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MYTHICAL NUMBERS

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Dedicated to Dušan Dudák

1. Introduction

In this paper we show that the three prime numbers 3, 7 and 13, which repeatedly occur in various myths, in the Bible, in fables and fairy tales, possess a remarkable property, distinguishing them from other integers.

The n-th prime is denoted as usual by p_n; additionally we put $p_0 = 1$. In case of a more complicated argument we sometimes use the alternative notation $P(n) = p_n$. Further we denote by

$$S(x) = \sum_{p \le x} p$$

the sum of all primes less than or equal to any real number x. Hence

$$S(x) = \sum_{i=1}^{n} p_i$$

where p_n is the biggest prime less than or equal to x.

Let us write in a table the initial segments of the following four sequences: the nonnegative integers n, the prime numbers p_n in their natural order, the sequence $P(p_n)$ of the prime numbers with prime subscripts, and the sequence $S(p_n)$ of sums of primes up to the n-th prime p_n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
p_n	1	2	3	5	7	11	16	17	19	23	29	31	37	41
$P(p_n)$	2	3	5	11	17	31	41	59	67	83	109	127	157	179
$S(p_n)$	0	2	5	10	17	28	41	58	77	100	129	160	297	138

Table 1

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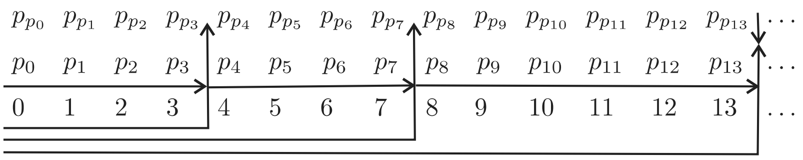
We can see that the items $P(p_n)$ and $S(p_n)$ in the third and fourth row of Table 1 coincide just for $n = 2$, $n = 4$ and $n = 6$. This means that by adding all the prime numbers up to $x = p_n$ in the second row of the table we obtain the prime number $p_x = P(x)$ in the third row just for $x = 3$, $x = 7$ and $x = 13$. Strengthening this observation to all positive integers x leads us to the formulation of the theorem, which will be proved in what follows.

Theorem. *The prime numbers* 3, 7 *and* 13 *are the only integers* $x \geq 1$ *satisfying the equation*

$$\sum_{p \leq x} p = p_x. \tag{1.1}$$

In view of the following scheme



it may seem natural to refer to integers x satisfying (1.1) as to "transcending". However, since these three solutions are exactly the primes 3, 7, 13, we shall call them "mythical". The verification that they satisfy (1.1) is straightforward:

$$\begin{aligned} 2 + 3 = 5 &= p_3, \\ 2 + 3 + 5 + 7 = 17 &= p_7, \\ 2 + 3 + 5 + 7 + 11 + 13 = 41 &= p_{13}. \end{aligned}$$

Before we prove the Theorem, let us discuss the "mythical trinity" more closely. Recall that $3 = p_2$, $7 = p_4$ and $13 = p_6$. This means that the number 3 is doubly distinguished by (1.1). Namely, it satisfies (1.1) (hence, it "transcends"), on the other hand, equation (1.1) is satisfied exactly by the first *three* primes with even indices p_2, p_4 and p_6.

Looking at the four sequences n, p_n, $P(p_n)$ and $S(p_n)$, again, we may try to iterate the idea of "transcending". More precisely, we modify (1.1) as follows:

$$\sum_{i=0}^{x} p_i = P(p_x), \tag{1.2}$$

where we now include the summand $p_0 = 1$ which was not included in (1.1). It is natural to consider (1.2) only for prime numbers x which satisfy (1.1):

$$\sum_{i=0}^{3} p_i = 1+2+3+5 = 11 = p_5 = P(p_3),$$
$$\sum_{i=0}^{7} p_i = 1+2+3+5+7+11+13+17 = 59 = p_{17} = P(p_7),$$
$$\sum_{i=0}^{13} p_i = 239 = p_{52} \neq 179 = p_{41} = P(p_{13}).$$

In conclusion, the number 13 does not satisfy the "second order transcendency" equation (1.2). Hence, from the trinity which advanced from the first round, the number 13 fails to transcend again. This perhaps could justify the belief in the "unlucky" 13.

To finish the introduction, let us have a look at the "third order transcendency". The corresponding equation reads as follows:

$$\sum_{i=0}^{x} p_{p_i} = P(p_{p_x}). \tag{1.3}$$

If we consider the validity of (1.3) for $x = 3$ and $x = 7$, we find that

$$\sum_{i=0}^{3} p_{p_i} = 2+3+5+11 = 21 \neq P(p_{p_3}) = p_{11} = 31, \tag{1.4}$$
$$\sum_{i=0}^{7} p_{p_i} = 2+3+5+11+17+31+41+59 = 169 \neq P(p_{p_7}) = p_{59} = 277. \tag{1.5}$$

This means that while only the number 13 fails at the "second transcendency", the remaining two numbers fail at the "third transcendency". However, it is worth noticing that the sums yield $21 = 3 \cdot 7$ in (1.4), and $169 = 13^2$ in (1.5). Thus the results can be expressed by means of numbers from our trinity, again.

2. Proof of the Theorem

Let $\pi(x)$ denote the number of prime numbers which are less than or equal to x, $\log x$ be the natural logarithm of x, and $\lfloor x \rfloor$ be the (lower) integer part of x. Then

$$\sum_{p \le x} p = \sum_{n=1}^{x} (\pi(n) - \pi(n-1))\, n = -\sum_{n=1}^{x} \pi(n) + \pi(x)(\lfloor x \rfloor + 1). \tag{2.1}$$

Using the following inequalities (see [**1**, page 228])

$$\begin{aligned} \pi(x) &< \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right) \qquad \text{for } x > 1, \\ \pi(x) &> \frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \qquad \text{for } x \ge 59, \end{aligned} \tag{2.2}$$

we obtain from (2.1) that

$$\begin{aligned}
\sum_{p\le x} p &= -\int_2^x \pi(u)\mathrm{d}u + \pi(x)(\lfloor x\rfloor + 1)\\
&> -\int_2^x \frac{u}{\log u}\left(1+\frac{3}{2\log u}\right)\mathrm{d}u + \pi(x)(\lfloor x\rfloor + 1)\\
&> -\int_2^x \frac{u\mathrm{d}u}{\log u} - \frac{3}{2}\int_2^x \frac{u\mathrm{d}u}{\log^2 u} + x\frac{x}{\log x}\left(1+\frac{1}{2\log x}\right). \qquad (2.3)
\end{aligned}$$

Integrating by parts we see that

$$\int_2^x \frac{u\mathrm{d}u}{\log u} = \frac{x^2}{2\log x} - \frac{2^2}{2\log 2} + \frac{1}{2}\int_2^x \frac{u\mathrm{d}u}{\log^2 u},$$

therefore (2.3) yields:

$$\sum_{p\le x} p > \frac{x^2}{2\log x} + \frac{x^2}{2\log^2 x} - 2\int_2^x \frac{u\mathrm{d}u}{\log^2 u} + \frac{2}{\log 2}. \qquad (2.4)$$

Since the function $u\log^{-2} u$ is decreasing for $1 < u \le \mathrm{e}^2$ and increasing for $u \ge \mathrm{e}^2$, it holds that

$$\int_2^x \frac{u\mathrm{d}u}{\log^2 u} = \int_2^{\mathrm{e}^2} \frac{u\mathrm{d}u}{\log^2 u} + \int_{\mathrm{e}^2}^x \frac{u\mathrm{d}u}{\log^2 u} < (\mathrm{e}^2 - 2)\frac{2}{\log^2 2} + (x-\mathrm{e}^2)\frac{x}{\log x}.$$

The last inequality and (2.4) imply now that

$$\sum_{p\le x} p > \frac{x}{2\log x} - \frac{3}{2}\frac{x^2}{\log^2 x} + \frac{2\,\mathrm{e}^2 x}{\log^2 x} - B \qquad \text{for } x \ge 59, \qquad (2.5)$$

where

$$B = \frac{2}{\log 2}\left(\frac{2}{\log 2}(\mathrm{e}^2 - 2) - 1\right).$$

Using the upper bound (see [**1**, page 247])

$$p_n < n\log n + n\log\log n \qquad \text{for } n \ge 6,$$

we have

$$\sum_{p\le x} p - p_{\lfloor x\rfloor} > x(f(x) - g(x)) \qquad \text{for } x \ge 59, \qquad (2.6)$$

where

$$f(x) = \frac{x}{2\log x} - \frac{3}{2}\frac{x}{\log^2 x} + \frac{2\,\mathrm{e}^2}{\log^2 x} - \frac{1}{x}B,$$

$$g(x) = \log x + \log\log x.$$

We shall show that there is an $x_0 > 59$ such that $f(x) - g(x) > 0$ for $x > x_0$. To this end it suffices to find an $x_0 > 0$ such that $f(x_0) > g(x_0)$ and the function $f(x) - g(x)$ is increasing for $x \geq x_0$. Since

$$f'(x) = \frac{1}{2\log x}\left(1 - \frac{4}{\log x}\right) + \frac{1}{\log^3 x}\left(6 - \frac{4\,\mathrm{e}^2}{x}\right) + \frac{1}{x^2}B,$$

we obtain that

$$f'(x) \geq \frac{1}{2\log x}\left(1 - \frac{4}{\log x}\right) + \frac{2}{\log^3 x} + \frac{1}{x^2}B \qquad \text{for } \; x \geq \mathrm{e}^2\,. \tag{2.7}$$

Using (2.7) we now see that

$$\begin{aligned} f'(x) - g'(x) &\geq \frac{1}{2\log x}\left(1 - \frac{4}{\log x}\right) + \frac{2}{\log^3 x} + \frac{B}{x^2} - \frac{1}{x}\left(1 + \frac{1}{\log x}\right) \\ &= \frac{1}{2\log x}\left[1 - \left(\frac{4}{\log x} + \frac{2}{x}\right)\right] + \left(\frac{2}{\log^3 x} - \frac{1}{x}\right) + \frac{B}{x^2} \qquad \text{for } x \geq \mathrm{e}^2\,. \end{aligned}$$

Therefore, if $x_0 \geq \mathrm{e}^2$ is such that

$$2x > \log^3 x \qquad \text{for } x \geq x_0, \tag{2.8}$$

and

$$\frac{2}{x} + \frac{4}{\log x} < 1 \qquad \text{for } x \geq x_0, \tag{2.9}$$

then

$$f'(x) - g'(x) > 0 \qquad \text{for } x \geq x_0,$$

It suffices to choose $x_0 = \mathrm{e}^5 \approx 143.413 > 59$ because then

$$f(x_0) \approx 362.436 > g(x_0) \approx 6.609,$$

as well as

$$\frac{2}{x_0} + \frac{4}{\log x_0} = \frac{2}{\mathrm{e}^5} + \frac{4}{5} < \frac{1}{2^4} + \frac{4}{5} < 1$$

and (2.9) holds. Clearly, (2.8) is satisfied too. Hence $f(x) - g(x)$ is increasing for $x \geq x_0$.

Thus for every integer $x \geq 149$ we have

$$\sum_{p \leq x} p > p_x.$$

Notice that $149 = p_{35}$. It remains to show that among the integers $1 \le x \le p_{35}$ just the primes 3, 7 and 13 satisfy (1.1). To this end notice that for each $n \ge 0$ the condition $p_n \le x < p_{n+1}$ implies

$$S(x) = \sum_{p \le x} p = \sum_{i=1}^{n} p_i = S(p_n).$$

Using a computer let us extend Table 1 up to $n = 35$ and by adding a fifth row containing the initial segment of the sequence $P(p_{n+1} - 1)$. Now, any column of the new Table 2 corresponds to the interval $p_n \le x < p_{n+1}$.

As readily seen, for $n \ge 10$, i.e. for $x \ge 29$, we already have

$$P(p_n) < P(p_{n+1} - 1) < S(p_n) = S(x),$$

whenever $p_n \le x < p_{n+1}$, exactly as for $n \ge 35$, i.e. for $x \ge 149$.

On the other hand, for $n \in \{0, 1, 3, 5, 7\}$, i.e. for $x \in \{1;\ 2;\ 5, 6;\ 11, 12;\ 17, 18\}$ we have

$$S(x) = S(p_n) < P(p_n) < P(p_{n+1} - 1),$$

excluding any counterexample $p_n \le x < p_{n+1}$ to (1.1), as well.

Finally, for $n \in \{8, 9\}$ we have

$$P(p_n) < S(x) = S(p_n) < P(p_{n+1} - 1),$$

so that a counterexample $p_n \le x < p_{n+1}$ could perhaps occur. Fortunately, for $n = 8$, we have $p_8 = 19$, so that all the primes

$$p_{19} = 67, \quad p_{20} = 71, \quad p_{21} = 73, \quad p_{22} = 79$$

differ from the sum $S(x) = 77$ for $19 \le x < 23$. Similarly, for $n = 9$, we have $p_9 = 23$, and all the primes

$$p_{23} = 83, \quad p_{24} = 89, \quad p_{25} = 97, \quad p_{26} = 101, \quad p_{27} = 103, \quad p_{28} = 107$$

are distinct from the sum $S(x) = 100$ for $23 \le x < 29$, again.

There remain the columns for $n \in \{2, 4, 6\}$, corresponding to our mythical numbers and their "prime interval companions" $x \in \{3, 4;\ 7, 8, 9, 10;\ 13, 14, 15, 16\}$.

□

Perhaps it is worthwhile to notice the "almost mythical" number $x = 26 = 2 \cdot 13$ for which the sum

$$\sum_{p \le 26} p = 100$$

and the prime $p_{26} = 101$ differ just by 1.

3. Supplement

In our opinion, the so-called natural numbers tell us about laws of this world a lot more than we are able to admit or comprehend. So for example, the recently proved Fermat's theorem on nonexistence of nontrivial integer solutions of the equation

$$x^n + y^n = z^n \tag{3.1}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
p_n	1	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
$P(p_n)$	2	3	5	11	17	31	41	59	67	83	109	127	157	179	191	211	241	277	283	331	353
$S(p_n)$	0	2	5	10	17	28	41	58	77	100	129	160	197	238	281	328	381	440	501	568	639
$P(p_{n+1}-1)$	2	3	7	13	29	37	53	61	79	107	113	151	173	181	199	239	271	281	317	349	359

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
p_n	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149
$P(p_n)$	367	401	431	461	509	547	563	587	599	617	709	739	773	797	859
$S(p_n)$	712	791	874	963	1060	1161	1264	1371	1480	1593	1720	1851	1988	2127	2276
$P(p_{n+1}-1)$	397	421	457	503	541	557	577	593	613	701	733	769	787	857	863

Table 2

for $n > 2$, together with the long ago known fact that there are infinitely many integer solutions of this equation for $n = 2$, seem apparently related in a strange or even mysterious way to the validity of the Pythagorean theorem which is essentially the basis of the Euclidean geometry.

Similarly, we can mention the theorem saying that the Diofantic equation

$$n = x^2 + y^2 + z^2 + u^2 \tag{3.2}$$

has an integer solution for every natural number n. Probably its most elegant proof makes use of the multiplicative property of the quaternion norm given by

$$|q|^2 = q\,q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2,$$

where $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ is an arbitrary quaternion and $q^* = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k}$ is its adjoint. On the other hand, this result seems essential to allow for the very existence of Hamilton's quaternions as a four-dimensional non-commutative and associative division algebra over reals with the above norm. A deep theorem states that there are up to isomorphisms just three continuous associative division algebras over the field of reals: the real numbers themselves, the complex numbers, and the quaternions, with dimensions 1, 2 and 4, respectively. Moreover, the quaternion multiplication, through the formula

$$p\,q = \langle p, q\rangle + p_0\vec{q} + q_0\vec{p} + (\vec{p}\times\vec{q}),$$

is closely related to the spatial vector product $\vec{p}\times\vec{q}$ of the vector parts $\vec{p} = p_1\mathrm{i} + p_2\mathrm{j} + p_3\mathrm{k}$, $\vec{q} = q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ of the quaternions p, q, and their pseudoscalar product

$$\langle p, q\rangle = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3,$$

determining the geometry of the Minkowski's four-dimensional time-space in Einstein's Special Theory of Relativity.

We find it very interesting that the equation (1.1) specifies precisely the three prime numbers 3, 7 and 13. As if equations (3.1) and (3.2) decided about geometry and physics and (1.1) about myths.

We add the following to the latter: Analogously to the definition of the factorial $n! = 1\cdot 2\cdot \ldots \cdot n$, we introduce the *summarial*

$$n!_{+} = 1 + 2 + \ldots + n.$$

If we then compute the summarials of the three solutions of (1.1), we obtain

$$3!_{+} = 6, \qquad 7!_{+} = 28, \qquad 13!_{+} = 91.$$

The first two summarials are the first two perfect numbers. The third one is not perfect, however, both the number as well as the sum of its proper divisors can be expressed as products

$$91 = 7\cdot 13, \qquad 1 + 7 + 13 = 21 = 3\cdot 7$$

of pairs of the mythical primes. This means that the last summarial does not give rise to perfection but just to some kind of "quasiperfection" — the number 13 returns in some sense, accompanied with 3 and 7.

We see that the three solutions of (1.1) satisfy many remarkable relations and this is perhaps the reason why they became selected.

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I dedicated this work to D. Dudák and I express my gratitude to P. Zlatoš.

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