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HYPERGEOMETRIC SERIES ASSOCIATED WITH THE HURWITZ-LERCH ZETA FUNCTION

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Abstract. The present work is a sequel to the papers [**3**] and [**4**], and it aims at introducing and investigating a new generalized double zeta function involving the Riemann, Hurwitz, Hurwitz-Lerch and Barnes double zeta functions as particular cases. We study its properties, integral representations, differential relations, series expansion and discuss the link with known results.

1. Introduction

The double zeta function of Barnes [**1**] is defined by

$$\zeta_2(z;a,w)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a+n+\omega m)^{-z}, \tag{1.1}$$

where $a>0$ and ω is a non-zero complex number with $|\arg(\omega)|<\pi$.

The series (1.1) is absolutely convergent for $\operatorname{Re} z>2$ and its continuation is holomorphic with respect to z except for the poles at $z=1$ and $z=2$.

For $m=0$, equation (1.1) reduces to Hurwitz zeta function

$$\zeta(z,a)=\sum_{n=0}^{\infty}(a+n)^{-z}, \qquad (a\neq\{0,-1,-2,-3,\ldots\};\ \ \operatorname{Re} z>1), \tag{1.2}$$

which is a generalization of the Riemann zeta function

$$\zeta(z)=\sum_{n=0}^{\infty}n^{-z}. \tag{1.3}$$

As a generalization of both Riemann and Hurwitz zeta functions the so-called Hurwitz-Lerch zeta function is defined by [**6**, p. 27 (1)]

$$\Phi(y,z,a)=\sum_{n=0}^{\infty}\frac{y^n}{(a+n)^z}, \qquad (a\in C\setminus\{0,-1,-2,-3,\ldots\};\ |y|<1). \tag{1.4}$$

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Φ is an analytic function in both variables y and z in a suitable region and it reduces to the ordinary Lerch zeta function when $y = \mathrm{e}^{2\pi\,\mathrm{i}\,\lambda}$:

$$\Phi(\mathrm{e}^{2\pi\,\mathrm{i}\,\lambda}, z, a) = \phi(\lambda, z, a) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2\pi\,\mathrm{i}\,n\lambda}}{(a+n)^z}. \tag{1.5}$$

Next, here we recall a further generalization of the Hurwitz-Lerch zeta function $\Phi(y, z, a)$ in the form (see [**10**, p. 100, eq. (1.5)])

$$\Phi_\mu^*(x, z, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n x^n}{(a+n)^z n!}, \tag{1.6}$$

where $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \mu(\mu+1)\ldots(\mu+n-1)$ denotes the Pochhammer's symbol, $\mu \in C$, $a \neq \{0, -1, -2, -3, \ldots\}$ and $|x| < 1$. Obviously, when $\mu = 1$, (1.6) reduces to (1.4).

In [**4**] Bin-Saad and Al-Gonah introduced two hypergeometric type generating functions of the generalized zeta function defined by (1.6) in the forms:

$$\zeta_\mu^*(x, y; z, a) = \sum_{m=0}^{\infty} \Phi_\mu^*(y, z+m, a)\frac{x^m}{m!}, \tag{1.7}$$

and

$$\zeta_{\mu,\nu}^*(x, y; z, a) = \sum_{m=0}^{\infty} (\nu)_m \Phi_\mu^*(y, z+m, a)\frac{x^m}{m!}, \tag{1.8}$$

which, in the special case when $\mu = 1$, are essentially known formulas of Bin-Saad [**3**]. Also, by noting that [**21**, p. 20, eq. (26)]

$$\lim_{|\lambda|\to\infty} \left\{(\lambda)_n \left(\frac{x^n}{\lambda}\right)\right\} = x^n, \tag{1.9}$$

we eventually end up with

$$\lim_{|\nu|\to 0} \zeta_{\mu,\nu}^*\left(\frac{x}{\nu}, y; z, a\right) = \zeta_\mu^*(x, y; z, a). \tag{1.10}$$

The present work is a sequel to the author's papers [**3**] and [**4**] and it aims at introducing and investigating a new kind of hypergeometric-type generating functions $\zeta_\lambda^\mu(x, y; z, a)$ or infinite series associated with the Hurwitz-Lerch zeta function $\Phi(y, z, a)$. The results we will obtain and discuss are a further contribution a long line developed in [**3**] and [**4**]. The layout of the paper is as follows. In Section 2 we introduce and describe some properties and relationships for the function ζ_λ^μ. Relevant connections of the function $\zeta_\lambda^\mu(x, y; z, a)$ with those considered in [**3**] and [**4**] are also indicated. In Section 3 we establish several integral representations for the function ζ_λ^μ involving integral representations of contour and Mellin-Barenes type of integrals. Section 4 is devoted to the differentiation of the function ζ_λ^μ with respect to the arguments x, y, z, λ, μ and a. In the final section, we present some series expansions for the function ζ_λ^μ involving Appell's function of two variables F_2 and the generalized hypergeometric function ${}_3F_2$.

2. The Generalized Double Zeta Function $\zeta_\lambda^\mu(x,y;z,a)$

Suggested by (1.1), (1.4) and (1.6) here we introduce a generalized double zeta function of the form

$$\zeta_\lambda^\mu(x,y;z,a)=\sum_{m=0}^{\infty}(\mu)_m\Phi(y,z,a+\lambda m)\frac{x^m}{m!}, \tag{2.1}$$

where $|x|<1$, $|y|<1$; $\mu\in C\setminus\{0,-1,-2,\ldots\}$, $\lambda\in C\setminus\{0\}$; $a\in C\setminus\{-(n+\lambda m)\}$, $\{n,m\}\in N\cup\{0\}$ and Φ is the Hurwitz-Lerch zeta function defined by (1.4).

The alternative representation

$$\zeta_\lambda^\mu(x,y;z,a)=\sum_{n=0}^{\infty}\Phi_\mu^*\left(x,z,\frac{a+n}{\lambda}\right)\frac{y^n}{\lambda^z}, \tag{2.2}$$

where Φ_μ^* is the generalized zeta function defined by (1.6), follows by changing the order of summations and considering equation (1.6). Clearly, we have the following relationships

$$\zeta_\lambda^\mu(0,1;z,a)=\zeta_1^1(1,0;z,a)=\zeta(z,a), \tag{2.3}$$

$$\zeta_\lambda^\mu(0,y;z,a)=\Phi(y,z,a), \tag{2.4}$$

$$\zeta_1^\mu(x,0;z,a)=\Phi_\mu^*(y,z,a), \tag{2.5}$$

$$\zeta_\lambda^1(1,1;z,a)=\zeta_2(z;a,\lambda)=\sum_{n=0}^{\infty}\zeta\left(z,\frac{a+n}{\lambda}\right)\lambda^{-z}. \tag{2.6}$$

Indeed, the function ζ_λ^μ is a hypergeometric-type generating function of the functions Φ and Φ_μ^* defined by (1.4) and (1.6), respectively. The case when $y=0$ of the definition (2.1) suggests us to define the following further generalization of the zeta function defined by (1.6)

$$\zeta_\lambda^\mu(x,0;z,a)=\Phi_{\mu,\lambda}^*(x,z,a)=\sum_{m=0}^{\infty}\frac{(\mu)_m x^m}{m!(a+\lambda m)^z}, \tag{2.7}$$

where $|x|<1$; $\mu\in C\setminus\{0,-1,-2,\ldots\}$; $a\in C\setminus\{-(\lambda m)\}$, $m\in N\cup\{0\}$.

In the case when $z=\lambda=1$, we have simply

$$\zeta_1^\mu(x,y;1,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!(a+m+n)}=a^{-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(\mu)_m(1)_n x^m y^n}{(a+1)_{m+n}m!n!},$$

which implies the next result.

Corollary 2.1. *Let* $\max\{|x|,|y|\}<1$, $\operatorname{Re}a>0$. *Then*

$$\zeta_1^\mu(x,y;\ 1,a)=a^{-1}F_1[a,\mu,1;\ a+1;\ x,y], \tag{2.8}$$

where F_1 is the Appell's function of two variables defined by the series [**21**, p. 22 (1)]

$$F_1\left[a,b,b';\,c;\,x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}.$$

According to the relationship (2.4), equation (2.8) yields the following known result [**6**, p. 30 (10)]

$$\Phi(y,1,a) = a^{-1}{}_2F_1[a,1;a+1;y],$$

where ${}_2F_1$ is the Gaussian hypergeometric function [**6**].

Corollary 2.2. *Let* $\lambda = 1$, $|x| < 1$ *and* $|y| < 1$. *Then*

$$\zeta_1^{\mu}(xy,y;z,a) = (1-x)^{-\mu}\ \Phi(y,z,a), \tag{2.9}$$

and

$$\zeta_1^{\mu}(x,yx;\ z,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^z}\ {}_2F_1\left[\begin{matrix} -n,1; \\ \\ 1-\mu-n; \end{matrix}\quad y\right]\frac{x^n}{n!}. \tag{2.10}$$

Proof. We have

$$\zeta_1^{\mu}(xy,y;\ z,a) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\mu)_m x^m y^{n+m}}{m!(a+n+m)^z}.$$

Then by letting $n \to n-m$ and considering the Hurwitz-Lerch zeta function $\Phi(y,z,a)$ defined by (1.4), we get (2.9). Similarly, one can prove the result (2.10). □

Remark 2.1. In view of the definition (1.4), the results (2.9) and (2.10) can be rewritten in the forms

$$\sum_{n=0}^{\infty}(\mu)_n\Phi(y,z,a+n)\frac{(xy)^n}{n!} = (1-x)^{-\mu}\ \Phi(y,z,a), \tag{2.11}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^z}\Phi(yx,z,a+n)\frac{x^n}{n!} \\ &\qquad = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^z}\ {}_2F_1\left[\begin{matrix} -n,1; \\ \\ 1-\mu-n; \end{matrix}\quad y\right]\frac{x^n}{n!}, \end{aligned} \tag{2.12}$$

respectively.

Further, for $y = 1$, the formulas (2.11) and (2.12) reduce to the interesting results

$$\sum_{n=0}^{\infty}(\mu)_n\zeta(z,a+n)\frac{x^n}{n!} = (1-x)^{-\mu}\ \zeta(z,a), \tag{2.13}$$

and

$$\sum_{n=0}^{\infty}(\mu)_n\Phi(x,z,a+n)\frac{x^n}{n!} = \Phi^{*}_{\mu+1}(x,z,a), \tag{2.14}$$

respectively.

Remark 2.2. For $\mu = 1$, the formula (2.12) yields the result

$$\sum_{n=0}^{\infty} \Phi(yx, z, a+n)x^n = (1-y)^{-1}\, \Phi(x, z, a). \tag{2.15}$$

Whereas, for $\mu = 1$ and $x \to \frac{1}{y}$, equation (2.11) yields

$$\sum_{n=0}^{\infty} \zeta(z, a+n)y^n = \left(1 - \frac{1}{y}\right)^{-1} \Phi(y, z, a). \tag{2.16}$$

A similar result as in (2.16) can be obtained from equation (2.12). Next, we present a series representation for the function ζ_λ^μ. First, we recall the following well-known expansion formula of the Hurwitz-Lerch zeta function [**6**, p. 29 (8)]

$$\Phi(y, z, a) = \frac{\Gamma(1-z)}{y^a}\left[\log\frac{1}{y}\right]^{z-1} + \frac{1}{y^a}\sum_{k=0}^{\infty}\zeta(z-k, a)\frac{(\log y)^k}{k!}, \tag{2.17}$$

valid for $|\log(y)| < 2\pi,\;\; z \neq 1, 2, 3, \ldots;\;\; a \neq \{0, -1, -2, -3, \ldots\}$.

Theorem 2.1. *Let $\lambda > 0,\;\; |\log(y)| < 2\pi$ and $\left|\frac{x}{y^\lambda}\right| < 1$. Then*

$$\begin{aligned}\zeta_\lambda^\mu(x, y;\, z, a) = \frac{1}{y^a}\Bigg[&\Gamma(1-z)\left[\log\frac{1}{y}\right]^{z-1}\left(1 - \frac{x}{y^\lambda}\right)^{-\mu} \\ &+ \sum_{k=0}^{\infty}\zeta_\lambda^\mu(x, 1;\, z-k, a)\frac{(\log y)^k}{k!}\Bigg],\end{aligned} \tag{2.18}$$

valid for $z \neq 1, 2, 3, \ldots;\;\; a \neq -(n + \lambda m),\;\; \{m, n\} \in N \cup \{0\}$.

Proof. Use the series representation (2.17) in the definition (2.1). $\square$

If $\mu = x = 1$, in formula (2.18), we get an expansion for the zeta function of Barnes (1.1):

$$\begin{aligned}\sum_{k=0}^{\infty}\zeta_2(z-k;\, a, \lambda)\frac{(\log y)^k}{k!} &= y^a\sum_{k=0}^{\infty}\Phi(y, z, a+\lambda m) \\ &\quad - \Gamma(1-z)\left[\log\frac{1}{y}\right]^{z-1}\left[1 - \frac{1}{y^\lambda}\right]^{-1},\end{aligned} \tag{2.19}$$

valid for $\left|\frac{1}{y^\lambda}\right| < 1,\;\; \lambda \neq 0,\;\; z \neq 1, 2, 3, \ldots;\;\; a \neq -(n + \lambda m),\;\; \{m, n\} \in N \cup \{0\}$.

Finally, putting $\mu = \alpha + \beta$ in (2.1) and using the classical formula of Nörlund for the Pochhammer symbol (cf. [**2**, Section 1, Chapter 3])

$$(a+b)_k = \sum_{m=0}^{k}\binom{k}{m}(a)_{k-m}\,(b)_m, \tag{2.20}$$

we find the form (2.1) that

$$\begin{aligned}\zeta_\lambda^{\alpha+\beta}(x,y;z,a) = &\sum_{m=0}^{\infty}(\alpha)_m\ \Phi(y,z,a+\lambda m)\\ &\times\ {}_2F_1\left[\begin{matrix}-m,\beta;\\ \\ 1-\alpha-m;\end{matrix}\quad 1\right]\frac{x^m}{m!}.\end{aligned}\tag{2.21}$$

By exploiting the results

$$(a+n+\lambda m)^{-z}\Gamma(z)=\int_0^\infty \mathrm{e}^{(a+n+\lambda m)t}\,t^{z-1}\mathrm{d}t,\tag{2.22}$$

and

$$(a+n)^{-(z+m)}(z)_m=\frac{1}{\Gamma(z)}\int_0^\infty \mathrm{e}^{(a+n)t}\,t^{z-1}\mathrm{d}t,\tag{2.23}$$

which follow from the Eulerian integral [**2**]

$$a^{-z}\Gamma(z)=\int_0^\infty \mathrm{e}^{(a)t}\,t^{z-1}\mathrm{d}t,\tag{2.24}$$

we can derive the following connection formula for the function ζ_λ^μ with the functions ζ_μ^* and $\zeta_{\mu,\nu}^*$ (see (1.7) and (1.8)).

Theorem 2.2. *Let* $\operatorname{Re} a>0$ *and* $\operatorname{Re} z>0$*, then*

$$\begin{aligned}&\zeta_\lambda^z(x,y;\,z,a)\\ &\quad=\frac{1}{(\Gamma(z))^2}\int_0^\infty\!\!\int_0^\infty \zeta_1^*(xu\,\mathrm{e}^{\lambda t},y\,\mathrm{e}^{u+t};-z,a)\,\mathrm{e}^{a(u+t)}\,u^{z-1}t^{z-1}\mathrm{d}u\mathrm{d}t,\end{aligned}\tag{2.25}$$

$$\begin{aligned}&\zeta_\lambda^z(x,y;\,z,a)\\ &\quad=\lim_{\nu\to 0}\left\{\frac{1}{(\Gamma(z))^2}\int_0^\infty\!\!\int_0^\infty \zeta_{1,\nu}^*(xu\nu^{-1}\,\mathrm{e}^{\lambda t},y\,\mathrm{e}^{u+t};-z,a)\,\mathrm{e}^{a(u+t)}\,u^{z-1}t^{z-1}\mathrm{d}u\mathrm{d}t\right\}.\end{aligned}\tag{2.26}$$

Proof. Denote, for convenience, the right-hand side of formula (2.25) by I. Then, in view of definition (1.7) it is easily seen that

$$I=\sum_{m,n=0}^{\infty}\frac{x^my^n}{(a+n)^{-(z+m)}}\int_0^\infty \mathrm{e}^{(a+n+\lambda m)t}\,t^{z-1}\mathrm{d}t\ \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{e}^{(a+n)t}\,t^{z-1}\mathrm{d}t.\tag{2.27}$$

Now, with the help of the results (2.22) and (2.23) and the definition (2.1), equation (2.27) gives us the left-hand side of formula (2.25). By empolying relation (1.9) and expoliting the same procedure leading to (2.25) one can derive the formula (2.26). □

In order to derive the inversion of Theorem 2.2, we first recall the definition of the integral operator D_x^{-1} (see [**17**] and [**5**])

$$D_x^{-m}x^\lambda=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m+1)}\,x^{\lambda+m},\qquad m\in N\cup\{0\},\tag{2.28}$$

and once acting on unity yields to

$$D_x^{-m}\{1\} = \frac{x^m}{m!}. \tag{2.29}$$

Now, it is not difficult to infer the following theorem.

Theorem 2.3. *Let* $\operatorname{Re} a > 0$ *and* $\operatorname{Re} z > 0$, *then*

$$\begin{aligned}&\zeta_z^*(x, y;\, z, a)\\ &= \frac{1}{(\Gamma(z))^2}\int_0^\infty\!\!\int_0^\infty \zeta_\lambda^1(u\,\mathrm{e}^{\lambda t}\, D_x^{-1}, \mathrm{e}^{u+t}\, D_y^{-1}; -z, a)\,\mathrm{e}^{a(u+t)}\, u^{z-1} t^{z-1}\mathrm{d}u\mathrm{d}t,\end{aligned} \tag{2.30}$$

$$\begin{aligned}&\zeta_{\mu,z}^*(x, y;\, z, a)\\ &= \frac{1}{(\Gamma(z))^2}\int_0^\infty\!\!\int_0^\infty \zeta_\lambda^\mu(xu\,\mathrm{e}^{\lambda t}, \mathrm{e}^{u+t}\, D_y^{-1}; -z, a)\,\mathrm{e}^{a(u+t)}\, u^{z-1} t^{z-1}\mathrm{d}u\mathrm{d}t.\end{aligned} \tag{2.31}$$

Proof. We refere to the proof of Theorem 2.2. □

3. Integral Representations

In many situations an integral representation of zeta function is more convenient to use than its series representation. First of all, we establish an integral representation for ζ_λ^μ that is derived directly from the corresponding integral representation of the Hurwitz-Lerch zeta function Φ [**6**, p. 27 (3)]

$$\Phi(y, z, a) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,\mathrm{e}^{-at}(1 - y\,\mathrm{e}^{-t})^{-1}\mathrm{d}t, \tag{3.1}$$

where $\operatorname{Re} a > 0$ and either $|y| \le 1$, $y \ne 1$, $\operatorname{Re} z > 0$ or $y = 1$, $\operatorname{Re} z > 1$.

Theorem 3.1. *Let* $\operatorname{Re} a > 0$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\lambda > 0$ *and either* $|x| \le 1$, $|y| \le 1$, $y \ne 1$, $x \ne 1$, $\operatorname{Re} z > 0$ *or* $x = 1$, $y = 1$, $\mu = 1$, $\operatorname{Re} z > 2$. *Then*

$$\begin{aligned}\zeta_\lambda^\mu(x, y; z, a) &= \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,\mathrm{e}^{-at}(1 - x\,\mathrm{e}^{-\lambda t})^{-\mu}(1 - y\,\mathrm{e}^{-t})^{-1}\mathrm{d}t\\ &= \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\frac{\mathrm{e}^{-(a-1)t}(1 - x\,\mathrm{e}^{-\lambda t})^{-\mu}}{(\mathrm{e}^t - y)}\mathrm{d}t.\end{aligned} \tag{3.2}$$

Proof. From (2.1) and (3.1) we have

$$\zeta_\lambda^\mu(x, y; z, a) = \sum_{m=0}^\infty \frac{(\mu)_m}{m!}\left[\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,\mathrm{e}^{-(a+\lambda m)t}(1 - y\,\mathrm{e}^{-t})^{-1}\mathrm{d}t\right]x^m.$$

The desired result now follows by changing the order of summation and integration and employing the binomial expansion. □

Another integral representation for the function ζ_λ^μ is based upon the simple observation that (see e.g. [**21**, p. 281 (25)])

$$(\lambda)_m = \frac{1}{\Gamma(\lambda)}\int_0^\infty \mathrm{e}^{-t}\, t^{\lambda+m-1}\mathrm{d}t, \qquad \operatorname{Re}\lambda > 0;\quad m = 0, 1, 2, \ldots, \tag{3.3}$$

which indeed follows immediately from (2.24).

Theorem 3.2. *Let* $\operatorname{Re} a > 0$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\lambda > 0$, $|x| < +\infty$ *and either* $|y| \leq 1$, $y \neq 1$, $\operatorname{Re} z > 0$ *or* $y = 1$, $\operatorname{Re} z > 1$. *Then*

$$\begin{aligned}
&\zeta_\lambda^\mu(x, y; z, a) \\
&= \frac{1}{\Gamma(z)\,\Gamma(\mu)} \int_0^\infty \int_0^\infty t^{z-1} s^{\mu-1}\, \mathrm{e}^{-at}\, \mathrm{e}^{-(1-x\,\mathrm{e}^{-\lambda t})s} (1 - y\,\mathrm{e}^{-t})^{-1} \mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{3.4}$$

Proof. The identities

$$(a + n + \lambda m)^{-z} x^m y^n = \frac{1}{\Gamma(z)} \int_0^\infty (x\,\mathrm{e}^{-\lambda})^m t^{z-1}\, \mathrm{e}^{-at} (y\,\mathrm{e}^{-t})^n \mathrm{d}t,$$

and

$$(\mu)_m = \frac{1}{\Gamma(\mu)} \int_0^\infty s^{\mu+m-1}\, \mathrm{e}^{-s}\, \mathrm{d}s,$$

follow from the integral representation (2.24). The result now follows from the definition of ζ_λ^μ. □

The Hurwitz-Lerch zeta function has the following contour integral representation [**6**, p. 28 (5)]

$$\Phi(y, z, a) = \frac{-\Gamma(1-z)}{2\pi i} \int_\infty^{0+} (-t)^{z-1}\, \mathrm{e}^{-at} (1 - y\,\mathrm{e}^{-t})^{-1} \mathrm{d}t, \tag{3.5}$$

valid for $\operatorname{Re} a > 0$, $z \in C$ and $|\arg(-t)|, \pi$, assuming as in [**6**] that the contour does not enclose any of the points $t = \log z \pm 2n\pi i$, $(n = 0, 1, 2, \ldots)$, which are the poles of the integrand of (3.5).

Similarly, for the function ζ_λ^μ we have the following contour integral representation.

Theorem 3.3. *Let* $\operatorname{Re} a > 0$, $\operatorname{Re}\lambda > 0$ *and* $|\arg(-t)| < \pi$. *Then*

$$\zeta_\lambda^\mu(x, y; z, a) = \frac{-\Gamma(1-z)}{2\pi i} \int_\infty^{0+} (-t)^{z-1}\, \mathrm{e}^{-at} (1 - x\,\mathrm{e}^{-\lambda t})^{-\mu} (1 - y\,\mathrm{e}^{-t})^{-1} \mathrm{d}t. \tag{3.6}$$

Proof. It follows from (2.1) and (3.5) that

$$\zeta_\lambda^\mu(x, y; z, a) = \sum_{m=0}^\infty \frac{(\mu)_m}{m!} \left[\frac{-\Gamma(1-z)}{2\pi i} \int_\infty^{0+} (-t)^{z-1}\, \mathrm{e}^{-(a+\lambda m)t} (1 - y\,\mathrm{e}^{-t})^{-1} \right] x^m.$$

The desired result now follows by changing the order of summation and integration and employing the binomial expansion. □

Now, we shall prove ζ_λ^μ as an application of the Mellin-Barnes type of integral. Our starting point is the same as the starting point of Katsurada's argument in ([**12**] and [**13**]), that is the formula [**22**, Section 14.51, p. 289, Corollary]

$$(1 - \omega)^{-z} = \frac{1}{2\pi\,\mathrm{i}} \int_c \frac{\Gamma(z + \nu)\Gamma(-\nu)(-\omega)^\nu}{\Gamma(z)} d\nu, \tag{3.7}$$

where z and ω are complex with $\operatorname{Re} z > 0$, $|\arg(\omega)| < \pi$, $\omega \neq 0$ and the path is the vertical line from $c - \mathrm{i}\infty$ to $c + \mathrm{i}\infty$. In [**21**] this formula is stated with $c = 0$, (with suitable modification of the path near the point $z = 0$), but it is clear that the formula is also valid for $-\operatorname{Re} z < c < 0$.

Theorem 3.4. *Let* $\operatorname{Re} z > 0$, $\operatorname{Re}(a - b) > 0$, $\operatorname{Re} b > 0$ *and* $\lambda \neq 0$. *Then*

$$\begin{aligned}\zeta_\lambda^\mu(x, y; z, a) = \frac{1}{2\pi i} \int_c \frac{(\Gamma(\nu + z)\Gamma(-\nu)}{\Gamma(z)} \Phi(y, z + \nu, b) \\ \times \Phi_\mu^*\left(x, -\nu, \frac{a-b}{\lambda}\right) \lambda^\nu d\nu, \qquad |x| < 1,\ |y| < 1.\end{aligned} \tag{3.8}$$

Proof. Let $\omega = (b - a - \lambda m)/(b + n)$ in (3.7) and multiply both sides by

$$\frac{(\mu)_m x^m y^n}{m!}, \qquad (m, n = 0, 1, 2, \ldots)$$

to obtain

$$\begin{aligned}\left(\frac{(\mu)_m x^m y^n}{m!}\right)(a + n + \lambda m)^{-z} = \int_c \frac{(\Gamma(\nu + z)\Gamma(-\nu)}{\Gamma(z)} \times \frac{(\mu)_m x^m}{m!\ (a - b + \lambda m)^{-\nu}} \\ \times \frac{y^n}{(b + n)^{z+\nu}} \mathrm{d}\nu, \qquad \text{for } m \geq 0,\ n > 0.\end{aligned}$$

Therefore, if we assume $(1 - \operatorname{Re}\nu) < c < -1$, then from (1.4) and (1.6) we get (3.8). □

Further, by using the definition (2.7) and (1.4), we can derive the following double integral representations for the function ζ_λ^μ.

Theorem 3.5. *Let* $\operatorname{Re} a > 0$, $\operatorname{Re} z > 0$ *and* $\operatorname{Re}\nu < 0$. *Then*

$$\zeta_\lambda^\mu(x, y; z, a)$$

$$= \frac{1}{\Gamma(z)\ \Gamma(1 - \nu)} \int_0^\infty \int_0^\infty \frac{t^{z-1} s^{-\nu}\, \mathrm{e}^{-a(t+s)}}{(1 - y\, \mathrm{e}^{-t})} \Phi_{\mu,\lambda}^*\left(x\, \mathrm{e}^{-\lambda(t+s)}, \nu - 1, a\right) \mathrm{d}s\ \mathrm{d}t, \tag{3.9}$$

$$= \frac{1}{\Gamma(z)\ \Gamma(1 - \nu)} \int_0^\infty \int_0^\infty \frac{t^{z-1} s^{-\nu}\, \mathrm{e}^{-a(t+s)}}{(1 - x\, \mathrm{e}^{-t})^\mu} \Phi\left(y\, \mathrm{e}^{-(t+s)}, \nu - 1, a\right) \mathrm{d}s\ \mathrm{d}t. \tag{3.10}$$

Proof. The results follow directly from the definitions (2.7), (1.4) and the integral representation of gamma function (3.3). □

Furthermore, we can easily prove the following inversion relations of the Theorem 3.5.

Theorem 3.6. *Let* $\operatorname{Re} a > 0$, $\operatorname{Re} z > 0$ *and* $\operatorname{Re}\nu < 0$. *Then*

$$\begin{aligned}\Phi_{\mu,\lambda}^*(x, z, a) = \frac{1}{\Gamma(z)\ \Gamma(1 - \nu)(1 - y)^{-1}} \int_0^\infty \int_0^\infty t^{-\nu} s^{z-1}\, \mathrm{e}^{-a(t+s)} \\ \times \zeta_\lambda^\mu\left(x\, \mathrm{e}^{-\lambda(s+t)}, y\, \mathrm{e}^{-t}; \nu - 1, a\right) \mathrm{d}s \mathrm{d}t,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\Phi(x,z,a) = \frac{1}{\Gamma(z)\,\Gamma(1-\nu)(1-x)^{-\mu}} \int_0^\infty \int_0^\infty t^{-\nu} s^{z-1}\,\mathrm{e}^{-a(t+s)} \\ \times \zeta_\lambda^\mu \left(x\,\mathrm{e}^{-\lambda t}, y\,\mathrm{e}^{-(t+s)}; \nu-1, a\right) \mathrm{d}s\mathrm{d}t.\end{aligned} \tag{3.12}$$

Proof. We refer to the proof of Theorem 3.5. □

Other integral representations of the functions $\Phi^*_{\mu,\lambda}$ and $\phi(\lambda,z,a)$ can be deduced from the formulas (3.2), (3.4), (3.5) and (3.6). For instance, when $y=0$, the formula (3.4) yields the integral representation

$$\Phi^*_{\mu,\lambda}(x,z,a) = \frac{1}{\Gamma(z)\Gamma(\mu)} \int_0^\infty \int_0^\infty t^{z-1} s^{\mu-1}\,\mathrm{e}^{-at}\,\mathrm{e}^{-\left(1-x\,\mathrm{e}^{-\lambda t}\right)s}\,\mathrm{d}s \quad \mathrm{d}t. \tag{3.13}$$

Similarly, for the Hurwitz-Lerch zeta function Φ equation (3.6) yields the following Mellin-Barenes integral formula

$$\Phi(y,z,a) = \frac{1}{2\pi\,\mathrm{i}} \int_c \frac{(\Gamma(\nu+z)\Gamma(-\nu)}{\Gamma(z)} \Phi(y,z+\nu,a-\lambda)\lambda^\nu \mathrm{d}\nu. \tag{3.14}$$

More interestingly, based on the relation (1.5), the representation (3.14) reduces further to a known result due to Katsurada [**14**, p. 168 (2.6)]

$$\phi(\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}, a+\lambda, z) = \frac{1}{2\pi\,\mathrm{i}} \int_c \frac{(\Gamma(\nu+z)\Gamma(-\nu)}{\Gamma(z)} \phi(\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}, a, z+\nu)\lambda^\nu d\nu. \tag{3.15}$$

4. Differential Relations

The generalized zeta function ζ_λ^μ as a function satisfies some differential recurrence relations. Fortunately these properties of ζ_λ^μ can be developed directly from the definition (2.1). First, by recalling the familiar derivative formula from calculus in terms of the gamma function [**17**]

$$D_x^m x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}, \qquad n-m \geq 0, \qquad D_x = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{4.1}$$

where $m \in N$, we aim now to derive the following differential relation for ζ_λ^μ.

Theorem 4.1. *Let* $\lambda \neq 0$ *and* $\mu - n \neq \{0,-1,-2,\ldots\}$. *Then*

$$\zeta_\lambda^\mu(x,y;z,a) = \sum_{n=0}^\infty \frac{(-1)^n y^n}{(1-\mu)_n} D_x^n \left[\Phi^*_{\mu-n}\left(x,z,\frac{a+(1-\lambda)n}{\lambda}\right)\lambda^{-z}\right]. \tag{4.2}$$

Proof. In view of (1.6) and (4.1) we have

$$\begin{aligned}&\sum_{n=0}^\infty \frac{(-1)^n y^n}{(1-\mu)_n} D_x^n \left[\Phi^*_{\mu-n}\left(x,z,\frac{a+(1-\lambda)n}{\lambda}\right)\lambda^{-z}\right] \\ &\qquad = \sum_{m=0}^\infty \sum_{n=0}^m \frac{(-1)^n y^n}{(1-\mu)_n} \frac{(\mu-n)_m x^{m-n}}{(m-n)!(a+\lambda m+(1-\lambda)n)^z}.\end{aligned} \tag{4.3}$$

Now, by using the identities

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k)$$

and

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n},$$

it leads to the result (4.2). □

Secondly, we show that the Hurwitz-Lerch zeta function Φ is related to the function ζ_λ^μ for $\mu \in N$ by the following differential relation.

Theorem 4.2. *Let $\lambda \neq 0$ and $\mu > 1$ be a positive integer number. Then*

$$\zeta_\lambda^\mu(x,y;z,a) = \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty} D_x^{\mu-1}\left[\Phi\left(x,z,\frac{a+n+\lambda(1-\mu)}{\lambda}\right)\lambda^{-z}\right]\ y^n. \tag{4.4}$$

Proof. We refer to the proof of Theorem 4.1. □

For $\lambda = 1$ with $y = 0$, equation (4.4) reduces to the following differential relation connecting the functions Φ and Φ_μ^*

$$D_x^{\mu-1}\left[\frac{1}{\Gamma(\mu)}\Phi(x,z,a-\mu+1)\right] = \Phi_\mu^*(x,z,a). \tag{4.5}$$

On the other hand, from (4.5) and with the aid of the formula (4.1), we can easily derive the following inversion relation of equation (4.5) in the form

$$D_x^{1-\mu}\left[\Gamma(\mu)\Phi_\mu^*(x,z,a+\mu-1)\right] = \Phi(x,z,a). \tag{4.6}$$

Next, we establish the derivative of the function ζ_λ^μ with respect to the argument λ.

Theorem 4.3. *Let $b \in R$. Then*

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\zeta_\lambda^\mu(x,y;z-1,a+\lambda b)\\ &\quad= (1-z)\left[x\mu\ \zeta_\lambda^{\mu+1}(x,y;z,a+\lambda(b+1)) + b\zeta_\lambda^\mu(x,y;z,a+\lambda b)\right].\end{aligned} \tag{4.7}$$

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial\lambda}\zeta_\lambda^\mu(x,y;z-1,a+\lambda b) = (1-z)&\left[\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{(m-1)!(a+n+\lambda(m+b))^z}\right.\\ &\left.+\ b\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!(a+n+\lambda(m+b))^z}\right].\end{aligned} \tag{4.8}$$

Now, let $m \to m+1$ in the first summation of (4.8) and then use the identity

$$(\mu)_{m+n} = (\mu)_n(\mu+n)_m$$

to obtain (4.7). □

The same type of differentiation gives the next result.

Theorem 4.4. *Let $q \in R$. Then*

$$\frac{\partial}{\partial q}\zeta_\lambda^\mu(x,y;z-1,a+bq) = b(1-z)\zeta_\lambda^\mu(x,y;z,a+bq). \tag{4.9}$$

Proof. We refer to the proof of Theorem 4.3. □

It is easily observed that the relations (4.7) and (4.9) are generalizations of the known results (see e.g. [**7**, p. 451]):

$$\frac{\partial}{\partial q}\zeta(z-1,a+\lambda b) = b(1-z)\zeta(z,a+qb),$$

and

$$\frac{\partial}{\partial \lambda}\zeta(z,\lambda) = -z\zeta(z+1,\lambda).$$

Further, we show that the function ζ_λ^μ satisfies the following theorem

Theorem 4.5. *Let $k \in N$. Then*

$$D_x^k\zeta_\lambda^\mu(x,y;z,a) = (\mu)_k\ \zeta_\lambda^{\mu+k}(x,y;z,a+\lambda k), \tag{4.10}$$

$$D_y^k\zeta_\lambda^\mu(x,y;z,a) = k!\sum_{m=0}^{\infty}\frac{(\mu)_m}{m!}\Phi_{k+1}^*(y,z,a+k+\lambda m)\ x^m, \tag{4.11}$$

$$D_a^k\zeta_\lambda^\mu(x,y;z,a) = (-1)^k(z)_k\zeta_\lambda^\mu(x,y;z+k,a). \tag{4.12}$$

Proof. Using (4.1), we get

$$D_x^k\zeta_\lambda^\mu(x,y;z,a) = \sum_{m=k}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^{m-k}y^n}{(m-k)!\ (a+n+\lambda m)^z}. \tag{4.13}$$

Now, letting $m \to m+k$ in (4.13) and considering the definition (2.1), we get the right-hand side of formula (4.10). Similarly, one can proof the formulas (4.11) and (4.12). □

Note that the results (4.10), (4.11) and (4.12) can be obtained directly from equation (4.2) by differentiating both sides of (4.2) with respect to x, y and a, respectively.

In view of the relationship (2.6), we find from equation (4.12) that

$$D_a^k\zeta_2(z;a,\lambda) = (-1)^k(z)_k\zeta_2(z+k;a,\lambda). \tag{4.14}$$

Similarly, according to the relation (2.3) formula (4.12) reduces to the result

$$D_a^k\zeta(z,a) = (-1)^k(z)_k\zeta(z+k,a), \tag{4.15}$$

which is a known result (see e.g. [**8**, p. 2 (1.8)]). A function closely associated with the derivative of the gamma function is the diagamma function, defined by

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad x \neq 0,-1,-2,\dots \tag{4.16}$$

Now, we wish to establish the derivative of the function ζ_λ^μ with respect to the parameter μ.

Theorem 4.6. *Let* $\mu \in C \setminus \{0, -1, -2, \ldots\}$. *Then*

$$\frac{\partial}{\partial\mu}\zeta_\lambda^\mu(x, y; z, a) = \sum_{m=0}^{\infty} (\mu)_m \Phi(y, z, a + \lambda m) \left[\psi(\mu + m) - \psi(\mu)\right] \frac{x^m}{m!}. \tag{4.17}$$

Proof. By noting that

$$\frac{\partial}{\partial\mu}\left[(\mu)_m\right] = \frac{\partial}{\partial\mu}\left[\frac{\Gamma(\mu + m)}{\Gamma(\mu)}\right] = (\mu)_m \left[\psi(\mu + m) - \psi(\mu)\right], \tag{4.18}$$

we obtain the result (4.17). □

According to the algebraic identity (cf. [**17**, p. 295 (6.7)]):

$$\psi(x + 1) - \psi(x + m + 1) = \sum_{k=1}^{m} \frac{(-1)^k m!\Gamma(x + 1)}{k(m - k)!\Gamma(x + k + 1)}, \tag{4.19}$$

the formula (4.17) can be rewritten in the following more compact form

$$\frac{\partial}{\partial\mu}\zeta_\lambda^\mu(x, y; z, a) = \sum_{m=0}^{\infty}\sum_{k=1}^{m} \frac{(-1)^{k+1}\Gamma(\mu + m)}{k\ (m - k)!\ \Gamma(\mu + k)} \Phi(y, z, a + \lambda m)x^m. \tag{4.20}$$

Finally, let us recall the definition of the Weyl fractional derivative of the exponential function e^{-at}, $a > 0$ of order ν in the form (see [**17**, p. 248 (7.4)])

$$D^\nu \mathrm{e}^{-at} = a^\nu \mathrm{e}^{-at}, \qquad (\nu \text{ is not restricted to be postive integer}). \tag{4.21}$$

We now proceed to find the fractional derivative of the function ζ_λ^μ with respect to z.

Theorem 4.7. *Let* $\nu > 0$. *Then*

$$D_z^\nu \left[\zeta_\lambda^\mu(x, y; z, a)\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!(a + n + \lambda m)^z} \times \left[\log(a + n + \lambda m)\right]^\nu. \tag{4.22}$$

Proof. Since

$$(a + n + \lambda m)^{-z} = \mathrm{e}^{-z\log(a+n+\lambda m)},$$

we have

$$\zeta_\lambda^\mu(x, y; z, a) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \mathrm{e}^{-z\log(a+n+\lambda m)}.$$

The desired result now follows by applying the formula (4.21) to the above identity. □

5. Series Expansions

Series expansions play an important role in the investigation of various useful properties of the sequences which they expand. This section aims at establishing some series relations for the double series zeta function ζ_λ^μ. First, based on two forms of Taylor's theorem for the deduction of addition and multiplication theorems for the confluent hypergeometric function (cf. [**9**, p. 63, eq. (2.8.8) and (2.8.9)] or [**20**, p. 21–22]):

$$f(x+y)=\sum_{m=0}^{\infty} f^{(m)}(x)\frac{y^m}{m!}, \tag{5.1}$$

and

$$f(xy)=\sum_{m=0}^{\infty} f^{(m)}(x)\frac{[(y-1)x]^m}{m!}, \tag{5.2}$$

where $|y|<\rho$, ρ being the radius of convergence of the analytic function $f(x)$, we aim to discuss certain addition and multiplication theorems of the generalized double zeta function ζ_λ^μ.

Theorem 5.1. *Let* $|\omega|<1$. *Then*

$$\zeta_\lambda^\mu(x+\omega,y;z.a)=\sum_{k=0}^{\infty}(\mu)_k\zeta_\lambda^{\mu+k}(x,y;z,a+\lambda k)\frac{\omega^k}{k!}, \tag{5.3}$$

$$\zeta_\lambda^\mu(x,y+\omega;z.a)=\sum_{m,n=0}^{\infty}(\mu)_m\Phi_{n+1}^*(\omega,z,a+n+\lambda m)\frac{x^m\ y^n}{m!}, \tag{5.4}$$

$$\zeta_\lambda^\mu(x\omega,y;z.a)=\sum_{k=0}^{\infty}(\mu)_k\zeta_\lambda^{\mu+k}\left((\omega-1)x,y;z,a+\lambda k\right)\frac{x^k}{k!}, \tag{5.5}$$

$$\zeta_\lambda^\mu(x,y\omega;z.a)=\sum_{m,n=0}^{\infty}(\mu)_m\Phi_{n+1}^*\left(y(\omega-1),z,a+n+\lambda m\right)\frac{x^m\ y^n}{m!}. \tag{5.6}$$

Proof. The proof is a direct application of the formulas (5.1), (5.2) and the first two results of Theorem 4.5. □

Next, we derive the Taylor expansion of ζ_λ^μ in the fourth variable a.

Theorem 5.2. *Let* $|\omega|<\mathrm{Re}\,(a)$. *Then*

$$\zeta_\lambda^\mu(x,y;z,a+\omega)=\sum_{k=0}^{\infty}(-1)^k\Phi(y,z+k,a)\times\Phi_{\mu,\lambda}^*(x,-k,\omega)\frac{(z)_k}{k!}. \tag{5.7}$$

Proof. We have

$$\zeta_\lambda^\mu(x,y;z,a+\omega)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}(a+n)^{-z}\left(1+\frac{\omega+\lambda m}{a+n}\right)^{-z}.$$

The result now follows from the binomial expansion and the definitions (1.4) and (2.7). □

In fact, equation (5.7) gives a number of known and new series expansions as particular cases. For instance, in view of the relation (2.4) we find from (5.7) that

$$\Phi(y,z,a-\omega)=\sum_{k=0}^{\infty}(z)_k\Phi(y,z+k,a)\,\frac{\omega^k}{k!},\qquad (z\neq 1,\ \ |\omega|<|a),\tag{5.8}$$

which is a known result due to Raina and Chhajed [**18**, p. 93 (3.3)]. Moreover according to the relationship (2.3), equation (5.7) yields

$$\zeta(z,a+\omega)=\sum_{k=0}^{\infty}(-1)^k(z)_k\zeta(z+k,a)\frac{\omega^k}{k!}.\tag{5.9}$$

Note that, formula (5.9) is a known result due to Kanemitsu et al. [**11**, p. 5, (2.6*)]. Further, in view of the relation (2.6), formula (5.7) yields

$$\zeta_2(z;a+\omega,\lambda)=\sum_{k=0}^{\infty}(-1)^k(z)_k\zeta(z+k,a)\times\zeta\left(-k,\frac{\omega}{\lambda}\right)\frac{\lambda^k}{k!}.\tag{5.10}$$

Furthermore, if in (5.8) we let $y=\mathrm{e}^{2\pi i\alpha}$ (in conjunction with (1.5)), formula (5.8) reduces to a known power series expansion due to Klusch [15]

$$\phi(\alpha,a+\omega,z)=\sum_{k=0}^{\infty}(-1)^k(z)_k\phi(\alpha,a,z+k)\omega^k,\qquad |\omega|<a.\tag{5.11}$$

Another expansion function for ζ_λ^μ can be derived by using the result [**16**, p. 374, exercise 9.4(7)]

$${}_2F_1\left[a,a+\frac{1}{2};\frac{1}{2};x\right]=\frac{1}{2}\left(1+\sqrt{x}\right)^{-2a}+\frac{1}{2}\left(1-\sqrt{x}\right)^{-2a}.\tag{5.12}$$

Theorem 5.3. *Let* $\mu\geq 1$, $\mathrm{Re}\,(a)>0$, $|x|<1$, $|y|<1$ *and* $|\omega|<|a|$. *Then*

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{z+2k-1}{2k}\zeta_\lambda^\mu(x,y;z+2k,a)\omega^{2k}=\ &\frac{1}{2}\left[\zeta_\lambda^\mu(x,y;z,a-\omega)\right.\\ &\left.+\ \zeta_\lambda^\mu(x,y;z,a+\omega)\right].\end{aligned}\tag{5.13}$$

Proof. We have

$$\begin{aligned}&\sum_{k=0}^{\infty}\binom{z+2k-1}{2k}\zeta_\lambda^\mu(x,y;z+2k,a)\omega^{2k}\\ &\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!(a+n+\lambda m)^z}\sum_{k=0}^{\infty}\frac{(z)_{2k}\omega^{2k}}{(2k)!(a+n+\lambda m)^{2k}}.\end{aligned}\tag{5.14}$$

By applying the formula (5.12) to the last summation in the right-hand side of equation (5.14), we come to the result (5.13). □

Next, we derive a series expansion for the function ζ_λ^μ involving Appell's function F_2 of two variables defined by the series (see e.g. [**21**, p. 23 (3)])

$$F_2\left[a,b,b';c,c';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m(c')_n m!\, n!}. \tag{5.15}$$

Theorem 5.4. *Let* $\max\{|x/b|,|y/b|\}<1$, $|b|<\operatorname{Re}a$ *and* $\lambda\neq 0$. *Then*

$$\begin{aligned}\sum_{k=0}^{\infty}(\nu)_k\zeta_\lambda^\mu(x,y;z+k,a+b)\frac{\omega^k}{k!}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\\ &\times F_2\left[z,\nu,1;z,1;\frac{\omega}{a+n+\lambda m},\frac{-b}{a+n+\lambda m}\right](a+n+\lambda m)^{-z}.\end{aligned}\tag{5.16}$$

Proof. Since

$$(a+n+\lambda m+b)^{-(z+k)}=(a+n+\lambda m)^{-(z+k)}\left(1+\frac{b}{a+n+\lambda m}\right)^{-(z+k)},$$

it follows that

$$\begin{aligned}\sum_{k=0}^{\infty}(\nu)_k\zeta_\lambda^\mu(x,y;z+k,a+b)\frac{\omega^k}{k!}=&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!(a+n+\lambda m)^z}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(z)_{k+s}(\nu)_k}{k!s!(z)_k}\\ &\times\left(\frac{\omega}{a+n+\lambda m}\right)^k\left(\frac{-b}{a+n+\lambda m}\right)^s.\end{aligned}\tag{5.17}$$

The result (5.16) now follows from the definition (5.15). □

Indeed, equation (5.16) is a generalization and unification of the well-known result of Ramanujan

$$\zeta(\nu,1+x)=\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\zeta(\nu+n)(-x)^n.$$

In view of the relations (2.3), (2.4) and (2.5) formula (5.16) yields the following interesting special cases:

$$\sum_{k=0}^{\infty}(\nu)_k\zeta(z+k,a+b)\frac{\omega^k}{k!}=\sum_{n=0}^{\infty}F_2\left[z,\nu,1;z,1;\frac{\omega}{a+n},\frac{-b}{a+n}\right](a+n)^{-z},\tag{5.18}$$

$$\sum_{k=0}^{\infty}(\nu)_k\Phi(y,z+k,a+b)\frac{\omega^k}{k!}=\sum_{n=0}^{\infty}F_2\left[z,\nu,1;z,1;\frac{\omega}{a+n},\frac{-b}{a+n}\right]\frac{y^n}{(a+n)^z},\tag{5.19}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}(\nu)_k\zeta_2(z+k;a+b,\lambda)\frac{\omega^k}{k!}\\ =\sum_{n=0}^{\infty}F_2\left[z,\nu,1;z,1;\frac{\omega}{a+n+\lambda m};\frac{-b}{a+n+\lambda m}\right](a+n+\lambda m)^{-z},\end{aligned}\tag{5.20}$$

respectively.

Finally, we recall here a generating function of the Hurwitz-Lerch zeta function due to Raina and Srivastava in the form (see [**19**, p. 302] or [**18**, p. 96 (3.11)])

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n}{(\gamma)_n}\Phi(y,\nu+\beta-\gamma+n,a)\frac{\omega^n}{n!}\\ =\sum_{k=0}^{\infty}\frac{y^k}{(a+k)^{\nu+\beta-\gamma}}{}_2F_1\left[\nu,\beta;\gamma;\frac{\omega}{(a+k)}\right].\end{aligned}\tag{5.21}$$

A further generalization of the above known formula (5.21) is given by the following theorem.

Theorem 5.5. *Let* $\operatorname{Re}\nu>0,\ \operatorname{Re}\beta>0,\ \operatorname{Re}\gamma>0,\ \operatorname{Re}\mu>0,\ \lambda\neq0$ *and* $|\omega/a|<1$. *Then*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n}{(\gamma)_n}\zeta_\lambda^{\mu+n}(x,y;\nu+\beta-\gamma+k,a)\frac{\omega^n}{n!}\\ =\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\mu)_m y^k}{(a+k+\lambda m)^{\nu+\gamma-\beta}}\times{}_3F_2\left[\begin{matrix}\nu,\beta,\mu+m; & \\ & \dfrac{\omega}{(a+k+\lambda m)}\\ \gamma,\mu; & \end{matrix}\right]\frac{x^m}{m!}.\end{aligned}\tag{5.22}$$

Proof. We refer to the proof of Theorem 5.4. □

Clearly, in view of the relationship (2.4) formula (5.22) reduces to (5.21).

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