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NONAUTONOMOUS LIVŠIC COLLIGATIONS AND HYPONORMAL OPERATORS

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Abstract. In this work, the model of a nonautonomous Livšic colligation for operators that are close to the normal is studied. The connection between the determining function for operator and the characteristic function for colligation is obtained.

In this paper the model of nonautonomous Livšic colligation is studied relying on the singular integral model for operators that are close to normal, and also its characteristic function is introduced. By means of the Plemel formula (extended to vector-valued functions) [**4**], [**1**], the Riemann-Hilbert problem that connects the boundary values of the determining function with the characteristic function of colligation is obtained.

1. Main notation and definitions

Let δ be a bounded closed set on the real axis, Λ be a set of all Borel sets that are contained in δ, m be the Lebesgue measure on δ, ν be a singular measure on Λ (i. e. there is a set $Q \in \Lambda$ such that $m(Q) = 0$ and $\nu(\delta \setminus Q) = 0$), and $\mu = m + \nu$ be σ-finite measure, $\Omega = (\delta, \Lambda, \mu)$ be a space with measure, D be a complex separable Hilbert space, $\langle a, b\rangle_D$ be a scalar product of vectors a and b in D, $L(D)$ be the algebra of all linear operators in D, $f(x) \in D$ be a vector valued (D-valued) measurable function defined for all $x \in \delta$,

$$L^2(\Omega, D) = \left\{ f \in D : \int\limits_{\delta} \|f(x)\|_D^2 \,\mathrm{d}\mu(x) < \infty \right\}$$

be the Hilbert space of measurable vector valued quadratically summable functions with the scalar product

$$\langle f, g\rangle_{L^2(\Omega,D)} = \int\limits_{\delta} \langle f(x), g(x)\rangle_D \,\mathrm{d}\mu(x), \qquad \forall f, g \in L^2(\Omega, D).$$

We identify functions $f(\cdot)$ and $g(\cdot)$ from $L^2(\Omega, D)$ if they differ only on a set of zero μ-measure.

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Let $\{0\} = D_0 \subset D_1 \subset \ldots \subset D_n \subset \ldots$ be a sequence of spaces of finite dimension in D, $F_0 \subset F_1 \subset \ldots \subset F_n \subset \ldots$ be a sequence of sets in Λ. An $L(D)$-valued measurable function $R(x)$, $x \in \delta$, is said to be the standard projector valued function [**4**] if

1. $R(x) = 0$, $x \in F_0$;
2. $R(x)$ is an orthoprojector from D into D_n if $x \in F_n \setminus F_{n-1}$, $n = 1, 2, \ldots$;
3. $R(x) = I$, $x \notin \bigcup\limits_{n \geq 0} F_n$.

If $\mu(\delta) = 1$, then the vector $a \in D$. One can identify with the vector valued function with the constant value of a. From this point of view, D is a subspace of $L^2(\Omega, D)$, and their scalar products coincide. In the general case, one can define the orthoprojector from $L^2(\Omega, D)$ into D as follows:

$$\widetilde{P}_0 : f(\cdot) \to \frac{1}{\mu(\delta)} \int\limits_{\delta} f(x)\,\mathrm{d}\mu(x).$$

Hereinafter, we will use the following operator:

$$P_0 = \mu(\delta)\,\widetilde{P}_0. \tag{1}$$

Introduce into consideration the following space:

$$H = L^2(\Omega, D, R(\cdot)) = \left\{ f \in L^2(\Omega, D) : R(\cdot) f(\cdot) = f(\cdot) \right\}. \tag{2}$$

It, as a subspace of $L^2(\Omega, D)$, also is a Hilbert space.

Let F be the vector valued Fourier transform of a function $f(\cdot)$ defined in $L^2(D)$ on the real axis, i. e. the next strong limit

$$(Ff)(x) = s - \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int\limits_{-N}^{N} f(t)\,\mathrm{e}^{\mathrm{i}\,tx}\,\mathrm{d}t.$$

It is well-known that F is a unitary operator in $L^2(D)$ on the axis. Define the following operator

$$P = F^* P_{[0,+\infty)} F$$

where $P_{[0,+\infty)}$ is the operator of multiplication by the characteristic function of semiaxis $[0, +\infty)$. By the Plemel formula, one can explicitly express the operator P [**4**], [**1**]:

$$(Pf)(x) = s - \lim_{\varepsilon \to -0} \frac{1}{2\pi\,\mathrm{i}} \int\limits_{-\infty}^{+\infty} \frac{f(t)\,\mathrm{d}t}{(x + \mathrm{i}\,\varepsilon) - t} = \frac{f(x)}{2} + \frac{1}{2\pi\,\mathrm{i}}\, v.p. \int\limits_{-\infty}^{+\infty} \frac{f(t)\,\mathrm{d}t}{x - t} \tag{3}$$

where the symbol $v.p.$ signifies the main value of singular integral.

2. Construction of singular integral model

Let $\alpha(\cdot)$ and $\beta(\cdot) = \beta^*(\cdot)$ be uniformly bounded measurable functions defined in Ω, with the values in $L(D)$, such that

$$\begin{aligned} &\alpha(\cdot) R(\cdot) = R(\cdot) \alpha(\cdot) = \alpha(\cdot), \\ &\beta(\cdot) R(\cdot) = R(\cdot) \beta(\cdot) = \beta(\cdot), \\ &\alpha(x) = 0, \qquad\qquad x \in Q, \end{aligned} \tag{4}$$

where Q is a set in δ, for which $m(Q) = 0$ and $\nu(\delta \setminus Q) = 0$.

Define the self-adjoint operator X in the Hilbert space H (2) in the following way:

$$(Xf)(x) = xf(x), \qquad \forall f \in H. \tag{5}$$

Let σ be a self-adjoint operator in D. It is easy to check that the following operator

$$(Yf)(x) = \beta(x) f(x) + \alpha^*(x) \sigma P(\alpha(\cdot) f(\cdot)), \qquad \forall f \in H, \tag{6}$$

where P is defined in (3), is self-adjoint in H (here we assume that $\alpha f \equiv 0$ outside δ, i. e. the singular integral we take by δ). Let the operators X and Y be respectively the real and the imaginary part of operator T, i. e.

$$T = X + \mathrm{i} Y. \tag{7}$$

Define the bounded linear operator from D in H:

$$Ka = \frac{1}{\sqrt{\pi}} \alpha^*(\cdot) a, \qquad \forall a \in D. \tag{8}$$

The adjoint operator K^* as an operator from H in D is given by

$$K^* f = \frac{1}{\sqrt{\pi}} P_0(\alpha(\cdot) f(\cdot)), \quad \forall f \in H, \tag{9}$$

where P_0 is defined in (1). Using the definitions of self-adjoint operators, one can calculate the self-commutator of operator T (7):

$$\langle [T^*, T] f, f \rangle_H = \frac{1}{\pi} \langle \sigma P_0(\alpha(\cdot) f(\cdot)), P_0(\alpha(\cdot) f(\cdot)) \rangle_D, \qquad \forall f \in H, \tag{10}$$

Besides, the properties of scalar product yield

$$([T^*, T] f)(x) = \frac{1}{\pi} \alpha^*(x) \sigma P_0(\alpha(\cdot) f(\cdot)), \qquad \forall f \in H. \tag{11}$$

In the terms of operator K from (8) and (9), the self-commutator of T is given by

$$[T^*, T] = 2\,\mathrm{i}\,[X, Y] = K \sigma K^*, \tag{12}$$

this implies that when $\sigma \geq 0$ the operator T is hyponormal.

In the book by Xia [**4**] it is proved that every hyponormal operator is unitarily equivalent to the operator T (5)–(7) with $\sigma = I_D$ and $\alpha(x) = \alpha^*(x) \geq 0$. Thus, the singular integral operator model is the functional model of the hyponormal operator.

Definition 2.1. An arbitrary operator A is said to be the completely non-normal (see [**4**]) if there is no subspace reducing A, on which A induces a normal operator.

Lemma 2.2. *If a singular integral operator T (7) is completely non-normal, then the singular measure $\nu \equiv 0$.*

Proof. From the contrary. Suppose that $\nu \not\equiv 0$ is concentrated in the set $Q \in \Lambda$ and $m(Q) = 0$. Consider the following subspace H:

$$N = \{f \in H : f(x) = 0, \forall x \notin Q\}.$$

Then from (4) $\alpha f \equiv 0$ for all $f \in N$, therefore

$$\begin{aligned}(Tf)(x) &= xf(x) + \mathrm{i}\,\beta(x) f(x), \qquad \forall f \in N,\\ (T^*f)(x) &= xf(x) - \mathrm{i}\,\beta(x) f(x), \qquad \forall f \in N.\end{aligned}$$

Consequently, N is a nontrivial normal subspace that reduces the operator T, it contradicts the complete non-normality of T. The lemma is proved.

3. Nonautonomous Livšic colligation

Introduce the open system $\{\Re_\Delta, S_\Delta\}$:

$$\left\{\begin{array}{ll}\Re_\Delta: & \mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}x}h(x) + \beta(x) h(x) = \alpha^*(x)\sigma u(x), \quad h(0) = h_0,\\ S_\Delta: & v(x) = u(x) + \alpha(x) h(x)\end{array}\right. \tag{13}$$

where $\alpha(x)$ and $\beta(x)$ are from the singular integral model for T.

The operators in this open system form the objects $(\beta(x), D, \beta(x), D, \sigma)$ for a Livšic node $\mathcal{C}_x$ [**2**] depending on the real parameter x. The associated input-state-output system (13) is nonautonomous [**2**].

By direct computation, it is easy to see that the following lemma is true.

Lemma 3.1. *The following energy conservation law is true:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\|h(x)\|_D^2 &= \langle \mathrm{i}\,\sigma v(x), u(x)\rangle_D - \langle \mathrm{i}\,\sigma u(x), v(x)\rangle_D\\ &= \left\langle \begin{pmatrix} 0 & \mathrm{i}\,\sigma \\ -\mathrm{i}\,\sigma & 0\end{pmatrix}\begin{pmatrix} u(x)\\ v(x)\end{pmatrix}, \begin{pmatrix} u(x)\\ v(x)\end{pmatrix}\right\rangle_{D\oplus D}.\end{aligned} \tag{14}$$

Similar to [**2**], for all the non-real z we can introduce into consideration the following D-valued function

$$S_\Delta(x,z) = I_D + \alpha(x)(\beta(x) - zI_D)^{-1}\alpha^*(x)\sigma, \tag{15}$$

which we will call the characteristic function of the colligation $\mathcal{C}_x$. Directly from definition (15), it is easy to see that the following relations are correct:

$$S_\Delta(x,z)\alpha(x) = \alpha(x)(\beta(x) - zI_D)^{-1}(\beta(x) + \alpha^*(x)\sigma\alpha(x) - zI_D), \tag{16}$$

$$\alpha^*(x)\sigma S_\Delta(x,z) = (\beta(x) + \alpha^*(x)\sigma\alpha(x) - zI_D)(\beta(x) - zI_D)^{-1}\alpha^*(x)\sigma, \tag{17}$$

$$S_\Delta(x,z)^{-1} = I_D - \alpha(x)(\beta(x) + \alpha^*(x)\sigma\alpha(x) - zI_D)^{-1}\alpha^*(x)\sigma; \tag{18}$$

$$\frac{1}{\bar{w}-z}\left[S_\Delta^*(x,w)\,\sigma-\sigma S_\Delta(x,z)\right]=\kappa^*(x,w)\,\kappa(x,z)\,, \qquad \kappa(x,z)=\left(\beta(x)-zI_D\right)^{-1}\alpha^*(x)\,\sigma. \tag{19}$$

Let the operator σ be invertible. For all the non-real z and w define the following operator valued function $E(z,w):D\to D$:

$$E(z,w)=\sigma^{-1}+\frac{1}{2\,\mathrm{i}}K^*\left(X-zI_H\right)^{-1}\left(Y-wI_H\right)^{-1}K, \tag{20}$$

which we will call the determining function of the operator T. It is easy to see that the determining function satisfies the following relations:

$$K\sigma E(z,w)=\left\{\left(X-zI_H\right),\left(Y-wI_H\right)\right\}K, \tag{21}$$

$$E(z,w)\,\sigma K^*=K^*\left\{\left(X-zI_H\right)^{-1},\left(Y-wI_H\right)^{-1}\right\}, \tag{22}$$

$$E(z,w)^{-1}=\sigma-\frac{1}{2\,\mathrm{i}}\sigma K^*\left(Y-wI_H\right)^{-1}\left(X-zI_H\right)^{-1}K\sigma, \tag{23}$$

where $\{A,B\}=ABA^{-1}B^{-1}$ is a multiplicative commutator of operators A and B.

Denote $f(x,w;a)=\left(Y-wI_H\right)^{-1}\alpha^*(\cdot)\,a$. It follows from (6) that for each $a\in D$

$$\left(Y-\beta(\cdot)\right)f(\cdot,w;a)=\alpha^*(\cdot)\,\sigma P_-\left[\alpha(\cdot)\,f(\cdot,w;a)\right],$$

where $P_-=P$ is the singular integral operator (3). Since

$$\left(Y-wI_H\right)f(\cdot,w;a)=\alpha^*(x)\,a,$$

then it is obvious that

$$\left(\beta(\cdot)-wI_D\right)f(\cdot,w;a)+\alpha^*(\cdot)\,\sigma P_-\left[\alpha(\cdot)\,f(\cdot,w;a)\right]=\alpha^*(\cdot)\,a.$$

Multiply both sides from the left by $\alpha(\cdot)\left(\beta(\cdot)-wI_D\right)^{-1}$ and denote $P_+=I_H-P_-$. Then

$$\begin{aligned}\left(P_++P_-\right)\left[\alpha(\cdot)\,f(\cdot,w;a)\right]+\alpha(\cdot)\left(\beta(\cdot)-wI_D\right)^{-1}\alpha^*(\cdot)\,\sigma\\ \times P_-\left[\alpha(\cdot)\,f(\cdot,w;a)\right]=\alpha(\cdot)\left(\beta(\cdot)-wI_D\right)^{-1}\alpha^*(\cdot)\,\sigma\sigma^{-1}a.\end{aligned}$$

Add $\sigma^{-1}a$ to the both sides and take (15) into account. Hence

$$\sigma^{-1}a+P_+\left[\alpha(\cdot)\,f(\cdot,w;a)\right]=S(\cdot,w)\left[\sigma^{-1}a-P_-\left[\alpha(\cdot)\,f(\cdot,w;a)\right]\right]. \tag{24}$$

Using (5), (8) and (9), one can write the determining function as an integral

$$E(z,w)\,a=\sigma^{-1}a+\frac{1}{2\pi\,\mathrm{i}}\int\limits_\delta\frac{\alpha(x)\,f(x,w;a)}{x-z}\mathrm{d}x, \tag{25}$$

whence by the Plemel formulae [**1**] one obtains that for all real x

$$E(x\pm\mathrm{i}\,0,w)\,a=\sigma^{-1}a\pm P_\pm\left[\alpha(\cdot)\,f(\cdot,w;a)\right]. \tag{26}$$

Thus, from (24) and (25) we finally obtain that the determining function satisfies the following Riemann-Hilbert problem :

$$E(x+\mathrm{i}\,0,w)=S(x,w)\,E(x-\mathrm{i}\,0,w)\,. \tag{27}$$

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