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A MEAN VALUE PROPERTY FOR PAIRS OF INTEGRALS

A. B. MINGARELLI, J. M. PACHECO AND A. PLAZA

Abstract. We show that for any two continuous real valued functions f, g on $[0, 1]$, the problem

$$\int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^c w(x)\,g(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x \cdot \int_0^c w(x)\,f(x)\,\mathrm{d}x,$$

always has at least one solution $c \in (0, 1)$, for a general class of weight-functions. Some applications are given.

Introduction

Mean value theorems for the integral calculus lie at the heart of analytical estimations of all kinds in mathematical analysis, see e.g., [**3**], [**5**]. In some cases they can be used to determine the sign of a given complicated looking integral without its actual evaluation, or estimation of the sizes of remainders in the study of infinite series, etc. (see [**4**, pp. 65 ff.]). Although many different kinds of mean value theorems for integrals now exist and have been generalized to all sorts of spaces and situations, we return to the basic one-dimensional real scenario and present one more such theorem with interesting applications.

The problem under consideration is this: there is given a real valued fixed continuous function $w(x)$ on $[0, 1]$ (thought of as a weight-function), to determine those real valued continuous functions f, g on $[0, 1]$ for which there exists a number $c \in (0, 1)$ such that

$$\int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^c w(x)\,g(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x \cdot \int_0^c w(x)\,f(x)\,\mathrm{d}x. \tag{1}$$

Note that (1) may be satisfied for given w, f, g: For example, if $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x = 0$ then (1) has infinitely many solutions, namely the whole c-interval $[0, 1]$. It may have exactly one solution as in the case, say, where for every $x \in [0, 1]$, $f(x) := 1$, $w(x) := x$, and $g(x) := x^2$, where the value of c is near 0.8. Finally,

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(1) may have no solution $c \in (0,1)$ whatsoever, e.g., in the case where $w(x) = 1$, $g(x) = 1$, $f(x) = x$.

In the original question [**1**] the weight-function is simply the identity function, $w(x) = x$, where it is alleged that (1) holds for all real valued continuous functions on $[0,1]$. We show that (1) always has a solution for any given pair f, g of continuous functions so long as the weight is suitably restricted (a situation which of course includes the case $w(x) = x$ in [**1**] and which was solved separately in [**2**]). We also observe that our restrictions on the weight are essentially best possible for (1) to hold for *all* continuous functions, in that counterexamples exist if the weight fails to be of the type given here.

In the sequel we let

$$W = \{w : [0,1] \to \mathbb{R}^+ | w \in C^1(0,1),\ w'(x) \geq 0,\ x \in [0,1]\}$$

and $S = C[0,1]$. One of the consequences of our main theorem, Theorem 1 below, is the following mean value theorem for integrals: Let $w \in W$ be a non-constant function on $[0,1]$. If $f \in S$ satisfies

$$\int_0^1 f(s)\,\mathrm{d}s = 0$$

then there is a point $c \in (0,1)$ such that

$$\int_0^c w(s)\,f(s)\,\mathrm{d}s = 0,$$

that is, the mean value of wf over $(0,c)$ is zero.

Another application of our main result is the investigation that the notion of orthogonality in the ordinary space of square integrable functions is pervasive (see Example 4 below). This is to be taken in the sense that such an orthogonality relation between two functions implies (for a general class of weights) orthogonality in an associated space of weighted square integrable functions, but on a subinterval of $(0,1)$. This is all a consequence of the following main result.

Theorem 1. *Let $w \in W$ be a non-constant function on $[0,1]$. For any $f, g \in S$ problem (1) always has at least one solution $c \in (0,1)$.*

Proof. To see this note that if $\int_0^1 f(x)\,\mathrm{d}x = 0$ and $\int_0^1 g(x)\,\mathrm{d}x = 0$ then (1) holds for every $c \in [0,1]$. So, without loss of generality we may assume that $\int_0^1 g(x)\,\mathrm{d}x \neq 0$. Write $h(t) = f(t) - mg(t)$ where the real number m is chosen so that

$$\int_0^1 h(s)\,\mathrm{d}s = 0. \tag{2}$$

Thus, $m = \int_0^1 f(x)\,\mathrm{d}x / \int_0^1 g(x)\,\mathrm{d}x$. Observe that (1) has a solution $c \in (0,1)$ if and only if

$$\int_0^c w(s)\,h(s)\,\mathrm{d}s = 0 \tag{3}$$

(for the same value of c). If possible, assume that (3) fails for all $c \in (0,1)$. Write $H(t) = \int_0^t h(s)\,\mathrm{d}s$. Then, by (2), $H(1) = 0$ while an integration by parts shows that

$$\int_0^t w(s)\,h(s)\,\mathrm{d}s \quad = \quad w(t)H(t) - \int_0^t w'(s)\,H(s)\,\mathrm{d}s. \tag{4}$$

Since the left side of (4) is necessarily of one sign by our assumption, we may assume that it is positive for every $t \in (0,1)$, that is

$$\int_0^t w(s)\,h(s)\,\mathrm{d}s > 0, \qquad t \in (0,1). \tag{5}$$

Thus,

$$H(t) > \frac{1}{w(t)} \int_0^t H(s)\,w'(s)\,\mathrm{d}s := H_1(t) \tag{6}$$

for $t \in (0,1)$. Writing $R(t) = \int_0^t w'(s)\,H(s)\,\mathrm{d}s$, (6) gives $w(t)\,H(t) > R(t)$. Since w is non-decreasing there follows $w(t)\,H(t)\,w'(t) \geq R(t)w'(t)$. On the other hand, $H(t)w'(t) = R'(t)$. Hence,

$$R'(s) \geq \frac{w'(s)}{w(s)}\,R(s) \tag{7}$$

for $s \in (0,1)$. For $0 < s < t$ we divide both sides of (7) by $w(s)$ to find, upon rearranging terms,

$$\frac{d}{\mathrm{d}s}\left(\frac{R(s)}{w(s)}\right) \geq 0.$$

Integrating the latter over the s-interval, $0 < \alpha < s < t$, and simplifying we obtain

$$R(t) - R(\alpha)\frac{w(t)}{w(\alpha)} \geq 0.$$

This implies that $R(t)/w(t) = H_1(t)$ is non-decreasing over $(0,1)$. Observe that $H_1(t)$ may be defined so as to be right-continuous at $t = 0$ by defining $H_1(0) = 0$. So, passing to the limit as $t \to 1^-$ in (6) we see that $H(1) \geq H_1(1) \geq 0$, or $H_1(1) = 0$ (since $H(1) = 0$). Hence $H_1(t) \equiv 0$ on $[0,1]$ which implies $R(t) \equiv 0$ on $[0,1]$. Using this fact in (4) it yields

$$\int_0^t w(x)\,h(x)\,\mathrm{d}x = w(t)\int_0^t h(s)\,\mathrm{d}s, \tag{8}$$

for all $t \in [0,1]$. Differentiating this identity and collecting terms we get that

$$w'(t)\int_0^t h(s)\,\mathrm{d}s \equiv 0, \quad t \in (0,1). \tag{9}$$

Since $w'(t)$ is not identically zero on $(0,1)$, there is a $t_0 \in (0,1)$ such that $w'(t_0) \neq 0$ and so, by (9), we must have $\int_0^{t_0} h(s)\,\mathrm{d}s = 0$. Therefore, the left side of (8) must vanish at $t = t_0$ and this contradicts (5).

If the left side of (4) is negative for all $t \in (0,1)$, then the previous argument may be used with the necessary changes to show that $H_1(t)$ is now non-increasing for all $t \in (0,1)$ and this leads to a contradiction once again. It follows that the left side of (4) cannot be of one sign for all $t \in (0,1)$ and so there must exist a point c such that (1) holds. On the other hand, if $\int_0^1 g(x)\,\mathrm{d}x = 0$, we simply redefine $h(x)$ by interchanging f and g and proceed as above. $\square$

Discussion

We consider the optimality of the conditions on the weight-function w in Theorem 1 and show that the conditions imposed upon w here are essentially best possible.

Firstly, $w(x)$ cannot be non-negative everywhere in $(0,1)$. It may be easily seen by considering the choice $f(x) = \cos \pi x$, $g(x) = 1$, and by defining w as a $C^1(0,1)$-function whose support is $[0,1/2]$ (hence $w(x) \equiv 0$ on $(1/2,1]$) with, in addition, $w(x) > 0$ on its support. In this case, since $w(x)f(x) > 0$ on $(0,1/2)$ and is identically zero on $(1/2,1)$, there is no $c \in (0,1)$ such that $\int_0^c w(s)f(s)\,\mathrm{d}s = 0$, hence (1) fails.

Secondly, consider the possibility that w may be negative somewhere in $(0,1)$. Then the functions $g(x) = \cos \pi x$, $f(x) = 1$, $w(x) = g(x)$, show that Theorem 1 is also false for this choice of functions.

Next, consider the possibility that w may be *decreasing* and positive. In this case observe that the choice $g(x) = \cos \pi x$, $f(x) = 1$, $w(x) = 1 - x$, shows that Theorem 1 fails, i.e., there is no value of $c \in (0,1)$ such that (1) holds. Thus, the condition on the sign of the derivative of w may not be removed in general.

We have seen by means of an example that if $w(x)$ is identically constant on $[0,1]$ then Theorem 1 may fail as well (see the Introduction). On the other hand, it is perfectly possible for $w(x)$ to be identically constant on a subinterval $J \subset (0,1)$ so long as $w'(x) \int_0^x h(t)\,dt \neq 0$ for $x \in [0,1]\backslash J$. For example, let $f(x) = \cos \pi x$, $g(x) = 1$. Define a weight-function $w \in W$ by

$$w(x) = \begin{cases} 1 - (x - 1/2)^2, & \text{if} \quad 0 \le x \le 1/2, \\ 1, & \text{if} \quad 1/2 < x \le 1. \end{cases}$$

Then $w'(x) \int_0^x h(t)\,\mathrm{d}t = 0$ for $x \in [1/2,1)$ and is non-zero in $(0,1/2)$. Since $w \in W$, Theorem 1 guarantees that the corresponding relation (1) given by

$$\int_0^c g(x)\,w(x)\,\mathrm{d}x = 0,$$

has at least one root $c \in (0,1)$. Using Newton iterations one can readily verify the location of such a root at $x \approx 0.9631$.

In another vein, the smoothness of w may be weakened to absolute continuity without affecting the conclusion of the main theorem although the proof needs further technical interpretations (e.g., inequalities and equalities are generally in the *almost everywhere* sense and integrals are to be thought of as Lebesgue integrals). In fact, the proof as presented here may be used with the stated interpretations to show that the theorem admits an extension to functions $f, g \in L^p_w(0,1)$, with a suitable interpretation for the weight.

Some applications

Our main result is in the spirit of a "mean value theorem" for integrals because of the following simple consequence of Theorem 1 (say, with $g(x) = 1$).

Example 2. Let $w \in W$ be a non-constant function on $[0, 1]$. If $f \in S$ satisfies

$$\int_0^1 f(s)\,\mathrm{d}s = 0$$

then there is a point $c \in (0, 1)$ such that

$$\int_0^c w(s)\,f(s)\,\mathrm{d}s = 0,$$

that is, the mean value of wf over $(0, c)$ is zero.

Example 3. Replacing $f,\ g \in S$ by their squares gives a more appealing inequality reminiscent of the theory of Hilbert spaces: For example, consider the equality

$$\int_0^1 f(x)^2\,\mathrm{d}x \cdot \int_0^c w(x)\,g(x)^2\,\mathrm{d}x = \int_0^1 g(x)^2\,\mathrm{d}x \cdot \int_0^c w(x)\,f(x)^2\,\mathrm{d}x$$

obtained by replacing f, g by their squares in (1). This really says that given any $f, g \in S$ (each non-identically zero), $w \in W$ a non-constant function, there always exists a point $c \in (0, 1]$ such that

$$\frac{\|f\|_{L^2{}_w(0,c)}}{\|g\|_{L^2{}_w(0,c)}} = \frac{\|f\|_{L^2(0,1)}}{\|g\|_{L^2(0,1)}}$$

where the quantities are the norms in the respective spaces of (weighted) square integrable functions. Thus, for instance, if $f, g \in S$ have equal $L^2(0, 1)$-norms, then for given $w \in W$ a non-constant function, there is a point $c \in (0, 1)$ such that their norms in the weighted space $L^2{}_w(0, c)$ are also equal. A more curious example deals with functions orthogonal in spaces of square integrable functions.

Example 4. For fixed $i \neq j$ let $f(x) = P_i(x)\,P_j(x)$ be the product of two orthogonal functions on $[0, 1]$ (e.g., orthogonal polynomials like the shifted Legendre polynomials), i.e., let

$$\int_0^1 P_i(x)\,P_j(x)\,\mathrm{d}x = 0.$$

Setting $g(x) = 1$, Theorem 1 implies that for every non-identically constant function $w(x) \in W$, there is a point $c_{ij} \in (0, 1]$ such that

$$\int_0^{c_{ij}} P_i(x)\,P_j(x)\,w(x)\,\mathrm{d}x = 0.$$

In other words, given any set of orthogonal functions in the space commonly known as $L^2(0, 1)$ (but using continuous functions for simplicity) and given any weight-function $w(x)$ satisfying the usual conditions above, there exists an interval

$[0, c_{ij}]$ such that these functions are also L^2-orthogonal with respect to the weight-function $w(x)$ on $[0, c_{ij}]$. Loosely speaking, we get that L^2-orthogonality always implies weighted L^2-orthogonality on some other intervals, or in other words, that orthogonality is pervasive in the weighted Lebesgue spaces of square integrable functions. This is a surprising result and one which is not easy to conceive. Of course, changing the pair of functions will usually change the interval (and thus the space where orthogonality prevails).

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A. B. Mingarelli, Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, 35017 Las Palmas de Gran Canaria, Spain, *e-mail*: amingare@math.carleton.ca

J. M. Pacheco, Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, 35017 Las Palmas de Gran Canaria, Spain, *e-mail*: pacheco@dma.ulpgc.es

A. Plaza, Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, 35017 Las Palmas de Gran Canaria, Spain, *e-mail*: aplaza@dmat.ulpgc.es