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CUBIC EDGE-TRANSITIVE GRAPHS OF ORDER $4p^2$

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ABSTRACT. A regular graph Γ is said to be semisymmetric if its full automorphism group acts transitively on its edge-set but not on its vertex-set. It was shown by Folkman [**5**] that a regular edge-transitive graph of order $2p$ or $2p^2$ is necessarily vertex-transitive, where p is a prime. In this paper, it is proved that there is no connected semisymmetric cubic graph of order $4p^2$, where p is a prime.

1. INTRODUCTION

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ its vertex set, edge set, arc set and full automorphism group, respectively. For u, $v \in V(\Gamma)$, denote by uv the edge incident to u and v in Γ, and by $N_\Gamma(u)$ the *neighborhood* of u in Γ, that is, the set of vertices adjacent to u in Γ. If a subgroup G of $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, we say that Γ is *G-vertex-transitive*, *G-edge-transitive* and *G-arc-transitive*, respectively. In the special case, when $G = \mathrm{Aut}(\Gamma)$ we say that Γ is vertex-transitive, edge-transitive and arc-transitive (or *symmetric*), respectively. A regular G-edge-transitive but not G-vertex-transitive graph, will be referred to as a *G-semisymmetric graph*. In particular, if $G = \mathrm{Aut}(\Gamma)$, then the graph Γ is said to be semisymmetric.

Let N be a subgroup of $\mathrm{Aut}(\Gamma)$. The *quotient graph* Γ/N or Γ_N of Γ relative to N is defined as the graph such that the set Σ of N-orbits in $V(\Gamma)$ is the vertex set of Γ/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $uv \in E(\Gamma)$.

A graph $\widetilde{\Gamma}$ is called a *covering* of a graph Γ with projection $\wp : \widetilde{\Gamma} \to \Gamma$, if $\wp$ is a surjection from $V(\widetilde{\Gamma})$ to $V(\Gamma)$ such that $\wp\,|_{N_{\widetilde{\Gamma}}(\tilde{v})} : N_{\widetilde{\Gamma}}(\tilde{v}) \to N_\Gamma(v)$ is a bijection for any vertices $v \in V(\Gamma)$ and $\tilde{v} \in \wp^{-1}(v)$. The *fibre* of an edge or a vertex is its preimage under $\wp$. If $\widetilde{\Gamma}$ is connected, then any two vertex or edge fibres are of the same cardinality n. This number is called the *fold number* of the covering and we say that $\wp$ is an n-fold covering. A covering $\widetilde{\Gamma}$ of Γ with a projection $\wp$ is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism

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group $\mathrm{Aut}(\widetilde{\Gamma})$ such that graph Γ is isomorphic to the quotient graph $\widetilde{\Gamma}/K$, say by h, and the quotient map $\widetilde{\Gamma} \to \widetilde{\Gamma}/K$ is the composition $\wp h$ of $\wp$ and h.

Covering techniques have been known as a powerful tool in topology and graph theory for a long time. The study of semisymmetric graphs was initiated by Folkman [**5**]. There is given a classification of semisymmetric graphs of order $2pq$ in [**4**], where p and q are distinct primes. Semisymmetric cubic graphs of orders $2p^3$ and $6p^2$ are classified in [**8, 7**], and also in [**1**] it is proved that every edge-transitive cubic graph of order $8p^2$, where p is a prime, is vertex-transitive. In [**3**], an overview of known families of semisymmetric cubic graphs is given.

In this paper, we investigate semisymmetric cubic graphs of order $4p^2$, where p is a prime. The following is the main result of this paper.

Theorem 1.1. *Let p be a prime. Then there is no connected semisymmetric cubic graph of order $4p^2$.*

2. Primary Analysis

The following proposition is a special case of [**7**, Lemma 3.2].

Proposition 2.1. *Let Γ be a connected G-semisymmetric cubic graph with bipartition sets $U(\Gamma)$ and $W(\Gamma)$, where $G \leq \mathrm{Aut}(\Gamma)$. Moreover, suppose that N is a normal subgroup of G. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(\Gamma)$ and $W(\Gamma)$, and Γ is an N-regular covering of an G/N-semisymmetric graph.*

We quote the following propositions.

Proposition 2.2. [**8**, Proposition 2.4] *The vertex stabilizers of a connected G-edge-transitive cubic graph Γ have order $2^r \cdot 3$, $r \geq 0$. Moreover, if u and v are two adjacent vertices, then $|G : \langle G_u, G_v\rangle| \leq 2$ and the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

Proposition 2.3 ([**9**]). *Every both edge-transitive and vertex-transitive cubic graph is symmetric.*

Proposition 2.4 ([**2**]). *If $\widetilde{\Gamma}$ is a bipartite covering of a non-bipartite graph Γ, then the fold number is even.*

3. Proof of Theorem 1.1

Lemma 3.1. *Suppose that Γ is a connected semisymmetric cubic graph of order $4p^2$, where $p \geq 11$ is an odd prime. Set $A := \mathrm{Aut}(\Gamma)$. Moreover, suppose that $Q := O_p(A)$ is the maximal normal p-subgroup of A. Then $|Q| = p^2$.*

Proof. Let Γ be a semisymmetric cubic graph of order $4p^2$ and set $A := \mathrm{Aut}(\Gamma)$. Then Γ is a bipartite graph. Denote by $U(\Gamma)$ and $W(\Gamma)$ the bipartition sets of Γ, where $|U(\Gamma)| = |W(\Gamma)| = 2p^2$. By Proposition 2.2, $|A| = 2^r 3p^2$, where $r \geq 1$ as A is transitive on the bipartition sets of Γ of size $2p^2$. We claim that A is solvable. Otherwise, by the classification of finite simple groups, its composition factors would have to be an A_5 or $PSL(2,7)$ (see [**6**]), which is a contradiction to order of A. Let $Q := O_p(A)$ be the maximal normal p-subgroup of A. We will show that $|Q| = p^2$.

First, suppose that $|Q| = 1$. Let N be a minimal normal subgroup of A. By solvability of A, N is solvable and so N is elementary Abelian. Therefore, N is intransitive on each of the both bipartition sets $U(\Gamma)$ and $W(\Gamma)$, and hence by Proposition 2.1, N acts semiregularly on $U(\Gamma)$ (also on $W(\Gamma)$). Therefore, $|N| = 2$. Now, we consider the quotient graph Γ_N of Γ relative to N, where Γ_N is A/N-semisymmetric. We have $|U(\Gamma_N)| = |W(\Gamma_N)| = p^2$. Let M/N is a minimal normal subgroup of A/N. Since A/N is solvable, M/N is also solvable and hence is elementary Abelian. It is easy to check that $|M/N| = p$ or p^2. So it follows that the order of normal subgroup M of A is equal to $2p$ or $2p^2$. Suppose that P is a Sylow p-subgroup of M. Then one can see that P is normal and hence is characteristic in M. Therefore, A has a normal subgroup of order p or p^2. It is a contradiction, and thus $|Q| \neq 1$.

Now, suppose that $|Q| = p$. Let Γ_Q be the quotient graph of Γ relative to Q, where Γ_Q is A/Q-semisymmetric. We have $|U(\Gamma_Q)| = |W(\Gamma_Q)| = 2p$. Suppose that N/Q is a minimal normal subgroup of A/N. Similar to before, N/Q is elementary Abelian. So by Proposition 2.1, N/Q is semiregular on each of the both bipartition sets $U(\Gamma_Q)$ and $W(\Gamma_Q)$ and hence $|N/Q| = 2$. Now, suppose that Γ_N is the quotient graph Γ relative to N with $|U(\Gamma_N)| = |W(\Gamma_N)| = p$, where Γ_N is A/N-semisymmetric. Further, let M/N be a minimal normal subgroup of A/N. Then as above, we must have $|M/N| = p$ and hence M is a normal subgroup of A of order $2p^2$. Therefore, A has a normal subgroup of order p^2. Now we can get a contradiction. The result now follows. □

Proof of Theorem 1.1. Suppose to the contrary that Γ is a (connected) semisymmetric cubic graph of order $4p^2$. By [**3**], there is no semisymmetric cubic graph of order $4p^2$, where $p \leq 7$. We can assume that $p \geq 11$ is an odd prime. By Lemma 3.1, $Q := O_p(A)$ is of order p^2. So by Proposition 2.1, Γ is a Q-covering of A/Q-semisymmetric graph Γ_Q, where Γ_Q is an edge-transitive cubic graph of order 4. But by [**3**] and Proposition 2.3, Γ_Q is symmetric. Hence Γ_Q is the complete graph K_4. Since Γ is bipartite, K_4 is non-bipartite and also p^2 is odd, we come to a contradiction to Proposition 2.4. Thus the proof of Theorem 1.1 is completed. □

By Theorem 1.1, Theorem 2 of [**5**], Theorem 1.1 of [**1**] and Proposition 2.4, we have the following corollary

Corollary 3.2. *Every connected edge-transitive cubic graph of order* $2^\alpha p^2$ *is symmetric, where* $\alpha \in \{1,2,3\}$ *and* p *is a prime.*

Now one may ask the following problem.

Problem 3.3. Classify all connected semisymmetric cubic graphs of order $2^\alpha p^n$, where p is a prime and $n, \alpha \geq 1$.

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