in N). \tag{7.21}$$

Corollary 10. *Let the function $f_1(z)$ defined by (7.1) be in the class $C(p,\alpha,j)$ and the function $f_2(z)$ defined by (7.1) be in the class $C(p,\tau,j)$, then $(f_1*f_2)(z)\in C(p,\theta,j)$, where*

$$\theta=(p-j+1) - \frac{(p-j+1-\alpha)(p-j+1-\tau)(p-j+1)}{(p-j+2-\alpha)(p-j+2-\tau)(p+1)-(p-j+1-\alpha)(p-j+1-\tau)(p-j+1)}. \tag{7.22}$$

The result is sharp for the functions

$$f_1(z)=z^p-\frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)}z^{p+1} \qquad (p\in N) \tag{7.23}$$

and

$$f_2(z)=z^p-\frac{(p-j+1-\tau)(p-j+1)}{(p-j+2-\tau)(p+1)}z^{p+1} \qquad (p\in N). \tag{7.24}$$

Theorem 12. *Let the functions $f_\nu(z)$ $(\nu=1,2)$ defined by (7.1) be in the class $T^*(p,\alpha,j)$. Then the function*

$$h(z)=z^p-\sum_{k=p+1}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^k \tag{7.25}$$

belongs to the class $T^*(p,\varphi,j)$, where

$$\varphi=(p-j+1) - \frac{2(p-j+1-\alpha)^2(p-j+2)}{(p-j+2-\alpha)^2(p+1)-2(p-j+1-\alpha)^2(p-j+2)}. \tag{7.26}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu=1,2)$ defined by (7.15).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=p+1}^{\infty}\left[\frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)}\right]^2 a_{k,\nu}^2 \le \left[\sum_{k=p+1}^{\infty}\frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)}a_{k,\nu}\right]^2 \le 1 \qquad (\nu=1,2). \tag{7.27}$$

It follows from (7.27) for $\nu=1$ and $\nu=2$ that

$$\sum_{k=p+1}^{\infty}\frac{1}{2}\left[\frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)}\right]^2 (a_{k,1}^2+a_{k,2}^2) \le 1. \tag{7.28}$$

Therefore, we need to find the largest φ such that

$$\frac{\delta(k,j-1)(k-j+1-\varphi)}{\delta(p,j-1)(p-j+1-\varphi)} \le \frac{1}{2}\left[\frac{\delta(k,j-1)(k-j+1-\alpha)}{\delta(p,j-1)(p-j+1-\alpha)}\right]^2 \qquad (k\ge p+1) \tag{7.29}$$

that is,

$$\varphi \le (p-j+1) - \frac{2\delta(p,j-1)(k-j+1-\alpha)^2(k-p)}{\delta(k,j-1)(k-j+1-\alpha)^2-2\delta(p,j-1)(p-j+1-\alpha)^2} \qquad (k\ge p+1). \tag{7.30}$$

Since

$$\Psi(k) = (p-j+1) - \frac{2\delta(p,j-1)(p-j+1-\alpha)^2(k-p)}{\delta(k,j-1)(k-j+1-\alpha)^2-2\delta(p,j-1)(p-j+1-\alpha)^2}$$

is an increasing function of k $(k\ge p+1)$, we readily have

$$\varphi \le \Psi(p+1) = (p-j+1) - \frac{2(p-j+1-\alpha)^2(p-j+2)}{(p-j+2-\alpha)^2(p+1)-2(p-j+1-\alpha)^2(p-j+2)},$$

and Theorem 12 follows at once. □

Corollary 11. *Let the functions $f_\nu(z)$ $(\nu=1,2)$ defined by (7.1) be in the class $C(p,\alpha,j)$. Then the function $h(z)$ defined by (7.25) belongs to the class $C(p,\xi,j)$, where*

$$\xi = (p-j+1) - \frac{2(p-j+1-\alpha)^2(p-j+1)}{(p-j+2-\alpha)^2(p+1)-2(p-j+1-\alpha)^2(p-j+1)}. \tag{7.31}$$

The result is sharp for the functions $f_\nu(z)(\nu = 1, 2)$ *defined by* (7.18).

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