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VALUATIONS ON THE RING OF ARITHMETICAL FUNCTIONS

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Abstract. In this paper we study a class of nontrivial independent absolute values on the ring A of arithmetical functions over the field $\mathbb{C}$ of complex numbers. We show that A is complete with respect to the metric structure obtained from each of these absolute values. We also consider an Artin-Whaples type theorem in this context.

1. Introduction

Let A denote the set of complex valued arithmetical functions $f : \mathbb{N} \to \mathbb{C}$, where $\mathbb{N}$ is the set of positive integers. For $f, g \in A$ their Dirichlet convolution is defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

for $n \in \mathbb{N}$. A is a ring with the usual addition of functions and Dirichlet convolution. It is known that A is a unique factorization domain. This was proved by Cashwell and Everett [**5**]. Schwab and Silberberg [**7**] constructed an extension of A which is a discrete valuation ring, and in [**8**], they showed that A is a quasi-noetherian ring. Yokom [**9**] investigated the prime factorization of arithmetical functions in a certain subring of the regular convolution ring. He also determined a discrete valuation subring of the unitary ring of arithmetical functions. Some questions on the structure of the ring of arithmetical functions in several variables have been recently investigated by Alkan and the authors in [**1**], [**2**], [**3**]. Our aim in the present paper is to construct an infinite class of valuations on A which are independent of each other. To keep the exposition short and simple, we will restrict to the case of arithmetical functions of one variable, with values in $\mathbb{C}$. We construct these valuations as follows. Let P be the set of prime numbers. Fix a weight function $w : P \to \mathbb{R}$ such that for all $p \in P$, $w(p) \geq 0$. Given $n \in \mathbb{N}$ with prime factorization $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, we define $\Omega_w(n) = \alpha_1 w(p_1) + \dots + \alpha_k w(p_k)$. Also for $f \in A$, let $\mathrm{supp}(f)$ denote the support of f, so $\mathrm{supp}(f) = \{n \in \mathbb{N} | f(n) \neq 0\}$,

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and define

$$V_w(f) = \inf_{n \in \mathrm{supp}(f)} \Omega_w(n),$$

with the convention $\min(\emptyset) = \infty$. Then V_w is a valuation on A. Next, we extend V_w to a valuation, also denoted by V_w, on the field of fractions $\mathbb{K} = \left\{\frac{f}{g} | f, g \in A, g \neq 0\right\}$ of A by letting $V_w\left(\frac{f}{g}\right) = V_w(f) - V_w(g)$. We also fix a number $\rho \in (0,1)$ and define an absolute value $|.|_w : \mathbb{K} \to \mathbb{R}$ by

$$|x|_w = \rho^{V_w(x)} \text{ if } x \neq 0, \qquad \text{and} \qquad |x|_w = 0 \text{ if } x = 0.$$

In Section 2 we show that V_w is indeed a valuation, and so $|.|_w$ is a non-archimedian absolute value. In Section 3 we show that A is complete with respect to the metric structure obtained from the absolute value $|.|_w$.

Lastly, we take a finite number $w_1, \ldots, w_s$ of weight functions on P for which the absolute values $|.|_{w_1}, \ldots, |.|_{w_s}$ are independent, and consider the completions $\mathbb{K}_{w_1}, \ldots, \mathbb{K}_{w_s}$ of $\mathbb{K}$ with respect to $|.|_{w_1}, \ldots, |.|_{w_s}$. Define the function $\psi : \mathbb{K} \to \mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ by $x \to \psi(x) = (x, \ldots, x)$. By the Artin-Whaples Theorem [**4**], we know that the topological closure of $\psi(\mathbb{K})$ in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ coincides with $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. Since we are more interested in the ring A than in its field of fractions $\mathbb{K}$, a natural question to ask is what the topological closure of the image $\psi(A)$ of A under ψ is in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. We show that this topological closure is $\psi(A)$ itself.

2. Absolute Values

Theorem 1.

(i) *For any $f, g \in A$, we have*

$$V_w(f+g) \geq \min(\{V_w(f), V_w(g)\}).$$

(ii) *For any $f, g \in A$, we have*

$$V_w(f * g) = V_w(f) + V_w(g).$$

Proof. (i) Let $f, g \in A$. Since $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$, we get that for any $n \in \mathrm{supp}(f+g)$, either $n \in \mathrm{supp}(f)$, or $n \in \mathrm{supp}(g)$. Thus we have that $\Omega_w(n) \geq V_w(f)$, or $\Omega_w(n) \geq V_w(g)$ for any $n \in \mathrm{supp}(f+g)$. So, it follows immediately that

$$V_w(f+g) \geq \min(\{V_w(f), V_w(g)\}).$$

(ii) Again let $f, g \in A$. Let $n \in \mathrm{supp}(f)$, and $m \in \mathrm{supp}(g)$. Suppose that k, and l satisfy the equations $\Omega_w(n) = k$, and $\Omega_w(m) = l$ respectively. Also assume that $V_w(f) = k$ and $V_w(g) = l$. Now,

$$V_w(f) + V_w(g) = k + l.$$

Let a be a positive integer such that $a \in \mathrm{supp}(f * g)$. Then,

$$0 \neq (f * g)(a) = \sum_{d|a} f(d) g\left(\frac{a}{d}\right).$$

Therefore $f(d) \neq 0$, and $g\left(\frac{a}{d}\right) \neq 0$ for some $d|a$. It follows that for any $a \in \operatorname{supp}(f * g)$,

$$\begin{aligned} V_w(f) + V_w(g) &\leq \Omega_w(d) + \Omega_w\left(\frac{a}{d}\right) \\ &= \Omega_w(a). \end{aligned}$$

So, $V_w(f) + V_w(g) \leq V_w(f * g)$.

To show the reverse inequality, we first define the following two sets.

$$\mathfrak{C}_f = \{a \in \mathbb{N} : f(a) \neq 0 \quad \text{and} \quad \Omega_w(a) = k\}$$

and

$$\mathfrak{C}_g = \{b \in \mathbb{N} : g(b) \neq 0 \quad \text{and} \quad \Omega_w(b) = l\}.$$

Let n be the smallest element of $\mathfrak{C}_f$. Also let m be the smallest element of $\mathfrak{C}_g$. Denote $u = nm$. We have that

$$V_w(f) + V_w(g) = \Omega_w(n) + \Omega_w(m) = \Omega_w(u).$$

So if we show that $(f*g)(u) \neq 0$, then we will be done. To show that $(f*g)(u) \neq 0$, we consider the identity

$$(f * g)(u) = \sum_{de=nm} f(d)g(e)$$

and show that all terms in this sum vanish except for the term $f(n)g(m)$ which is nonzero. Suppose that $f(d)g(e)$ is a nonzero term of the sum. Then note that none of the inequalities $\Omega_w(d) < k$ and $\Omega_w(e) < l$ can hold since otherwise the term $f(d)g(e)$ is zero. Also observe that if $\Omega_w(d) > k$, then $\Omega_w(e) < l$ and the latter inequality cannot hold as we have seen above. Similarly if $\Omega_w(e) > l$, then $\Omega_w(d) < k$ and again the latter inequality cannot hold. We conclude that $\Omega_w(d) = k$ and $\Omega_w(e) = l$. It follows that $d \in \mathfrak{C}_f$, and $e \in \mathfrak{C}_g$. Since we have $d \leq n$ and $e \leq m$, it is clear from the definition of n and n that $d = n$, and $e = m$. Hence, if $f(d)g(e)$ is a nonzero term of the sum, then it follows that $d = n$, and $e = m$. Thus (ii) holds, and this completes the proof of the theorem. $\square$

It follows from the above theorem and [**6**, Proposition 3.1.10] that $|.|_w$ is a non-archimedean absolute value on $\mathbb{K}$.

3. Completeness and Topological Closure

Define a distance d_w on $\mathbb{K}$ by putting for $x, y \in \mathbb{K}$, $d_w(x, y) = |x-y|_w$, and consider also the restriction of this distance to A.

Theorem 2. *The metric space (A, d_w) with respect to the distance d_w defined above is complete.*

Proof. Let $(f_n)_{n\geq 0}$ be a Cauchy sequence in A. Then for each $\varepsilon > 0$, there exists $N = N_\epsilon \in \mathbb{N}$ such that $|f_m - f_n|_w < \epsilon$ for all $m, n \geq N_\varepsilon$. For each $k \in \mathbb{N}$, taking $\varepsilon = \rho^k$, there exists $N_k \in \mathbb{N}$ such that $|f_m - f_n|_w < \rho^k$ for all

$m, n \geq N_k$. Equivalently, $V_w(f_m - f_n) > k$ for all $m, n \geq N_k$, i.e., we have that for all $m, n \geq N_k$,

$$f_m(l) = f_n(l)$$

whenever $\Omega_w(l) \leq k$, for all $l \in \mathbb{N}$. We choose for each $k \in \mathbb{N}$, the smallest natural number N_k with the above property such that

$$N_1 < N_2 < \ldots < N_k < N_{k+1} < \ldots.$$

Let us define $f : \mathbb{N} \to \mathbb{C}$ as follows. Given $l \in \mathbb{N}$, let k be the smallest positive integer such that $k > \Omega_w(l)$. We set $f(l) = f_{N_k}(l)$. Then f is the limit of the sequence $(f_n)_{n \geq 0}$. This completes the proof of Theorem 2. $\square$

Remark 1. *Let w, w' be weight functions on P. If the absolute values $|.|_w, |.|_{w'}$, correspondingly the valuations $V_w, V_{w'}$, arising from w and w' respectively are dependent, then there exists a constant C such that $w(p) = Cw'(p)$ for all primes p.*

Proof. Let w, w' be weight functions on P. If $V_w, V_{w'}$ are dependent on $\mathbb{K}$, then there exists a constant C such that $V_w(x) = CV_{w'}(x)$ for all $x \in \mathbb{K}$. For each prime number p, define $\delta_p \in A$ by

$$\delta_p(n) = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{else} \end{cases}$$

for all $n \in \mathbb{N}$. Then $V_w(\delta_p) = w(p)$, and $V_{w'}(\delta_p) = w'(p)$. Hence $w(p) = Cw'(p)$ for all primes p as claimed. $\square$

We have seen that each weight function $w : P \to \mathbb{R}$ on the set P of prime numbers gives rise, via the function $|.|_w$, to an absolute value on $\mathbb{K}$. We denote by $\mathbb{K}_w$ the completion of $\mathbb{K}$ with respect to the absolute value $|.|_w$.

Let $s > 0$ be an integer. Let $w_1, \ldots, w_s$ be weight functions on P. Suppose that the absolute values $|.|_{w_1}, \ldots, |.|_{w_s}$, corresponding to the valuations $V_{w_1}, \ldots, V_{w_s}$ on $\mathbb{K}$, are independent. Consider the product topology on $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. Define the function $\psi : \mathbb{K} \to \mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ by $x \to \psi(x) = (x, \ldots, x)$. Then the topological closure of $\psi(\mathbb{K})$ in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ coincides with $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. We would like to identify the topological closure of the image $\psi(A)$ of A under ψ in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. For a subset F of $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ we denote by $\overline{F}$ the topological closure of F in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. Then $\overline{\psi(A)} \subseteq A \times \cdots \times A$ since A is complete with respect to each absolute value $|.|_{w_i}$ $(i = 1, \ldots, s)$.

Theorem 3. *The topological closure of $\psi(A)$ in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ is $\psi(A)$ itself.*

Proof. We have already seen that $\overline{\psi(A)} \subseteq A \times \cdots \times A$. Let $f_1, \ldots, f_s \in A \times \cdots \times A$, and assume that $(f_1, \ldots, f_s) \in \overline{\psi(A)}$. We want to show that $(f_1, \ldots, f_s) \in \psi(A)$. By the above assumption we know that there exists a sequence $(h_n)_{n \in \mathbb{N}}$ in A such that $\psi(h_n)$ converges to $(f_1, \ldots, f_s)$. So for each $i \in \{1, \ldots, s\}$, we have that $|h_n - f_i|_{w_i} \to 0$ as $n \to \infty$, correspondingly, $V_{w_i}(h_n - f_i) \to \infty$ as $n \to \infty$.

Fix $m = p_1^{\alpha_1} \cdots P_u^{\alpha_u}$. Also fix $j \in \{1, \dots, s\}$. Since $V_{w_i}(h_n - f_i) \to \infty$ as $n \to \infty$, there exists $N_j \in \mathbb{N}$ such that

$$V_{w_j}(h_n - f_j) > \alpha_1 w_j(p_1) + \alpha_2 w_j(p_2) + \cdots + \alpha_u w_j(p_u) \qquad \text{for all } n > N_j.$$

for all $n > N_j$. Let N be the maximum of $N_1, \dots, N_s$. Then, for all $j \in \{1, \dots, s\}$, and all $n > N$, we have that

$$V_{w_j}(h_n - f_j) > \alpha_1 w_j(p_1) + \alpha_2 w_j(p_2) + \cdots + \alpha_u w_j(p_u).$$

Thus, $m \notin \operatorname{supp}(h_n - f_j)$ and therefore, $h_n(m) = f_j(m)$ for all $j \in \{1, \dots, s\}$, and all $n > N$. Hence, $f_1(m) = f_2(m) \cdots = f_s(m)$. Since m is arbitrary, it follows that the arithmetical functions $f_1, f_2, \dots, f_s$ are identical, and hence $(f_1, \dots, f_s) \in \psi(A)$. This completes the proof of the theorem. $\square$

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