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NEW CLASSES OF k-UNIFORMLY CONVEX AND STARLIKE FUNCTIONS WITH RESPECT TO OTHER POINTS

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Abstract. In this paper we introduce new subclasses of k-uniformly convex and starlike functions with respect to other points. We provide necessary and sufficient conditions, coefficient estimates, distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for these classes. We also obtain integral means inequalities with the extremal functions for these classes.

1. Introduction, Definitions and Preliminaries

Let A denote the class of functions given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are regular in the unit disc $D = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions that are regular and univalent in D. Let S^* be the subclass of S consisting of functions starlike in D. It is known that $f \in S^*$ if and only if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ z \in D$.

In [**6**], Sakaguchi defined the class of starlike functions with respect to symmetric points as follows:

Let $f \in S$. Then f is said to be starlike with respect to symmetric points in D if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \qquad z \in D.$$

We denote this class by S_s^*. Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [**6**]. EL-Ashwah and Thomas in [**2**] introduced two other classes, namely the class S_c^* consisting of functions starlike with respect to conjugate points and S_{sc}^* consisting of functions starlike with respect to symmetric conjugate points.

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Motivated by S_s^*, many authors discussed the following class C_s^* of functions convex with respect to symmetric points and its subclasses (See [**4, 5, 7, 11**]).

Let $f \in S$. Then f is said to be convex with respect to symmetric points in D if and only if

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)+f'(-z)}\right\} > 0, \quad z \in D.$$

Let T denote the class consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

where a_n is a non-negative real number.

Silverman [**8**] introduced and investigated the following subclasses of T:

$$T^*(\alpha) := S^*(\alpha) \cap T \qquad \text{and} \qquad C(\alpha) := K(\alpha) \cap T \qquad (0 \le \alpha < 1).$$

In this paper we introduce the class $S_s(\lambda, k, \beta)$ of functions regular in D given by (1) and defined as follows

Definition 1.1. A function $f(z) \in A$ is said to be in the class $S_s(\lambda, k, \beta)$ if for all $z \in D$,

$$\begin{aligned} &\operatorname{Re}\left[\frac{2zf'(z)+2\lambda z^2 f''(z)}{(1-\lambda)(f(z)-f(-z))+\lambda z(f'(z)+f'(-z))}\right] \\ &\quad > k\left|\frac{2zf'(z)+2\lambda z^2 f''(z)}{(1-\lambda)(f(z)-f(-z))+\lambda z(f'(z)+f'(-z))} - 1\right| + \beta, \end{aligned} \tag{3}$$

for some $0 \le \lambda \le 1$, $0 \le \beta < 1$ and $k \ge 0$.

For suitable values of λ, k, β the class of functions $S_s(\lambda, k, \beta)$ reduces to various new classes of regular functions. We also observe that

$$S_s(0,0,0) \equiv S_s^* \qquad \text{and} \qquad S_s(1,0,0) \equiv C_s^*.$$

We now let $TS_s(\lambda, k, \beta) = S_s(\lambda, k, \beta) \cap T$.

In the present investigation of the function class $TS_s(\lambda, k, \beta)$ we obtain necessary and sufficient conditions, coefficient estimates, distortion bounds, extreme points, radii of close-to-convexity, starlikeness and convexity. We also obtain integral means inequality for the functions belonging to this class. Analogous results are also obtained for the class of functions $f \in T$ and k-uniformly convex and starlike with respect to conjugate points. The class is defined below:

Definition 1.2. A function $f(z) \in A$ is said to be in the class $S_c(\lambda, k, \beta)$ if for all $z \in D$,

$$\begin{aligned} &\operatorname{Re}\left[\frac{2zf'(z)+2\lambda z^2 f''(z)}{(1-\lambda)(f(z)+\overline{f(\bar{z})})+\lambda z(f'(z)+\overline{f'(\bar{z})})}\right] \\ &\quad > k\left|\frac{2zf'(z)+2\lambda z^2 f''(z)}{(1-\lambda)(f(z)+\overline{f(\bar{z})})+\lambda z(f'(z)+\overline{f'(\bar{z})})} - 1\right| + \beta, \end{aligned} \tag{4}$$

for some $0 \le \lambda \le 1$, $0 \le \beta < 1$ and $k \ge 0$.

Here we let $TS_c(\lambda,k,\beta) = S_c(\lambda,k,\beta) \cap T$.

We now state two lemmas which we may need to establish our results in the sequel.

Lemma 1.3. *If β is a real number and w is a complex number, then*

$$\mathrm{Re}\,(w) \geq \beta \Leftrightarrow |w + (1-\beta)| - |w - (1+\beta)| \geq 0.$$

Lemma 1.4. *If w is a complex number and β, k are real numbers, then*

$$\mathrm{Re}\,(w) \geq k|w-1| + \beta \Leftrightarrow \mathrm{Re}\,\{w(1 + k\,\mathrm{e}^{i\theta}) - k\,\mathrm{e}^{i\theta}\} \geq \beta, \qquad -\pi \leq \theta \leq \pi.$$

2. Coefficient Inequalities

We employ the technique adopted by Aqlan et al. [**1**] to find the coefficient estimates for the function class $TS_s(\lambda,k,\beta)$.

Theorem 2.1. *A function $f \in TS_s(\lambda,k,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[2(1+k)n - (k+\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_n \leq 2(1-\beta) \tag{5}$$

for $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$ *and* $k \geq 0$.

Proof. Let a function $f(z)$ of the form (2) in T satisfy the condition (5). We will show that (3) is satisfied and so $f \in TS_s(\lambda,k,\beta)$. Using Lemma 1.4 it is enough to show that

$$\mathrm{Re}\left\{\frac{2zf'(z) + 2\lambda z^2 f''(z)}{(1-\lambda)(f(z)-f(-z)) + \lambda z(f'(z)+f'(-z))}(1 + k\,\mathrm{e}^{i\theta}) - k\,\mathrm{e}^{i\theta}\right\} > \beta, \tag{6}$$
$$-\pi \leq \theta \leq \pi.$$

That is, $\mathrm{Re}\left\{\frac{A(z)}{B(z)}\right\} \geq \beta$, where

$$\begin{aligned} A(z) &:= [2zf'(z) + 2\lambda z^2 f''(z)](1 + k\,\mathrm{e}^{i\theta}) - k\,\mathrm{e}^{i\theta}[(1-\lambda)(f(z)-f(-z)) \\ &\qquad\qquad + \lambda z(f'(z)+f'(-z))], \\ B(z) &:= (1-\lambda)(f(z)-f(-z)) + \lambda z(f'(z)+f'(-z)). \end{aligned}$$

In view of Lemma 1.3, we only need to prove that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0.$$

For $A(z)$ and $B(z)$ as above, we have

$$\begin{aligned} &|A(z) + (1-\beta)B(z)| \\ &\quad = \Bigg|(4-2\beta)z - \sum_{n=2}^{\infty}[2n + (1-\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_n z^n \\ &\qquad\qquad - k\,\mathrm{e}^{i\theta}\sum_{n=2}^{\infty}[2n - (1-(-1)^n)](1-\lambda+\lambda n)a_n z^n\Bigg| \end{aligned}$$

$$\geq (4-2\beta)|z| - \sum_{n=2}^{\infty}[2n+(1-\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_n|z|^n$$
$$- k\sum_{n=2}^{\infty}[2n-(1-(-1)^n)](1-\lambda+\lambda n)a_n|z|^n.$$

Similarly, we obtain

$$\begin{aligned}|A(z)-(1+\beta)B(z)| \\ &\leq 2\beta|z| + \sum_{n=2}^{\infty}[2n-(1+\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_n|z|^n \\ &\quad + k\sum_{n=2}^{\infty}[2n-(1-(-1)^n)](1-\lambda+\lambda n)a_n|z|^n.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|A(z)+(1-\beta)B(z)| - |A(z)-(1+\beta)B(z)| \\ &\geq 4(1-\beta)|z| - 2\sum_{n=2}^{\infty}[2(1+k)n-(k+\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_n|z|^n \\ &\geq 0,\end{aligned}$$

by the given condition (5). Conversely, suppose $f \in TS_s(\lambda,k,\beta)$. Then by Lemma 1.4 we have (6). Choosing the values of z on the positive real axis the inequality (6) reduces to

$$\mathrm{Re}\left\{\frac{2(1-\beta)-\sum_{n=2}^{\infty}[2n-\beta(1-(-1)^n)](1-\lambda+\lambda n)a_nz^{n-1}}{2-\sum_{n=2}^{\infty}(1-\lambda+\lambda n)(1-(-1)^n)a_nz^{n-1}}\right.$$
$$\left.-\frac{k\,\mathrm{e}^{i\theta}\sum_{n=2}^{\infty}[2n-(1-(-1)^n)](1-\lambda+\lambda n)a_nz^{n-1}}{2-\sum_{n=2}^{\infty}(1-\lambda+\lambda n)(1-(-1)^n)a_nz^{n-1}}\right\} \geq 0.$$

In view of the elementary identity $\mathrm{Re}\,(-\mathrm{e}^{i\theta}) \geq -|\,\mathrm{e}^{i\theta}\,| = -1$, the above inequality becomes

$$\mathrm{Re}\left\{\frac{2(1-\beta)-\sum_{n=2}^{\infty}[2(1+k)n-(k+\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_nr^{n-1}}{2-\sum_{n=2}^{\infty}(1-\lambda+\lambda n)(1-(-1)^n)a_nr^{n-1}}\right\} \geq 0.$$

Letting $r \to 1^-$ we get the desired inequality (5). □

The following coefficient estimate for $f \in TS_s(\lambda,k,\beta)$ is an immediate consequence of Theorem 2.1.

Theorem 2.2. *If $f \in TS_s(\lambda,k,\beta)$, then*

$$a_n \leq \frac{2(1-\beta)}{\Phi(\lambda,k,\beta,n)}, \qquad n \geq 2$$

where $\Phi(\lambda,k,\beta,n) = (1-\lambda+\lambda n)[2(1+k)n-(k+\beta)(1-(-1)^n)]$.
The equality holds for the function

$$f(z) = z - \frac{2(1-\beta)}{\Phi(\lambda,k,\beta,n)}z^n.$$

We now state coefficient properties for the functions belonging to the class $TS_c(\lambda,k,\beta)$. Method of proving Theorem 2.3 is similar to that of Theorem 2.1.

Theorem 2.3. *A function* $f \in TS_c(\lambda,k,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty}[(1+k)n-(k+\beta)](1-\lambda+\lambda n)a_n \le (1-\beta) \tag{7}$$

for $0 \le \lambda \le 1$, $0 \le \beta < 1$ *and* $k \ge 0$.

Theorem 2.4. *If* $f \in TS_c(\lambda,k,\beta)$, *then*

$$a_n \le \frac{(1-\beta)}{\Theta(\lambda,k,\beta,n)}, \qquad n \ge 2,$$

where $\Theta(\lambda,k,\beta,n) = (1-\lambda+\lambda n)[(1+k)n-(k+\beta)]$.
The equality holds for the function

$$f(z) = z - \frac{(1-\beta)}{\Theta(\lambda,k,\beta,n)}z^n.$$

3. Distortion and Covering Theorems

Theorem 3.1. *Let* f *be defined by* (2). *If* $f \in TS_s(\lambda,k,\beta)$ *and* $|z| = r < 1$, *then we have the sharp bounds*

$$r - \frac{1-\beta}{2(1+k)(1+\lambda)}r^2 \le |f(z)| \le r + \frac{1-\beta}{2(1+k)(1+\lambda)}r^2 \tag{8}$$

and

$$1 - \frac{1-\beta}{(1+k)(1+\lambda)}r \le |f'(z)| \le 1 + \frac{1-\beta}{(1+k)(1+\lambda)}r.$$

Proof. We only prove the right side inequality in (8), since the other inequalities can be justified using similar arguments.
First, it is obvious that

$$4(1+k)(1+\lambda)\sum_{n=2}^{\infty}a_n \le \sum_{n=2}^{\infty}[2(1+k)n-(k+\beta)(1-(-1)^n)](1-\lambda+\lambda n)a_n$$

and as $f \in TS_s(\lambda,k,\beta)$, the inequality (5) yields

$$\sum_{n=2}^{\infty}a_n \le \frac{1-\beta}{2(1+k)(1+\lambda)}.$$

From (2) with $|z| = r(r < 1)$, we have

$$|f(z)| \le r + \sum_{n=2}^{\infty} a_n r^n \le r + \sum_{n=2}^{\infty} a_n r^2 \le r + \frac{1-\beta}{2(1+k)(1+\lambda)} r^2.$$

The distortion bounds in Theorem 3.1 are sharp for

$$f(z) = z - \frac{1-\beta}{2(1+k)(1+\lambda)} z^2, \qquad z = \pm r. \tag{9}$$

□

Theorem 3.2. *If $f \in TS_s(\lambda, k, \beta)$, then $f \in T^*(\delta)$, where*

$$\delta = 1 - \frac{1-\beta}{2(1+k)(1+\lambda) - (1-\beta)}$$

The result is sharp for the function given by (9).

Proof. It is sufficient to show that (5) implies

$$\sum_{n=2}^{\infty} (n-\delta) a_n \le 1 - \delta$$

that is

$$\frac{n-\delta}{1-\delta} \le \frac{[2(1+k)n - (k+\beta)(1-(-1)^n)](1-\lambda+\lambda n)}{2(1-\beta)}, \quad n \ge 2. \tag{10}$$

Since, (10) is equivalent to

$$\delta \le 1 - \frac{2(n-1)(1-\beta)}{[2(1+k)n - (k+\beta)(1-(-1)^n)](1-\lambda+\lambda n) - 2(1-\beta)} = \psi(n), \quad n \ge 2$$

and $\psi(n) \le \psi(2)$, (10) holds true for any $n \ge 2$, $k \ge 0$ and $0 \le \beta < 1$. This completes the proof of Theorem 3.2. □

For completeness, we now state the following results with regards to the class $TS_c(\lambda, k, \beta)$.

Theorem 3.3. *Let f be defined by* (2) *and $f \in TS_c(\lambda, k, \beta)$. Then for $\{z : 0 < |z| = r < 1\}$ we have the sharp bounds*

$$r - \frac{1-\beta}{(2+k-\beta)(1+\lambda)} r^2 \le |f(z)| \le r + \frac{1-\beta}{(2+k-\beta)(1+\lambda)} r^2 \tag{11}$$

and

$$1 - \frac{2(1-\beta)}{(2+k-\beta)(1+\lambda)} r \le |f'(z)| \le 1 + \frac{2(1-\beta)}{(2+k-\beta)(1+\lambda)} r.$$

The result in (11) *is sharp for the function*

$$f(z) = z - \frac{1-\beta}{(2+k-\beta)(1+\lambda)} z^2, \qquad z = \pm r. \tag{12}$$

Theorem 3.4. *If $f \in TS_c(\lambda, k, \beta)$, then $f \in T^*(\delta)$, where*

$$\delta = 1 - \frac{1-\beta}{(2+k-\beta)(1+\lambda)-(1-\beta)}.$$

The result is sharp for the function given by (12).

4. Extreme Points

Theorem 4.1. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, n)} z^n \qquad (n \geq 2),$$

where $\Phi(\lambda, k, \beta, n)$ is defined in Theorem 2.2. Then $f(z)$ is in $TS_s(\lambda, k, \beta)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Adopting the same technique used by Silverman [**8**], we first assume that

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \lambda_n \left[\frac{2(1-\beta)}{\Phi(\lambda, k, \beta, n)} z^n \right].$$

$$\sum_{n=2}^{\infty} \lambda_n \left\{ \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, n)} \right\} \cdot \left\{ \frac{\Phi(\lambda, k, \beta, n)}{2(1-\beta)} \right\} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

Therefore by Theorem 2.1, $f \in TS_s(\lambda, k, \beta)$.
Conversely, suppose $f \in TS_s(\lambda, k, \beta)$. Then by Theorem 2.2

$$a_n \leq \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, n)}, \qquad n \geq 2.$$

Now, by letting

$$\lambda_n = \left\{ \frac{\Phi(\lambda, k, \beta, n)}{2(1-\beta)} \right\} a_n, \qquad n \geq 2$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$ we conclude the theorem, since

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z).$$

□

Now, we give extreme points for functions belonging to $TS_c(\lambda, k, \beta)$. We omit the proof of Theorem 4.2 as it is similar to that of Theorem 4.1.

Theorem 4.2. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(1-\beta)}{\Theta(\lambda, k, \beta, n)} z^n \qquad (n \geq 2),$$

where $\Theta(\lambda, k, \beta, n)$ is defined in Theorem 2.4. Then $f(z)$ is in $TS_c(\lambda, k, \beta)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

5. Radii of Close-To-Convexity, Starlikeness and Convexity

Theorem 5.1. *If $f(z) \in TS_s(\lambda, k, \beta)$, then f is close-to-convex of order γ $(0 \le \gamma < 1)$ in $|z| < r_1(\lambda, k, \beta, \gamma)$, where*

$$r_1(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Phi(\lambda, k, \beta, n)}{2n(1-\beta)} \right\}^{\frac{1}{n-1}}, \qquad n \ge 2 \tag{13}$$

and $\Phi(\lambda, k, \beta, n)$ is defined in Theorem 2.2.

Proof. By a computation, we have

$$\left|f'(z) - 1\right| = \left| -\sum_{n=2}^{\infty} n a_n z^{n-1} \right| \le \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Now, f is close-to-convex of order γ if we have the condition

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\gamma} \right) a_n |z|^{n-1} \le 1. \tag{14}$$

Considering the coefficient conditions required by Theorem 2.1, the above inequality (14) is true if

$$\left(\frac{n}{1-\gamma} \right) |z|^{n-1} \le \frac{\Phi(\lambda, k, \beta, n)}{2(1-\beta)}$$

or if

$$|z| \le \left\{ \frac{(1-\gamma)\Phi(\lambda, k, \beta, n)}{2n(1-\beta)} \right\}^{\frac{1}{n-1}}, \qquad n \ge 2.$$

This last expression yields the bound required by the above theorem. □

Theorem 5.2. *If $f(z) \in TS_s(\lambda, k, \beta)$, then f is starlike of order γ $(0 \le \gamma < 1)$ in $|z| < r_2(\lambda, k, \beta, \gamma)$, where*

$$r_2(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Phi(\lambda, k, \beta, n)}{2(n-\gamma)(1-\beta)} \right\}^{\frac{1}{n-1}}, \qquad n \ge 2 \tag{15}$$

and $\Phi(\lambda, k, \beta, n)$ is defined in Theorem 2.2.

Proof. By a computation, we have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{-\sum\limits_{n=2}^{\infty} (n-1) a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \right| \\ &\le \frac{\sum\limits_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n |z|^{n-1}}. \end{aligned}$$

Now, f is starlike of order γ if we have the condition

$$\sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1-\gamma}\right)a_n|z|^{n-1}\leq 1. \tag{16}$$

Considering the coefficient conditions required by Theorem 2.1, the above inequality (16) is true if

$$\left(\frac{n-\gamma}{1-\gamma}\right)|z|^{n-1}\leq\frac{\Phi(\lambda,k,\beta,n)}{2(1-\beta)}$$

or if

$$|z|\leq\left\{\frac{(1-\gamma)\Phi(\lambda,k,\beta,n)}{2(n-\gamma)(1-\beta)}\right\}^{\frac{1}{n-1}},\qquad n\geq 2.$$

This last expression yields the bound required by the above theorem. □

Theorem 5.3. *If $f(z)\in TS_s(\lambda,k,\beta)$, then f is convex of order γ $(0\leq\gamma<1)$ in $|z|<r_3(\lambda,k,\beta,\gamma)$, where*

$$r_3(\lambda,k,\beta,\gamma)=\inf_n\left\{\frac{(1-\gamma)\Phi(\lambda,k,\beta,n)}{2n(n-\gamma)(1-\beta)}\right\}^{\frac{1}{n-1}},\qquad n\geq 2 \tag{17}$$

and $\Phi(\lambda,k,\beta,n)$ is defined in Theorem 2.2.

Proof. By a computation, we have

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum\limits_{n=2}^{\infty}n(n-1)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_nz^{n-1}}\right|\leq\frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n|z|^{n-1}}.$$

Now, f is convex of order γ if we have the condition

$$\sum_{n=2}^{\infty}\frac{n(n-\gamma)}{1-\gamma}a_n|z|^{n-1}\leq 1. \tag{18}$$

Considering the coefficient conditions required by Theorem 2.1, the above inequality (18) is true if

$$\left(\frac{n(n-\gamma)}{1-\gamma}\right)|z|^{n-1}\leq\frac{\Phi(\lambda,k,\beta,n)}{2(1-\beta)}$$

or if

$$|z|\leq\left\{\frac{(1-\gamma)\Phi(\lambda,k,\beta,n)}{2n(n-\gamma)(1-\beta)}\right\}^{\frac{1}{n-1}},\qquad n\geq 2.$$

This last expression yields the bound required by the above theorem. □

For completeness, we give, without proof, theorem concerning the radii of close-to-convexity, starlikeness and convexity for the class $TS_c(\lambda,k,\beta)$.

Theorem 5.4. *If $f(z) \in TS_c(\lambda, k, \beta)$, then f is close-to-convex of order γ $(0 \leq \gamma < 1)$ in $|z| < r_4(\lambda, k, \beta, \gamma)$, where*

$$r_4(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Theta(\lambda, k, \beta, n)}{n(1-\beta)} \right\}^{\frac{1}{n-1}}, \qquad n \geq 2 \tag{19}$$

and $\Theta(\lambda, k, \beta, n)$ is defined in Theorem 2.4.

Theorem 5.5. *If $f(z) \in TS_c(\lambda, k, \beta)$, then f is starlike of order γ $(0 \leq \gamma < 1)$ in $|z| < r_5(\lambda, k, \beta, \gamma)$, where*

$$r_5(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Theta(\lambda, k, \beta, n)}{(n-\gamma)(1-\beta)} \right\}^{\frac{1}{n-1}}, \qquad n \geq 2 \tag{20}$$

and $\Theta(\lambda, k, \beta, n)$ is defined in Theorem 2.4.

Theorem 5.6. *If $f(z) \in TS_c(\lambda, k, \beta)$, then f is convex of order γ $(0 \leq \gamma < 1)$ in $|z| < r_6(\lambda, k, \beta, \gamma)$, where*

$$r_6(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Theta(\lambda, k, \beta, n)}{n(n-\gamma)(1-\beta)} \right\}^{\frac{1}{n-1}}, \qquad n \geq 2 \tag{21}$$

and $\Theta(\lambda, k, \beta, n)$ is defined in Theorem 2.4.

6. Integral means Inequalities

In [**8**], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [**9**] and settled in [**10**], that

$$\int_0^{2\pi} |f(r\,\mathrm{e}^{i\theta})|^\eta \mathrm{d}\theta \leq \int_0^{2\pi} |f_2(r\,\mathrm{e}^{i\theta})|^\eta \mathrm{d}\theta,$$

for all $f \in T, \eta > 0$ and $0 < r < 1$. In [**10**], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman's conjecture for the class of functions $TS_s(\lambda, k, \beta)$. An analogous result is also obtained for the family of functions $TS_c(\lambda, k, \beta)$.

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [**3**].

Two given functions f and g, which are analytic in D, the function f is said to be subordinate to g in D if there exists a function w analytic in D with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in D),$$

such that

$$f(z) = g(w(z)) \qquad (z \in D).$$

We denote this subordination by $f(z) \prec g(z)$.

Lemma 6.1. *If the functions f and g are analytic in D with $f(z) \prec g(z)$, then for $\eta > 0$ and $z = r\,\mathrm{e}^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |g(r\,\mathrm{e}^{i\theta})|^\eta \mathrm{d}\theta \leq \int_0^{2\pi} |f(r\,\mathrm{e}^{i\theta})|^\eta \mathrm{d}\theta.$$

Now, we discuss the integral means inequalities for functions f in $TS_s(\lambda, k, \beta)$.

Theorem 6.2. *Let $f \in TS_s(\lambda, k, \beta)$, $\quad 0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $k \geq 0$ and $f_2(z)$ be defined by*

$$f_2(z) = z - \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, 2)} z^2,$$

where $\Phi(k, \beta, \lambda, n)$ is defined in Theorem 2.2. Then for $z = r\,\mathrm{e}^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \mathrm{d}\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \mathrm{d}\theta. \tag{22}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (22) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\eta \mathrm{d}\theta \leq \int_0^{2\pi} \left|1 - \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, 2)} z\right|^\eta d\theta.$$

By Lemma 6.1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, 2)} z.$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{2(1-\beta)}{\Phi(\lambda, k, \beta, 2)} w(z),$$

and using (5), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Phi(\lambda, k, \beta, 2)}{2(1-\beta)} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\lambda, k, \beta, n)}{2(1-\beta)} a_n \\ &\leq |z|. \end{aligned}$$

This completes the proof by Theorem 2.1. ∎

For completeness, we now give the integral means inequality for the class $TS_c(\lambda, k, \beta)$. The method of proving Theorem 6.3 is similar as that of Theorem 6.2.

Theorem 6.3. *Let $f \in TS_c(\lambda, k, \beta)$, $0 \le \lambda \le 1$, $0 \le \beta < 1$, $k \ge 0$ and $f_2(z)$ be defined by*

$$f_2(z) = z - \frac{(1-\beta)}{\Theta(\lambda, k, \beta, 2)} z^2,$$

where $\Theta(\lambda, k, \beta, n)$ is defined in Theorem 2.4. Then for $z = r\,\mathrm{e}^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \mathrm{d}\theta \le \int_0^{2\pi} |f_2(z)|^\eta \mathrm{d}\theta. \tag{23}$$

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