Acta Math. Univ. Comenianae
Vol. LXXVIII, 1(2009), pp. 29–36

ANALYTIC EXTENSION OF A MAXIMAL SURFACE IN $\mathbb{L}^3$ ALONG ITS BOUNDARY

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Abstract. We prove that a maximal surface in Lorentz-Minkowski space $\mathbb{L}^3$ can be extended analytically along its boundary if the boundary lies in a plane meeting the surface at a constant angle.

1. Introduction

A maximal surface in Lorentz-Minkowski space $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. Maximal surfaces share many interesting properties with their counterparts, minimal surfaces, in $\mathbb{R}^3$. For example, they are critical points (the maxima) of area variations and also admit Enneper-Weierstrass representations. It is well known that a minimal surface in $\mathbb{R}^3$ can be extended (symmetrically) along its boundary if the boundary lies in a plane meeting the minimal surface orthogonally. This fact also holds for maximal surfaces in $\mathbb{L}^3$ (see [**1**]), where the plane is assumed to be timelike since spacelike and lightlike planes can not meet a maximal surface orthogonally, except at singular points, see the Remark in Section 3.

In 1996, J. Choe ([**2**]) proved that a minimal surface in $\mathbb{R}^3$ can be extended analytically along its boundary if the boundary lies in a plane meeting the minimal surface at a constant angle. The main idea is based on Enneper-Weierstrass representation of a minimal surface in terms of a holomorphic function f and a meromorphic function g. The meromorphic function g can be viewed as the Gauss map of the minimal surface. Since the plane meets the minimal surface at a constant angle, the image of the boundary under the Gauss map g lies in a circle and hence we can apply Schwartz reflection principle to extend both f and g along the boundary.

In this paper, we show that the above idea can be applied for the case of maximal surfaces in $\mathbb{L}^3$. The complication in this situation is that a plane can be spacelike, timelike or lightlike.

Received September 20, 2007; revised April 13, 2008.
2000 *Mathematics Subject Classification.* Primary 53C50; Secondary 58D10, 53C42.
Key words and phrases. Maximal surfaces; conelike singularities.
This work was supported by the National Basis Reseach Program 101706, Vietnam.

2. Preliminaries

The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is the 3-dimensional vector space $\mathbb{R}^3 = \{(x_1, x_2, x_3,) : x_i \in \mathbb{R}, i = 1, 2, 3\}$ endowed with the indefinite $(2, 1)$-metric

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{L}^3$.

We say that a nonzero vector $x \in \mathbb{L}^3$ is *spacelike*, *lightlike* or *timelike* if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. The vector zero is always considered as a spacelike one.

The norm of a vector $x \in \mathbb{L}^3$, denoted by $\|x\|$, is defined by $\sqrt{|\langle x, x\rangle|}$. The definition of the cross-product of two vectors $a = (a_1, a_2, a_3)$; $b = (b_1, b_2, b_3)$, denoted by $a \wedge b$ is given as follows

$$a \wedge b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2).$$

For a nonzero vector $n \in \mathbb{L}^3$, a plane with (pseudo) normal n is the set

$$P(n, c) = \{x \in \mathbb{L}^3 : \langle x, n\rangle = c, c \in \mathbb{R}\}.$$

The plane $P(n, c)$ is called *spacelike*, *lightlike* or *timelike* if n is timelike, lightlike or spacelike, respectively.

It is easy to see that $P(n, c)$ is spacelike if any vector $x \in P(n, c)$ is spacelike; $P(n, c)$ is lightlike if $P(n, 0)$ is tangent to the lightcone; $P(n, c)$ is timelike if it contains timelike vectors.

The set

$$\mathbb{H}^2 = \{x \in \mathbb{L}^3 : \langle x, x\rangle = -1\}$$

is called the hyperbolic. It has two connected components $\mathbb{H}^2_+ = \{x \in \mathbb{H}^2 : x_3 \geq 1\}$ and $\mathbb{H}^2_- = \{x \in \mathbb{H}^2 : x_3 \leq -1\}$. For studying spacelike surfaces in Lorentz-Minkowski spaces, $\mathbb{H}^2$, $\mathbb{H}^2_-$, $\mathbb{H}^2_+$ play the same roles as the unit sphere $\{|x|^2 = 1\}$ in Euclidean spaces.

Let $X : M \longrightarrow \mathbb{L}^3$ be an immersion of a 2-dimensional connected manifold. X (or $X(M)$) is called spacelike if the induced metric on M via X is a Riemannian metric. That means the tangent plane $T_pM \subset T_p\mathbb{L}^3$ is spacelike, for every $p \in M$. In this case, the manifold M is orientable. Now, suppose that $X : M \longrightarrow \mathbb{L}^3$ is a spacelike immersion and (u, v) is a local coordinate system. The (local) unit normal vector field is defined as follows

$$N(u, v) = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}.$$

Because M is spacelike, $N(u, v)$ is always timelike.

Since M is connected, we can define the unit normal timelike vector field N on M globally and the image of N lies in one of components of $\mathbb{H}^2$. Because of that we can consider N as a map $N : M \longrightarrow \mathbb{H}^2_+$. The map N is called the Gauss map of the immersion. The shape operator is the map $A := -dN$ defined for all vector fields on manifold M and the mean curvature H is a half of the trace of A

$$H := \frac{1}{2}\,\mathrm{tr}(A).$$

A spacelike immersion $X : M \longrightarrow \mathbb{L}^3$ is said to be a maximal immersion if its mean curvature (at every point) is equal to zero, that is $H = 0$.

In 1983, Kobayashi ([**5**]) showed Enneper-Weierstrass representations for maximal immersions in $\mathbb{L}^3$. Such representations for maximal immersions are quite similar to that for minimal immersions in Euclidean space $\mathbb{R}^3$. It is clear that we can define local isothermal coordinate systems whose changes of coordinates preserve the orientation for maximal immersions. The existence of such coordinate systems is proved quite similar to that for minimal immersions. Thus, since every spacelike immersion is orientable, M admits a structure of a Riemann surface.

Now, suppose that $X = (x_1, x_2, x_3)$ and $z = u + \mathrm{i}\, v$ is the local complex parameter on M. We set

$$\phi_k := \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - \mathrm{i}\,\frac{\partial x_k}{\partial v}\right), \qquad k = 1, 2, 3.$$

Since M is maximal, x_k, $k = 1, 2, 3$ are harmonic and hence ϕ_k, $k = 1, 2, 3$ are holomorphic. Direct computation shows that

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = 0, \tag{1}$$

and

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 > 0. \tag{2}$$

We see that $ds^2 = |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^3 > 0$ is the Riemannian metric on M induced by the immersion X and ϕ_k, $k = 1, 2, 3$ have no real periods and hence the immersion X can be represented as

$$X(z) = \mathrm{Re}\int (\phi_1, \phi_2, \phi_3)\mathrm{d}z, \tag{3}$$

where the integral is taken on an arbitrary path from a fixed point to z.

Conversely, if ϕ_1, ϕ_2, ϕ_3 are holomorphic functions on M and they have no real periods and satisfy (1) and (2), then (3) determines a maximal surface.

If $\phi_1 - \mathrm{i}\,\phi_2 = 0$, then $\phi_3 = 0$. In this case, M is a plane. Now, suppose that $\phi_1 - \mathrm{i}\,\phi_2 \neq 0$, we set

$$f = \phi_1 - \mathrm{i}\,\phi_2, \qquad g = \frac{\phi_3}{\phi_1 - i\phi_2}.$$

We have

$$\begin{cases} \phi_1 = \dfrac{1}{2} f(1 + g^2) \\ \phi_2 = \dfrac{\mathrm{i}}{2} f(1 - g^2) \\ \phi_3 = fg \end{cases}, \tag{4}$$

and then (3) can be writen as follows

$$X(z) = \mathrm{Re}\left(\frac{1}{2}\int_{z_0}^{z} f(1 + g^2)d\omega, \frac{\mathrm{i}}{2}\int_{z_0}^{z} f(1 - g^2)d\omega, \int_{z_0}^{z} fg d\omega\right). \tag{5}$$

From (4), we have $\phi_1 + \mathrm{i}\,\phi_2 = fg^2$. Thus, we can conclude that the poles of g coincide with the zeroes of f in such a way that a pole of order m of g corresponds to a zero of order $2m$ of f. Conversely, if such g and f are given, then (5) determines a maximal immersion.

Since $(X_u - \mathrm{i}\, X_v) = 2(\phi_1, \phi_2, \phi_3)$, we have

$$X_u \wedge X_v = 4\mathrm{Im}(\phi_2\overline{\phi}_3, \phi_3\overline{\phi}_1, \phi_2\overline{\phi}_1) \quad = |f|^2(1-|g|^2)\left(2\mathrm{Re}(g), 2\mathrm{Im}(g), 1+|g|^2\right).$$

Thus, the Gauss map N can be expressed as follows

$$N = \left(\frac{2\mathrm{Re}(g)}{1-|g|^2}, \frac{2\mathrm{Im}(g)}{1-|g|^2}, \frac{1+|g|^2}{1-|g|^2}\right).$$

Since $N(z) \in \mathbb{H}^2_+$, we conclude that $|g| < 1$.

It is clear that $z \longmapsto \left(\dfrac{2\mathrm{Re}(z)}{1-|z|^2}, \dfrac{2\mathrm{Im}(z)}{1-|z|^2}, \dfrac{1+|z|^2}{1-|z|^2}\right)$ is a conformal isomorphism π between $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{H}^2_+$. The map π^{-1} is the stereographic projection from the point $(0,0,-1)$. The formula of π^{-1} is

$$\pi^{-1}(x_1, x_2, x_3) = \frac{x_2 + \mathrm{i}\, x_2}{1+x_3}.$$

We can view g as a map from M into D and $\pi^{-1} \circ N = g$. Because of that, we also call g the Gauss map of M.

3. Extension of a maximal surface

Let Ω be a domain in $\mathbb{R}^2$. We will call a maximal immersion $x : \Omega \longrightarrow \mathbb{L}^3$ a maximal surface and always assume that the parameters u, v on Ω are isothermal and set $z = u + \mathrm{i}\, v$.

It is well known that every maximal immersion can be locally written as a maximal surface and by Uniformization theorem a simply connected maximal immersion can be expressed as a maximal surface globally.

Denote $D = \{u^2+v^2 < 1\}$, $D_+ = \{u^2+v^2 < 1;\ v > 0\}$, $D_- = \{u^2+v^2 < 1;\ v < 0\}$ and $D_0 = D \cap \{v = 0\}$; we have the main theorem of this paper.

Theorem 1. *Let $X_+ : D_+ \longrightarrow \mathbb{L}^3$ be a maximal surface with isothermal parameters u, v and Π be a plane. Suppose that γ is an analytic curve in Π, $X_+(u,v)$ tends to $\gamma(u)$ whenever $v \to 0$, and*

$$\lim_{v\to 0} \langle N(z), n\rangle = c \neq 0,$$

where N is the Gauss map of X_+ and n is the unit normal vector of Π. Then X_+ can be analytically extended along γ to a maximal surface $X : D \longrightarrow \mathbb{L}^3$ such that $X|_{D_+} = X_+$ and $X(D_0) = \gamma$.

Proof. The main idea for the proof is showing that both g and f can be extended analytically on D and hence by (5) we get the extended maximal surface. We will consider three cases: Π is spacelike, Π is timelike and Π is lightlike. In each case we will use the following fact: if g can be continuously extended to $D_+ \cup D_0$ and $g(D_0)$ lies in a circle, then after using a Möbius transformation that maps $g(D_0)$ to the real axis, Schwartz reflection principle can be applied to extend g on D.

1. Π **is spacelike.** By using a suitable Lorentzian transformation, we can assume that Π is the plane $x_3 = 0$. Let n be a timelike vector $(0, 0, 1)$, then
$$\lim_{v\to 0} \langle -N(z), n\rangle = \lim_{v\to 0} \frac{1+|g|^2}{1-|g|^2} = -c.$$
Setting $c = \cosh\theta$ we conclude that
$$\lim_{v\to 0} |g(z)| = \coth\frac{\theta}{2}.$$
Then the meromorphic function g can be continuously extended on $D_+ \cup D_0$ such that
$$|g(z)| = \coth\frac{\theta}{2}, \qquad \forall z \in D_0.$$
That means g maps D_0 into the circle with the center O and radius $r = \coth\frac{\theta}{2}$, and therefore, g can be extended analytically on D. The extension of g also denoted as g is expressed as follows:
$$g(z) = \coth^2\left(\frac{\theta}{2}\right)(\overline{g(\overline{z})})^{-1}, \qquad z \in D_-.$$
Next, we extend f on D. First we observe that x_3 is extended to a harmonic function, also denoted as x_3, on D by setting
$$x_3(z) = -x_3(\overline{z}), \qquad z \in D_-.$$
Then ϕ_3 can be extended to a holomorphic function, also denoted as ϕ_3, on D by setting
$$\phi_3(z) = -\overline{\phi_3(\overline{z})}, \qquad z \in D_-.$$
Finally, f is extended analytically on D by setting
$$f(z) = \frac{-\overline{f(\overline{z})g^2(\overline{z})}}{\coth^2\left(\frac{\theta}{2}\right)} = \frac{-\overline{\phi_3(\overline{z})g(\overline{z})}}{\coth^2\left(\frac{\theta}{2}\right)} = \frac{-\overline{\phi_3(\overline{z})}}{\coth^2\left(\frac{\theta}{2}\right)(\overline{g(\overline{z})})^{-1}} = \frac{\phi_3(z)}{g(z)}, \quad \forall z \in D_-.$$

2. Π **is timelike.** We can assume that Π is the plane $x_2 = 0$. Set $c = \frac{1}{\lambda}, \lambda \neq 0$ and choose $n = (0, 1, 0)$. The assumption $\lim_{v\to 0} \langle N(z), n\rangle = \frac{1}{\lambda}$ implies that
$$\lim_{v\to 0} \frac{2\mathrm{Im}(g)}{1-|g|^2} = \frac{1}{\lambda}.$$
Thus, g is extended continuously on $D_+ \cup D_0$ such that the following is satisfied
$$\frac{2\mathrm{Im}(g)}{1-|g|^2} = \frac{1}{\lambda}. \tag{6}$$
Equation (6) gives
$$[\mathrm{Re}(g)]^2 + [\mathrm{Im}(g) + \lambda]^2 = 1 + \lambda^2.$$

Therefore, g maps D_0 into the circle with the center at $(0, -\lambda)$ and radius $r = \sqrt{1+\lambda^2}$ and then the meromorphic function g is extended as follows

$$g(z) = -\mathrm{i}\,\lambda + (1+\lambda^2)\left(\overline{g(\overline{z})} - \mathrm{i}\,\lambda\right)^{-1}, \qquad \forall z \in D_-.$$

Because $X_+ = (x_1, x_2, x_3)$ is maximal and u, v are isothermal parameters, x_2 is harmonic on D_+. The assumption

$$\lim_{v\to 0} X_+(z) = \gamma(u) \in \Pi,$$

implies that x_2 can be continuously extended on $D_+ \cup D_0$ by setting

$$x_2(z) = 0, \qquad \forall z \in D_0.$$

Schwartz reflection principle says that x_2 can be extended on D as follows

$$x_2(z) = -x_2(\overline{z}), \qquad \forall z \in D_-.$$

Therefore, ϕ_2 is extended on D by setting

$$\phi_2(z) = -\overline{\phi_2(\overline{z})}, \qquad z \in D_-.$$

Since $g^2(z) \neq 1$, the holomorphic f is extended analytically on D by setting

$$f(z) = \frac{2\phi_2(z)}{\mathrm{i}(1-g^2(z))}, \qquad z \in D.$$

3. Π **is lightlike.**

Assume that the equation of Π is $x_1 - x_3 = 0$. We set $c = 1+\lambda$ and choose $n = (1, 0, 1)$.

If $\lambda = 0$, then by the assumption

$$\lim_{v\to 0} \langle N(z), n\rangle = 1,$$

we have

$$\lim_{v\to 0}\left[\frac{2\mathrm{Re}(g)}{1-|g|^2} - \frac{1+|g|^2}{1-|g|^2}\right] = 1$$

or equivalently, $\mathrm{Re}(g)$ tends to 1 whenever v tends to 0. The meromorphic function g can be extended continuously on $D \cup D_0$ such that $\mathrm{Re}g(z) = 1$, $\forall z \in D_0$. That means $g(D_0) \subset \{\mathrm{Re}(z) = 1\}$. In this case g is extended analytically on D by setting

$$g(z) = 2 - \overline{g(\overline{z})}, \forall z \in D_-.$$

If $\lambda \neq 0$, by the assumption

$$\lim_{v\to 0} \langle N(z), n\rangle = \lim_{v\to 0}\left[\frac{2\mathrm{Re}(g)}{1-|g|^2} - \frac{1+|g|^2}{1-|g|^2}\right] = 1+\lambda,$$

we conclude that g can be continuously extended on $D \cup D_0$ in such a way that

$$\left(\mathrm{Re}(g) + \frac{1}{\lambda}\right)^2 + (\mathrm{Im}(g))^2 = \left(1 + \frac{1}{\lambda}\right)^2, \qquad \forall z \in D_0.$$

That means the image of D_0 under g lies in the circle with the center $(-\frac{1}{\lambda}, 0)$ and the radius $r = \left|1 + \frac{1}{\lambda}\right|$. Then g is extended analytically on D by setting

$$g(z) = -\frac{1}{\lambda} + (1 + \frac{1}{\lambda})^2 \left(\overline{g(\overline{z})} + \frac{1}{\lambda}\right)^{-1}, \qquad \forall z \in D_-.$$

In order to extend f we first observe that $\psi = x_1 - x_3$ is a harmonic function on D_+ and by the assumption

$$\lim_{v \to 0} X_+(z) \to \gamma(u) \in \Pi,$$

it can be extended to a continuous function on $D_+ \cup D_0$ by setting $\psi(z) = 0$, $\forall z \in D_0$. Then by Schwartz reflection principle for harmonic function, ψ can be extended to a harmonic function on D by setting

$$\psi(z) = -\psi(\overline{z}), \forall z \in D_-.$$

Let ψ^* be the harmonic conjugation of ψ, then $\dfrac{d(\psi + \mathrm{i}\,\psi^*)}{dz}$ is a holomorphic function on D. It is clear that $\left.\dfrac{d(\psi + \mathrm{i}\,\psi^*)}{dz}\right|_{D_+} = \phi_1 - \phi_3$. So $\dfrac{d(\psi + \mathrm{i}\,\psi^*)}{dz}$ is the extension of $\phi_1 - \phi_3$ and we can write $\phi_1 - \phi_3$ instead of $\dfrac{d(\psi + \mathrm{i}\,\psi^*)}{dz}$. Then, the analytic extension of f can be writen as follows

$$f(z) = \frac{2(\phi_1 - \phi_3)}{(1 - g(z))^2}, \qquad \forall z \in D_-.$$

$\square$

Remarks.

1. It is clear that if $z = u + \mathrm{i}\,v$ is a pole of order m of g then $z = u + \mathrm{i}\,v$ is a zero of order $2m$ of f.
2. If $\langle N(z), n\rangle = 0$ along $S \cap \Pi$, then we say that the plane Π meets the maximal surface S orthogonally. Suppose that Π is spacelike, then we can conclude that $1 + |g|^2 = 0$, is a contradiction. Thus, a spacelike plane can not meet a maximal surface orthogonally. If Π is lightlike, we can suppose that the equation of Π is $x_1 - x_3 = 0$. Then we can conclude that $g = -1$ along $S \cap \Pi$. Therefore, $X_u \wedge X_v = 0$. Thus a lightlike plane can not meet a maximal surface orthogonally, except at singular points.
3. We can see the extension clearly on Lorentzian Catenoid. Let

$$X(u, v) = (\sinh u \cos v, \sinh u \sin v, u); \quad (u, v) \in U = \{(u, v) \in \mathbb{R}_+ \times (-\pi, \pi)$$

be the Lorentzian Catenoid with only conelike sigularity at the origin. Let Π_1 be spacelike plane $x_3 = a > 0$, Π_2 be spacelike plane $x_3 = b > a$ and Π_3 be spacelike plane $x_3 = 2b - a$. The extension about Π_2 as in proof of Theorem 1 maps $X(U) \cap \Pi_1$ to $X(U) \cap \Pi_3$ and also maps the component bounded by Π_1 and Π_2 to the component bounded by Π_2 and Π_3.
4. (**Extension about a conelike singularity**) Nevertheless, there are important differences between maximal surfaces and minimal surfaces. The fact that the only complete maximal surfaces in $\mathbb{L}^3$ are spacelike planes is

an example in global theory. On the other hand, maximal surfaces may have isolated singularities that never happen for minimal surfaces (see [**6**]).

Let S be a maximal surface and $p \in S$ is a conelike singularity. For more detail about conelike singularities, we refer readers to [**6**]. Now, let $X : \overline{D} \longrightarrow \mathbb{L}^3$ be a neighbourhood of a conelike singularity where $X(0,0)$ is the conelike singularity and suppose that $X(\partial D)$ meets spacelike plane $x_3 = a$ at a constant angle. In this situation, the image of g is an annulus bounded by circles $\{|z| = 1\}$ and $\{|z| = r < 1\}$ and hence conformally identified with $D - \{(0,0)\}$. Obviously, we can extend both ϕ_3 and g analitycally to the whole $\mathbb{C}$ by using the inversion about circle $\{|z| = 1\}$. The result is a complete maximal surface with one conelike singular point and one end and therefore there is an embedding entire graph (see [**4**, Proposition 2.1]). It must be the Lorentzian catenoid according to the a result of Ecker (see [**3**]).

5. (**Extension about an end**) The same argument as in item 4 also holds for $X : \overline{D} - \{(0,0)\} \longrightarrow \mathbb{L}^3$ being a neighbourhood of an end of a maximal surface, and $X(\partial D)$ meets spacelike plane $x_3 = a$ at a constant angle. In this case, the image of Gauss map g is the disk $\{0 < |x| < r;\ r < 1\}$ and also can be extended analitycally to $D - \{(0,0)\}$.

Acknowledgment. We thank the referee for introducing the paper of Sung-Eun Koh and pointing out some typos.

References

1. Alias B.J., Chaves R. M. B. and Mira P., *Björling problem for maximal surfaces in Lorentz-Minkowski space*, Math. Proc. Camb. Soc. (2003), 289–316.

2. Choe J., *On the analytic reflection of a minimal surface*, Pacific Journal of Mathematics, **157**(1) (1993), 29–36.

3. Ecker K., *Area maximimizing hypersurfaces in Minkowski space having an isolated singularity*, Manuscripta Math., **56** (1986), 375–397.

4. Fernandez I., Lopez F. J. and Souam R., *The space of complete embedded maximal surfaces with isolated singularities in the 3-dimensional Lorentz-Minkowski space L^3*. Mathematische Annalen, **332** (2005), 605–643.

5. Kobayashi O., *Maximal surfaces in the 3-dimensional Minkowski space* $\mathbb{L}^3$, Tokyo J. Math., **6** (1983), 297–309.

6. ______, *Maximal surfaces with conelike singularities*, J. Math. Soc. Japan, **4** (1984), 609–617.

7. Sung-Eun Koh, *An analytic extension of a spacelike maximal surface*, Bull. Austral. Math. Soc. **76**(1) (2007), 43–47.

8. Osserman R., *A survey of minimal surfaces* , Dover Publications, New York, 1986.

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