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STRONG STABLY FINITE RINGS AND SOME EXTENSIONS

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Dedicated to Professor Ahmad Haghany

Abstract. A ring R is called right strong stably finite (r.ssf) if for all $n \geq 1$, injective endomorphisms of $R_R^{(n)}$ are essential. If R is an r.ssf ring and e is an idempotent of R such that eR is a retractable R-module, then eRe is an r.ssf ring. A direct product of rings is an r.ssf ring if and only if each factor is so. The R.ssf condition is investigated for formal triangular matrix rings. In particular, if M is a finitely generated module over a commutative ring R such that for all $n \geq 1$, $M_R^{(n)}$ is co-Hopfian, then $\left[\begin{smallmatrix} \mathrm{End}_R(M) & M \\ 0 & R \end{smallmatrix}\right]$ is an r.ssf ring. If X is a right denominator set of regular elements of R, then R is an r.ssf ring if and only if RX^{-1} is so.

1. Introduction

All rings are associative with a unit element and all modules are unitary right modules. Rings in which right-invertibility of elements implies left-invertibility are called *directly-finite* or *Dedekind finite*. A ring R is *stably finite* (*sf* for short) if the matrix rings $M_n(R)$ are directly finite for all $n \geq 1$. The stable finiteness property is of interest in various parts of mathematics, see [**10**, § 1B]. In [**4**], Goodearl gave a characterization of rings R for which every surjective endomorphism of a finitely generated R-module is injective. Such rings form a proper subclass of sf-rings [**10**, Proposition 1.7]. Another proper subclass of sf-rings is the class of *right strong stably finite* rings (*r.ssf* for short) [**8**]. A ring R is said to be r.ssf if for every $n \geq 1$, injective endomorphisms of $R_R^{(n)}$ are essential. In [**8**], it was shown that the class of r.ssf rings is closed under Morita equivalence and r.ssf rings R satisfy the *right strong rank condition* (*r.src*) (i.e., a right R-module monomorphism $R^{(n)} \to R^{(m)}$ can exist only when $n \leq m$). The main results about r.ssf rings of [**8**], can be summarized in Figure 1. All of the implications here are not reversible.

In this paper, we will study the r.ssf condition for rings R and S where $R \subseteq S$ is a ring extension. Direct products, formal triangular matrix rings and some localization extensions are investigated. Any unexplained terminology, and all the basic results on rings and modules that are used in the sequel can be found in [**5**] and [**10**].

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$$\begin{array}{c} (\text{sf}) \\ \Uparrow \\ (\text{u.dim}(R_R) < \infty) \Rightarrow (\text{r.ssf}) \Rightarrow (\text{r.src}) \\ \Uparrow \\ \text{Commutative} \end{array}$$

Figure 1

2. Results

Recall that a module is *Hopfian* (resp. *co-Hopfian*) if any of its surjective (resp. injective) endomorphisms is an automorphism. Following [**8**], an R-module M is called *weakly co-Hopfian* (*wcH*) if every injective endomorphism of M_R is essential. We call a ring R *right wcH* if R as right R-module is wcH. We first state some results from [**8**], [**9**] which are generalizations of the fact that a ring R is right wcH if and only if every right regular element of R generates an essential right ideal in R. Recall that a module M_R is *semi-projective* if for every surjective homomorphism $f : M_R \to N_R$ with $N \leq M_R$ and every homomorphism $g : M_R \to N_R$ there exists $h \in \text{End}_R(M)$ such that $fh = g$. Also M_R is called *retractable* if $\text{Hom}_R(M, N) \neq 0$ for all $0 \neq N \leq M_R$.

Theorem 2.1. *Let M be a semi-projective retractable R-module. Then M_R is wcH if and only if $\text{End}_R(M)$ is a right wcH ring.*

Proof. By [**9**, Theorem 2.6]. □

The following result is a useful characterization of r.ssf rings whose proof is immediate from the above Theorem and [**8**, Proposition 2.7].

Theorem 2.2. *The following statements are equivalent for a ring R.*

(i) *R is an r.ssf ring.*

(ii) *For any $n \geq 1$, if $u_1, \cdots, u_n$ are R-linearly independent elements in $R_R^{(n)}$ then $u_1R + \cdots + u_nR$ is an essential submodule of $R_R^{(n)}$.*

(iii) *For any $n \geq 1$, $M_n(R)$ is a right wcH ring.*

A ring S is said to be *right Ore* if for every $a, b \in S$, with b regular, there exist $c, d \in S$, with d regular, such that $ad = bc$. Clearly, every right Ore ring in which right regular elements are regular is a right wcH ring. In [**2**, Theorem 2.5] it is proved that if $R[x]$ is right Ore, then R is an sf ring. But from some results of [**2**] and Theorem 2.2, we observe that R is in fact an r.ssf ring. We record this as below.

Theorem 2.3 (Cedo and Herbera)**.** *Let R be a ring such that $R[x]$ is right Ore. Then R is an r.ssf ring.*

Proof. Let $n \geq 1$ and set $S = \text{M}_n(R)$. By [**2**, Lemma 2.4], for all $A, B \in S$ there exist $p(x), q(x) \in S[x]$, with $q(x)$ regular, such that $(1_S - Ax)p(x) = Bq(x)$. Hence by [**2**, Lemma 2.1(ii)], S is a right wcH ring. The result is now clear by Theorem 2.2. □

Using Theorem 2.2, we will show that the class of r.ssf rings is closed under direct products. The following Lemma is needed and may be found in the literature, we give a proof for completeness.

Lemma 2.4. *Let $\{R_i\}_{i\in I}$ be any family of rings and $T = \prod R_i$ their direct product. Then for any $n \geq 1$, the rings $M_n(T)$ and $\prod_{i\in I} M_n(R_i)$ are isomorphic.*

Proof. Let $n \geq 1$, $Q = \mathrm{M}_n(T)$ and $S = \prod_{i\in I}\mathrm{M}_n(R_i)$. For any $A=[a_{rs}]_{n\times n} \in Q$, suppose that $a_{rs} = \prod_i a_{irs} \in T$ for any $r, s \in \{1, \ldots, n\}$. For each $i \in I$, let $A_i = [a_{irs}]_{n\times n} \in \mathrm{M}_n(R_i)$. Then it is easy to verify that the map $\varphi : Q \to S$ with $\varphi(A) = \prod_i A_i$ is an additive group isomorphism. To see that φ is indeed a ring homomorphism, let $B = [b_{rs}]_{n\times n} \in Q$ and set $AB = C$. Then $C = [c_{rs}]_{n\times n} \in Q$ where

$$c_{rs} = \sum_{t=1}^{n}\left(\prod_i a_{irt}\right)\left(\prod_i b_{its}\right) = \prod_i\left(\sum_{t=1}^{n} a_{irt}b_{its}\right) \in T.$$

Hence $c_{irs} = \sum_{t=1}^{n} a_{irt}b_{its}$ for all $i \in I$, $r, s \in \{1, \ldots, n\}$. Thus by definition of φ, we have

$$\varphi(C) = \prod_i C_i$$

where

$$C_i = [c_{irs}]_{n\times n} = \left[\sum_{t=1}^{n} a_{irt}b_{its}\right]_{n\times n} \in \mathrm{M}_n(R_i)$$

for all $i \in I$. On the other hand,

$$\varphi(A)\varphi(B) = \prod_i (A_iB_i) = \prod_i [a_{irs}][b_{irs}] = \prod_i \left[\left(\sum_{t=1}^{n} a_{irt}b_{its}\right)\right].$$

It follows that

$$\varphi(AB) = \varphi(A)\varphi(B).$$

Therefore φ is a ring isomorphism. □

Theorem 2.5. *Let $\{R_i\}_{i\in I}$ be any family of rings and $T = \prod R_i$ their direct product. Then T is r.ssf if and only if R_i is r.ssf for each $i \in I$.*

Proof. In view of Theorem 2.2(iii) and Lemma 2.4, we need to prove that if $\{S_i\}_{i\in I}$ is a family of rings and $Q = \prod S_i$ their direct product, then Q is a right wcH ring if and only if each S_i is a right wcH ring. Suppose that Q is a right wcH ring and $1_Q = \{e_i\}_{i\in I}$. Let $j \in I$ and x_j be a right regular element of S_j. Then the element $x = \{x'_i\}_{i\in I}$ with $x'_j = x_j$ and $x'_i = e_i$ for every $i \neq j$, is a right regular element in Q. Thus by our assumption, xQ is an essential right ideal in Q. It follows that $(x_j)S_j$ is also an essential right ideal in S_j. Hence S_j is a right wcH ring.

Conversely, let S_i be a right wcH ring for all $i \in I$. If $q = \{q_i\}_{i\in I}$ is a right regular element in Q then each q_i is a right regular element in S_i. Hence for every $i \in I$, the right ideal x_iS_i is essential in S_i. It follows that qQ is an essential right ideal in Q, proving that Q is a right wcH ring. □

Let M be an R-module. We call M, Σ-*co-Hopfian* if $M_R^{(n)}$ is co-Hopfian for all $n \geq 1$. Every quasi-injective Dedekind finite module is Σ-co-Hopfian; see for example [**8**, Proposition 1.4 and Corollary 1.5(i)]. Note that if M is any non-zero module, then any infinite direct sum of copies of M is neither Hopfian nor co-Hopfian. We investigate in Theorem 2.12, the r.ssf condition of formal triangular matrix rings $\left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ where either M_B is Σ-co-Hopfian or ${}_AM$ is flat. Such bimodules naturally arise among localizations of a ring. Let X be a right denominator set in a ring R, then RX^{-1} is a flat left R-module [**5**, Corollary 10.13]. If $X = \mathrm{C}_R(0)$, the set of all regular elements of R, then the ring RX^{-1} is called the *classical right quotient* ring of R [**10**, 10.17]. Suppose that R is a ring having a classical right quotient ring Q. Then [**6**, Theorem 2.4] shows that if Q_Q is (Σ-)co-Hopfian, then R is an r.ssf ring. The converse of this will be investigated in Theorem 2.7 where the r.ssf condition is characterized for RX^{-1} with $X \subseteq \mathrm{C}_R(0)$.

Proposition 2.6. *Let X be a right denominator set of regular elements in a ring R. Let $S = RX^{-1}$ and $n \geq 1$. Then the following statements hold.*

(i) *If $u_1, \cdots, u_n \in R^{(n)}$ are R-linearly independent, then for every $x \in X$, $u_1x^{-1}, \cdots, u_nx^{-1}$ are S-linearly independent.*
(ii) *Let $u_1x^{-1}, \cdots, u_nx^{-1} \in S^{(n)}$ be S-linearly independent, where $x \in X$ and $u_i \in R^{(n)}$. Then $u_1, \cdots, u_n$ are R-linearly independent.*
(iii) *If $u_1, \cdots, u_n \in R^{(n)}$ such that $\sum_{i=1}^n u_iR$ is an essential submodule of $R_R^{(n)}$, then for every $x \in X$, $\sum_{i=1}^n u_ix^{-1}S$ is an essential submodule of $S_S^{(n)}$.*
(iv) *Let $\sum_{i=1}^n (u_ix^{-1})S$ be an essential submodule of $S_S^{(n)}$, where $x \in X$ and $u_i \in R^{(n)}$. Then $\sum_{i=1}^n u_iR$ is an essential in $R_R^{(n)}$.*

Proof. We only prove (iv). Let $W = \sum_{i=1}^n u_iR$ and $0 \neq u \in R^{(n)}$. By hypothesis, there exists $s \in S$ such that

$$0 \neq us = u_1x^{-1}s_1 + \cdots ... + u_nx^{-1}s_n.$$

Now by [**5**, Lemma 6.1(b)], there is $y \in X$ such that sy and $(x^{-1}s_i)y$ $(1 \leq i \leq n)$ are in R. Hence, $0 \neq u(sy) \in W$. It follows that W is essential in $R^{(n)}$. □

Theorem 2.7. *Let X be a right denominator set of regular elements in a ring R. Then R is an r.ssf ring if and only if RX^{-1} is an r.ssf ring.*

Proof. Note first that for every $v_1, \cdots v_n$ elements in $S^{(n)}$, by using the common denominator property [**5**, Lemma 6.1(b)], there exist $u_i \in R^{(n)}$ and $x \in X$ such that $v_i = u_ix^{-1}$. Hence, the result is proved by Theorem 2.2(ii) and Proposition 2.6. □

Corollary 2.8.

(i) *Let $R \subseteq S$ be rings such that R is a right order in S. Then R is an r.ssf ring if and only if S is an r.ssf ring.*
(ii) *For any ring R, the ring $R[x]$ is r.ssf if and only if $R[x, x^{-1}]$ is so.*

Proof. By Theorem 2.7. □

We are now going to study formal triangular matrix rings. In the next results, A, B are two rings, and ${}_AM_B$ is a left A right B-bimodule and T is the formal triangular matrix ring $\left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$. Let $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = b \in T$, then it is easy to verify that $b = b^2$, bT is a retractable (right) T-module and $bTb \simeq B$ as rings. In [**8**, Theorem 2.6], it is proved that being right strong stably finite is a Morita invariant property. Hence, if R is an r.ssf ring and e is a *full idempotent* in R (i.e., $e^2 = e$ and $ReR = R$), then the ring eRe is r.ssf. In Proposition 2.10, we extend this fact to *retractable idempotents* "e" (i.e., eR is a retractable R-module). Note that if $0 \neq I \leq (eR)_R$ and $ReR = R$, then $I = I(ReR) = I(eR)$ is non-zero and so $\mathrm{Hom}_R(eR, I) \neq 0$. Hence, eR is a retractable R-module. As we mentioned above, in general, full idempotent elements of a ring form a proper subset of the set of retractable idempotents. The following result is needed and it is stated in [**8**] as a corollary of [**8**, Theorem 1.1], we give a direct proof for reader's convenience.

Lemma 2.9. *Any direct summand of a wcH module is a wcH module.*

Proof. Let $M = N \oplus K$ be a direct sum of modules and let M be wcH. If $f : N \to N$ is a monomorphism, then $f \oplus 1 : (N \oplus K) \to (N \oplus K)$ is also monomorphism. Hence by hypothesis, the image of $f \oplus 1$ is an essential submodule of M. It follows that $f(N)$ is an essential submodule of N, proving that N is wcH. $\square$

Proposition 2.10. *If R is r.ssf then so is eRe for every retractable idempotent $e \in R$.*

Proof. Let $n \geq 1$, $S = \mathrm{M}_n(R)$ and $M = (eR)^{(n)}$. Since e is a retractable idempotent, it is easy to verify that M_R is retractable. Under the standard Morita equivalence of R with S, the n-generated right R-module M corresponds to a cyclic retractable projective right S-module N. It follows that $N \simeq qS$ for some retractable idempotent $q \in S$. Because R is r.ssf, S_S is wcH by Theorem 2.2. Thus qS is a wcH right S-module by Lemma 2.9. Now

$$\mathrm{M}_n(eRe) \simeq \mathrm{End}_R(M) \simeq \mathrm{End}_S(N) \simeq \mathrm{End}_S(qS)$$

is a right wcH ring by Theorem 2.1. Thus eRe is an r.ssf ring by Theorem 2.2. $\square$

The following Proposition is needed for our main result about formal triangular matrix rings.

Proposition 2.11.
(i) *Let $R \subseteq S$ be rings such that ${}_RS$ is faithfully flat. If S is an r.ssf ring, then so is R.*
(ii) *Let M be a non-zero R-module and N be a submodule of M_R which is invariant under any injective endomorphism of M_R. If N_R is co-Hopfian and $(M/N)_R$ is wcH then M_R is wcH.*

Proof. (i) Let $n \geq 1$, $M = R^{(n)}$. We shall show that M is a wcH right R-module. Suppose that $f : M_R \to M_R$ is an injective homomorphism and $N \cap f(M) = 0$ for some $N \leq M_R$. Hence $N \oplus M$ embeds in M_R. Since S is flat as a left

R-module, the functor $-\otimes_R S$: Mod-$R \to$ Mod-S preserves monomorphisms. Thus $(N\otimes_R S)\oplus(M\otimes_R S)$ can be embedded in $M\otimes_R S$. It follows that $N\otimes_R S=0$ by the wcH condition on $S^{(n)}\simeq M\otimes_R S$. Therefore $N=0$ because ${}_RS$ is faithfully flat. Consequently, $f(M)$ is essential in M_R, proving that M_R is wcH.

(ii) Let $f: M\to M$ be an R-module monomorphism. By hypothesis $f(N)\subseteq N$ and hence $f(N)=N$ by the co-Hopfian condition on N_R. It follows that the map $\bar{f}: M/N\to M/N$ is an R-module monomorphism. Since M/N is wcH, the image of $\bar{f}=f(M)/N$ is an essential R-submodule of M/N. It follows that $f(M)$ is an essential submodule of M_R, proving that M_R is wcH. □

Theorem 2.12.

(i) *If A and B are r.ssf rings and M_B is Σ-co-Hopfian, then T is an r.ssf ring.*
(ii) *Let T be an r.ssf ring. Then B is an r.ssf ring. If further ${}_AM$ is flat, then A is an r.ssf ring.*

Proof. (i) Let $I=\left[\begin{smallmatrix}0&M\\0&0\end{smallmatrix}\right]$ and let $n\geq 1$. Then $T/I\simeq A\oplus B$ as rings. By hypothesis and Theorem 2.5, T/I is an r.ssf ring. Thus $(T/I)^{(n)}$ is a wcH (T/I)-module and hence as a T-module. On the other hand, $(T/I)^{(n)}\simeq T^{(n)}/I^{(n)}$ as right T-modules and $I^{(n)}$ is a fully invariant T-submodule of $T^{(n)}$. Using the hypothesis, we can also conclude that $I^{(n)}$ is a co-Hopfian right T-module. Therefore, $T^{(n)}$ is a wcH right T-module by Proposition 2.11(ii). Proving that T is an r.ssf ring.

(ii) First note that $B\simeq eTe$ for the retractable idempotent $e=\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$, hence B is r.ssf by Proposition 2.10. For the second part, using the unital embedding $(a,b)\to\left[\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right]$ of $A\times B$ in T, we can regard T as a left $A\times B$-module. Thus in view of Proposition 2.11(i) and Theorem 2.5, it is enough to show that T is a faithfully flat left $A\times B$-module. Since M is flat as a left A-module, T is flat as a left $A\times B$-module [**7**, Proposition 4.7]. Now let $R=A\times B$ and $N\otimes_R T=0$ for some R-module N. From $\left[\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right]\left[\begin{smallmatrix}0&m\\0&0\end{smallmatrix}\right]=\left[\begin{smallmatrix}0&am\\0&0\end{smallmatrix}\right]$ for any $a\in A$, $b\in B$ and $m\in M$, we see that M is a left R-submodule of T. It follows that $T\simeq R\oplus M$ as left R-modules. Hence the condition $N\otimes_R T=0$ implies that $N\otimes_R R=0$ and so $N=0$. This shows that ${}_RT$ is faithfully flat, as wanted. □

Let R be a ring and $n\geq 1$. The upper triangular $n\times n$-matrix ring over R is denoted by $\mathrm{T}_n(R)$.

Corollary 2.13. *Consider the following statements for a ring R.*

(i) *For all $n\geq 1$, $R_R^{(n)}$ is co-Hopfian.*
(ii) *For all $n\geq 1$, $T_n(R)$ is an r.ssf ring.*
(iii) *There exists $n\geq 1$ such that $T_n(R)$ is an r.ssf ring.*
(iv) *R is an r.ssf ring.*

Then (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) *and all the statements are equivalent if R is right self injective.*

Proof. By Theorem 2.12 and the fact that $\mathrm{T}_n(R) \simeq \begin{bmatrix} R & R^{(n-1)} \\ 0 & \mathrm{T}_{n-1}(R) \end{bmatrix}$. For the last statement note that if R is a right self injective r.ssf ring then for all $n \geq 1$, $R_R^{(n)}$ is injective and wcH and hence co-Hopfian. □

In [**3**], rings over which all finitely generated modules are Σ-co-Hopfian are characterized, see also [**1**, Theorem 1.1]. Hence, the following result provides extensive examples of r.ssf formal triangular matrix rings.

Theorem 2.14. *Let R be any commutative ring and let M be a finitely generated Σ-co-Hopfian R-module with $S = \mathrm{End}_R(M)$. Then the ring $\left[\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right]$ is r.ssf.*

Proof. By [**8**, Theorem 2.8], R is an r.ssf ring. Hence, by Theorem 2.12, it is enough to show that S is an r.ssf ring. We use Theorem 2.2(iii). Let $n \geq 1$, we shall show that $\mathrm{M}_n(S)$ is a right wcH ring. Set $L = M^{(n)}$ and let f be a right regular element in the ring $\mathrm{End}_R(L) \simeq \mathrm{M}_n(S)$. We will show that f is a unit element of $\mathrm{End}_R(L)$. Suppose that $K = \ker f$ is non-zero. Since f is right regular, $\mathrm{Hom}_R(L, K) = 0$. Now let $E = \mathrm{E}(K)$ be the injective hull of K. Then the inclusion map $K \to L$ can be extended to an R-module homomorphism g from L to E. Since L is a finitely generated R-module, we have $g(L) = e_1R + \cdots + e_mR$ for some positive integer m and some $e_i \in E$. On the other hand, $K^{(n)}$ is an essential submodule of $E_R^{(n)}$, see for example [**5**, Proposition 5.6]. Thus there exists $r \in R$ such that $0 \neq (e_1, \cdots, e_m)r \in K^{(n)}$. It follows that the multiplication by r defines a non-zero R-homomorphism from $g(L)$ to K. Consequently, $0 \neq \mathrm{Hom}_R(L, K)$, that is a contradiction. Therefore, $K = 0$ and hence f should be an isomorphism by the Σ-co-Hopfian condition on M, as wanted. □

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