Acta Math. Univ. Comenianae
Vol. LXXVIII, 2(2009), pp. 223–233

SPECIAL REPRESENTATIONS OF THE BOREL AND MAXIMAL PARABOLIC SUBGROUPS OF $G_2(q)$

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Abstract. A square matrix over the complex field with a non-negative integral trace is called a quasi-permutation matrix. For a finite group G, the minimal degree of a faithful representation of G by quasi-permutation matrices over the complex numbers is denoted by $c(G)$, and $r(G)$ denotes the minimal degree of a faithful rational valued complex character of G. In this paper $c(G)$ and $r(G)$ are calculated for the Borel and maximal parabolic subgroups of $G_2(q)$.

1. Introduction

Let G be a finite linear group of degree n, that is, a finite group of automorphisms of an n-dimensional complex vector space. We shall say that G is a quasi-permutation group if the trace of every element of G is a non-negative rational integer. The reason for this terminology is that, if G is a permutation group of degree n, its elements, considered as acting on the elements of a basis of an n-dimensional complex vector space V, induce automorphisms of V forming a group isomorphic to G. The trace of the automorphism corresponding to an element x of G is equal to the number of letters left fixed by x, and so is a non-negative integer. Thus, a permutation group of degree n has a representation as a quasi-permutation group of degree n (See [**12**]). In [**4**] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results.

By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. By a rational valued character we mean a complex character χ of G such that $\chi(g) \in Q$ for all $g \in G$. As the values of the characters of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of

Received April 6, 2008; revised December 16, 2008.
2000 *Mathematics Subject Classification.* Primary 20C15.
Key words and phrases. Borel and parabolic subgroups; rational valued character; quasi-permutation representations.

G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module.

We will call a homomorphism from G to $GL(n, Q)$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G.

Finding the above quantities has been carried out in some papers, for example in [**5**], [**6**], [**7**] and [**10**] we found them for the groups $GL(2, q)$, $SU(3, q^2)$, $PSU(3, q^2)$, $SL(3, q)$, $PSL(3, q)$ and $G_2(2^n)$ respectively. In [3] we found the rational character table and above values for the group $PGL(2, q)$.

In this paper we will calculate $c(G)$ and $r(G)$ where G is a Borel subgroup or the maximal parabolic subgroups of $G_2(q)$.

2. Notation and preliminaries

Let $G = G_2(q)$ be the Chevalley group of type G_2 defined over K. An excellent description of the group can be found in [**11**]. We summarize some properties of the group. Let Σ be the set of roots of a simple Lie algebra of type G_2. In some fixed ordering the set of positive roots of Σ can be written as

$$\Sigma^+ = \{a, b, a + b, 2a + b, 3a + b, 3a + 2b\}$$

and Σ consists of the elements of Σ^+ and their negatives. For each $r \in \Sigma$, let $x_r(t), x_{-r}(t)$ and ω_r be as in [**11**]. Moreover we denote the element $h(\chi)$ of [**11**] by $h(z_1, z_2, z_3)$, where $\chi(\xi_i) = z_i$ with $z_1z_2z_3 = 1$. Note that $a = \xi_2$, $b = \xi_1 - \xi_2$ and $\xi_1 + \xi_2 + \xi_3 = 0$. For simplicity of notation $h(x^i, x^j, x^{-i-j})$ is also denoted by $h_x(i, j, -i-j)$ for $x = \gamma, \theta, \omega$, etc. Let $X_r = \{x_r(t) \mid t \in K\}$ be the one-parameter subgroup corresponding to a root r. Set

$$\begin{aligned} &H = \{h(z_1, z_2, z_3) \mid z_i \in K^\times, z_1z_2z_3 = 1\},\\ &U = X_aX_bX_{a+b}X_{2a+b}X_{3a+b}X_{3a+2b},\\ &B = HU, \quad P =< B, \omega_a >, \quad Q =< B, \omega_b > . \end{aligned}$$

Then $B = N_G(U)$ is a Borel subgroup and P and Q are the maximal parabolic subgroups containing B.

By [**1**], [**8**], [**9**], every irreducible character of B will be defined as an induced character of some linear character of a subgroup. This implies that B is an M-group. The character tables of the Borel subgroup B for different q are given in Tables I of [**1**], [**8**], [**9**].

The character tables of parabolic subgroups

$$P =< B, \omega_a >= B \cup B\omega_aB, \qquad Q =< B, \omega_b >= B \cup B\omega_bB$$

for different q are given in Tables [A.4, A.6], [III, IV], [II-2, III-2] of [**1**], [**8**], [**9**] respectively.

Now we give algorithms for calculation of $r(G)$ and $c(G)$.

Definition 2.1. Let χ be a character of G such that, for all $g \in G$, $\chi(g) \in Q$ and $\chi(g) \geq 0$. Then we say that χ is a non-negative rational valued character.

Let η_i for $0 \le i \le r$ be Galois conjugacy classes of irreducible complex characters of G. For $0 \le i \le r$ let φ_i be a representative of the class η_i with $\varphi_o = 1_G$. Write $\Psi_i = \sum_{\chi_i \in \eta_i} \chi_i$, $m_i = m_Q(\varphi_i)$ and $K_i = \ker \varphi_i$. We know that $K_i = \ker \Psi_i$. For $I \subseteq \{0, 1, 2, \cdots, r\}$, put $K_I = \cap_{i \in I} K_i$. By definition of $r(G)$ and $c(G)$ and using the above notations we have:

$$r(G) = \min\{\xi(1) : \xi = \sum_{i=1}^{r} n_i \Psi_i, \quad n_i \ge 0,\ K_I = 1 \text{ for } I = \{i, i \ne 0, n_i > 0\}\}$$

$$c(G) = \min\{\xi(1) : \xi = \sum_{i=0}^{r} n_i \Psi_i, \quad n_i \ge 0,\ K_I = 1 \text{ for } I = \{i, i \ne 0, n_i > 0\}\}$$

where $n_0 = -\min\{\xi(g) | g \in G\}$ in the case of $c(G)$.

In [**2**] we defined $d(\chi)$, $m(\chi)$ and $c(\chi)$ (see Definition 3.4). Here we can redefine it as follows:

Definition 2.2. Let χ be a complex character of G such that $\ker \chi = 1$ and $\chi = \chi_1 + \cdots + \chi_n$ for some $\chi_i \in \mathrm{Irr}(G)$. Then define

(1) $d(\chi) = \sum_{i=1}^{n} |\Gamma_i(\chi_i)| \chi_i(1)$,

(2) $m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G, \\ |\min\{\sum_{i=1}^{n} \sum_{\alpha \in \Gamma_i(\chi_i)} \chi_i^{\alpha}(g) : g \in G\}| & \text{otherwise}, \end{cases}$

(3) $c(\chi) = \sum_{i=1}^{n} \sum_{\alpha \in \Gamma_i(\chi_i)} \chi_i^{\alpha} + m(\chi) 1_G$.

So

$$r(G) = \min\{d(\chi) : \ker \chi = 1\}$$

and

$$c(G) = \min\{c(\chi)(1) : \ker \chi = 1\}.$$

We can see all the following statements in [**2**].

Corollary 2.3. *Let $\chi \in \mathrm{Irr}(G)$, then $\sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}$ is a rational valued character of G. Moreover $c(\chi)$ is a non-negative rational valued character of G and $c(\chi) = d(\chi) + m(\chi)$.*

Lemma 2.4. *Let $\chi \in \mathrm{Irr}(G)$, $\chi \ne 1_G$. Then $c(\chi)(1) \ge d(\chi) + 1 \ge \chi(1) + 1$.*

Lemma 2.5. *Let $\chi \in \mathrm{Irr}(G)$. Then*

(1) *$c(\chi)(1) \ge d(\chi) \ge \chi(1)$;*

(2) *$c(\chi)(1) \le 2d(\chi)$. Equality occurs if and only if $Z(\chi)/\ker \chi$ is of even order.*

3. Quasi-permutation representations

In this section we will calculate $r(G)$ and $c(G)$ for Borel and parabolic subgroups of $G_2(q)$. First we shall determine the above quantities for a Borel subgroup.

Theorem 3.1. *Let B be a Borel subgroup of $G_2(q)$, then*

(1) $r(B) = \begin{cases} 2q(q-1)|\Gamma(\chi_7(k))| & \text{if } q = 3^n, \\ q^2(q-1)|\Gamma(\chi_7(k))| & \text{otherwise}, \end{cases}$

(2) $c(B) = \begin{cases} 2q^2|\Gamma(\chi_7(k))| & \text{if } q = 3^n, \\ q^3|\Gamma(\chi_7(k))| & \text{otherwise}, \end{cases}$

(3) $\lim_{q\to\infty} \dfrac{c(B)}{r(B)} = 1.$

Proof. Since there are similar proofs for $q = 2^n$, $q = p^n$; $p \neq 3$, we will prove only the case $q = 2^n$.

In order to calculate $r(B)$ and $c(B)$, we need to determine $d(\chi)$ and $c(\chi)(1)$ for all characters that are faithful or $\bigcap_\chi \ker \chi = 1$.

Now, by Corollary 2.3 and Lemmas 2.4, 2.5 and [**9**, Table I-1], for the Borel subgroup B we have

$$\begin{aligned}
d(\chi_1) &= |\Gamma(\chi_1(k,l))|\chi_1(k,l)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q^2(q-1)+1 \\
&\quad \text{and} \quad c(\chi_1)(1) \geq q^3+2, \\
d(\chi_2) &= |\Gamma(\chi_2(k))|\chi_2(k)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq (q-1)(q^2+1) \\
&\quad \text{and} \quad c(\chi_2)(1) \geq q(q^2+1), \\
d(\chi_3) &= |\Gamma(\chi_3(k))|\chi_3(k)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq (q-1)(q^2+1) \\
&\quad \text{and} \quad c(\chi_3)(1) \geq q(q^2+1), \\
d(\chi_4) &= |\Gamma(\chi_4(k))|\chi_4(k)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q(q^2-1) \\
&\quad \text{and} \quad c(\chi_4)(1) \geq q^2(q+1), \\
d(\chi_5) &= |\Gamma(\chi_5(k))|\chi_5(k)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q(q^2-1) \\
&\quad \text{and} \quad c(\chi_5)(1) \geq q^2(q+1), \\
d(\chi_6) &= |\Gamma(\chi_6(k))|\chi_6(k)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q(q^2-1) \\
&\quad \text{and} \quad c(\chi_6)(1) \geq q^2(q+1), \\
d(\chi_7) &= |\Gamma(\theta_1)|\theta_1(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq (q-1)(q^2+q-1) \\
&\quad \text{and} \quad c(\chi_7)(1) \geq q(q^2+q-1), \\
d(\chi_8) &= |\Gamma(\theta_3)|\theta_3(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q(q-1)(2q-1) \\
&\quad \text{and} \quad c(\chi_8)(1) \geq q^2(2q-1), \\
d(\chi_9) &= |\Gamma(\Sigma_{l=0}^2\theta_3(k,l))|(\Sigma_{l=0}^2\theta_3(k,l))(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q(q-1)(2q-1) \\
&\quad \text{and} \quad c(\chi_9)(1) \geq q^2(2q-1),
\end{aligned}$$

$$
\begin{aligned}
d(\chi_{10}) &= |\Gamma(\theta_4(r,s))|\theta_4(r,s)(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq \frac{q(q-1)(3q-1)}{2} \\
&\text{and} \qquad c(\chi_{10})(1) \geq \frac{q^2(3q-1)}{2}, \\
d(\chi_{11}) &= |\Gamma(\Sigma_{x\in K}\theta_5(x))|(\Sigma_{x\in K}\theta_5(x))(1) + |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q^3(q-1) \\
&\text{and} \qquad c(\chi_{11})(1) \geq q^4, \\
d(\chi_{12}) &= |\Gamma(\chi_1(k,l))|\chi_1(k,l)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq q^2(q-1)^2+1 \\
&\text{and} \qquad c(\chi_{12})(1) \geq q^3(q-1)+2, \\
d(\chi_{13}) &= |\Gamma(\chi_2(k))|\chi_2(k)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq (q-1)(q^3-q^2+1) \\
&\text{and} \qquad c(\chi_{13})(1) \geq q(q^3-q^2+1), \\
d(\chi_{14}) &= |\Gamma(\chi_3(k))|\chi_3(k)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq (q-1)(q^3-q^2+1) \\
&\text{and} \qquad c(\chi_{14})(1) \geq q(q^3-q^2+1), \\
d(\chi_{15}) &= |\Gamma(\chi_4(k))|\chi_4(k)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq q(q-1)(q^2-q+1) \\
&\text{and} \qquad c(\chi_{15})(1) \geq q^2(q^2-q+1), \\
d(\chi_{16}) &= |\Gamma(\chi_5(k))|\chi_5(k)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq q(q-1)(q^2-q+1) \\
&\text{and} \qquad c(\chi_{16})(1) \geq q^2(q^2-q+1), \\
d(\chi_{17}) &= |\Gamma(\chi_6(k))|\chi_6(k)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq q(q-1)(q^2-q+1) \\
&\text{and} \qquad c(\chi_{17})(1) \geq q^2(q^2-q+1), \\
d(\chi_{18}) &= |\Gamma(\theta_1)|\theta_1(1) + |\Gamma(\theta_2)|\theta_2(1) \geq (q-1)^2(q^2+1) \\
&\text{and} \qquad c(\chi_{18})(1) \geq q(q-1)(q^2+1), \\
d(\chi_{19}) &= |\Gamma(\theta_3)|\theta_3(1) + |\Gamma(\theta_2)|\theta_2(1) \geq q(q-1)^2(q+1) \\
&\text{and} \qquad c(\chi_{19})(1) \geq q^2(q-1)(q+1), \\
d(\chi_{20}) &= \left|\Gamma\left(\Sigma_{l=0}^2\theta_3(k,l)\right)\right|\left(\Sigma_{l=0}^2\theta_3(k,l)\right)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq q(q-1)^2(q+1) \\
&\text{and} \qquad c(\chi_{20})(1) \geq q^2(q-1)(q+1), \\
d(\chi_{21}) &= |\Gamma(\theta_4(r,s))|\theta_4(r,s)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq \frac{q(q-1)^2(2q+1)}{2} \\
&\text{and} \qquad c(\chi_{21})(1) \geq \frac{q^2(q-1)(2q+1)}{2}, \\
d(\chi_{22}) &= \left|\Gamma\left(\Sigma_{x\in K}\theta_5(x)\right)\right|\left(\Sigma_{x\in K}\theta_5(x)\right)(1) + |\Gamma(\theta_2)|\theta_2(1) \geq 2q^2(q-1)^2 \\
&\text{and} \qquad c(\chi_{22})(1) \geq 2q^3(q-1), \\
d(\chi_7(k)) &= |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q^2(q-1) \\
&\text{and} \qquad c(\chi_7)(k)(1) \geq q^3, \\
d(\theta_2) &= |\Gamma(\theta_2)|\theta_2(1) = q^2(q-1)^2 \\
&\text{and} \qquad c(\theta_2(1) = q^3(q-1),
\end{aligned}
$$

An overall picture is provided by the Table I on the next page.

For the character $\chi_7(k)$, $k \in R_0$ as $|R_0| = q-1$, so $|\Gamma(\chi_7(k))| \leq q-1$, where $\Gamma(\chi_7(k)) = \Gamma(Q(\chi_7(k)) : Q)$. Therefore we have

$$q^2(q-1) \leq d(\chi_7(k)) \leq q^2(q-1)^2.$$

Now by Table I and the above equality we have

$$\min\{d(\chi) : \ker\chi = 1\} = d(\chi_7(k)) = q^2(q-1)|\Gamma(\chi_7(k))|$$

and

$$\min\{c(\chi)(1) : \ker\chi = 1\} = c(\chi_7(k))(1) = q^3|\Gamma(\chi_7(k))|.$$

The quasi-permutation representations of Borel subgroup of $G_2(3^n)$ are constructed by the same method. In this case by [**8**, Table I] we have

$$\ker\chi_7(k) \bigcap \ker\chi_6(k) = 1.$$

Now, it is not difficult to calculate the values of $d(\chi)$ and $c(\chi)(1)$, so

$$\begin{aligned}\min\{d(\chi) : \ker\chi = 1\} &= |\Gamma(\chi_7(k))|\chi_7(k)(1) + |\Gamma(\chi_6(k))|\chi_6(k)(1)\\ &= 2q(q-1)|\Gamma(\chi_7(k))| = 2q(q-1)|\Gamma(\chi_6(k))|\end{aligned}$$

and

$$\begin{aligned}\min\{c(\chi)(1) : \ker\chi = 1\} = 2q^2|\Gamma(\chi_7(k))| &= 2q^2|\Gamma(\chi_6(k))|\\ &(\text{Since } |\Gamma(\chi_7(k))| = |\Gamma(\chi_6(k))|).\end{aligned}$$

By parts (1) and (2) we have

$$\frac{c(B)}{r(B)} = \begin{cases} \dfrac{q^2}{q(q-1)} & \text{if } q = 3^n,\\ \dfrac{q^3}{q^2(q-1)} & \text{otherwise.}\end{cases}$$

Hence $\lim\limits_{q\to\infty} \dfrac{c(B)}{r(B)} = 1$. Therefore the result follows. □

The following theorem gives the quasi-permutation representations of a maximal parabolic subgroup P.

Theorem 3.2.

A. *Let P be a maximal parabolic subgroup of $G_2(p^n)$, $p \neq 3$, then*

(1) $r(P) = q^2(q-1)$,

(2) $c(P) = q^3$,

(3) $\lim\limits_{q\to\infty} \dfrac{c(P)}{r(P)} = 1$.

Table I

χ	$d(\chi)$	$c(\chi)(1)$
χ_1	$\geq q^2(q-1)+1$	$\geq q^3+2$
χ_2	$\geq (q-1)(q^2+1)$	$\geq q(q^2+1)$
χ_3	$\geq (q-1)(q^2+1)$	$\geq q(q^2+1)$
χ_4	$\geq q(q^2-1)$	$\geq q^2(q+1)$
χ_5	$\geq q(q^2-1)$	$\geq q^2(q+1)$
χ_6	$\geq q(q^2-1)$	$\geq q^2(q+1)$
χ_7	$\geq (q-1)(q^2+q-1)$	$\geq q(q^2+q-1)$
χ_8	$\geq q(q-1)(2q-1)$	$\geq q^2(2q-1)$
χ_9	$\geq q(q-1)(2q-1)$	$\geq q^2(2q-1)$
χ_{10}	$\geq q(q-1)(3q-1)/2$	$\geq q^2(3q-1)/2$
χ_{11}	$\geq q^3(q-1)$	$\geq q^4$
χ_{12}	$\geq q^2(q-1)^2+1$	$\geq q^3(q-1)+2$
χ_{13}	$\geq (q-1)(q^3-q^2+1)$	$\geq q(q^3-q^2+1)$
χ_{14}	$\geq (q-1)(q^3-q^2+1)$	$\geq q(q^3-q^2+1)$
χ_{15}	$\geq q(q-1)(q^2-q+1)$	$\geq q^2(q^2-q+1)$
χ_{16}	$\geq q(q-1)(q^2-q+1)$	$\geq q^2(q^2-q+1)$
χ_{17}	$\geq q(q-1)(q^2-q+1)$	$\geq q^2(q^2-q+1)$
χ_{18}	$\geq (q-1)^2(q^2+1)$	$\geq q(q-1)(q^2+1)$
χ_{19}	$\geq q(q-1)^2(q+1)$	$\geq q^2(q-1)(q+1)$
χ_{20}	$\geq q(q-1)^2(q+1)$	$\geq q^2(q-1)(q+1)$
χ_{21}	$\geq q(q-1)^2(2q+1)/2$	$\geq q^2(q-1)(2q+1)/2$
χ_{22}	$\geq 2q^2(q-1)^2$	$\geq 2q^3(q-1)$
$\chi_7(k)$	$\geq q^2(q-1)$	$\geq q^3$
θ_2	$= q^2(q-1)^2$	$= q^3(q-1)$

B. *Let P be a maximal parabolic subgroup P of $G_2(3^n)$, then*

(1) $r(P) = q(q-1)(q+2)$,

(2) $c(P) = q^2(q+1)$,

(3) $\lim\limits_{q\to\infty} \dfrac{c(P)}{r(P)} = 1$.

Proof. **A.** Similar to the proof of Theorem 3.1, in order to calculate $r(P)$ and $c(P)$ we need to determine $d(\chi)$ and $c(\chi)(1)$ for all characters that are faithful or $\bigcap_\chi \ker\chi = 1$.

Now, in this case, since the degrees of faithful characters are minimal, so we consider just the faithful characters and by Corollary 2.3, Lemmas 2.4, 2.5 and

[**9**, Table (II-2)], for the maximal parabolic subgroup P of $G_2(2^n)$ we have

$$\begin{aligned} d(\chi_7(k)) &= |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q^2(q^2-1) && \text{and} && c(\chi_7)(k)(1) \geq q^3(q+1),\\ d(\chi_8(k)) &= |\Gamma(\chi_8(k))|\chi_8(k)(1) \geq q^2(q-1)^2 && \text{and} && c(\chi_8)(k)(1) \geq q^3(q-1),\\ d(\theta_7) &= |\Gamma(\theta_7)|\theta_7(1) = q^2(q-1) && \text{and} && c(\theta_7(1)) = q^3,\\ d(\theta_8) &= |\Gamma(\theta_8)|\theta_8(1) = q^3(q-1) && \text{and} && c(\theta_8(1)) = q^4. \end{aligned}$$

The values are set out in the following table

Table II

χ	$d(\chi)$	$c(\chi)(1)$
$\chi_7(k)$	$\geq q^2(q^2-1)$	$\geq q^3(q+1)$
$\theta_8(k)$	$\geq q^2(q-1)^2$	$\geq q^3(q-1)$
θ_7	$= q^2(q-1)$	$= q^3$
θ_8	$= q^3(q-1)$	$= q^4$

Now, by Table II we have

$$\min\{d(\chi) : \ker\chi = 1\} = d(\chi_7(k)) = q^2(q-1)) \qquad \text{and}$$

$$\min\{c(\chi)(1) : \ker\chi = 1\} = c(\chi_7(k))(1) = q^3.$$

By the same method for the maximal parabolic subgroup P of $G_2(p^n)$, $p \neq 3$ and by [**1**, Table A.6], Table III is constructed.

Table III

χ	$d(\chi)$	$c(\chi)(1)$
${}_P\chi_7(k)$	$\geq q^2(q^2-1)$	$\geq q^3(q+1)$
${}_P\theta_8(k)$	$\geq q^2(q-1)^2$	$\geq q^3(q-1)$
${}_P\theta_7$	$= q^2(q-1)$	$= q^3$
${}_P\theta_8$	$= q^3(q-1)$	$= q^4$
${}_P\theta_9$	$= q^2(q-1)^2/2$	$= q^3(q-1)/2$
${}_P\theta_{10}$	$= q^2(q-1)^2/2$	$= q^3(q-1)/2$
${}_P\theta_{11}$	$= q^2(q^2-1)/2$	$= q^3(q+1)/2$
${}_P\theta_{12}$	$= q^2(q^2-1)/2$	$= q^3(q+1)/2$

Now by Table III we have

$$\min\{d(\chi) : \ker\chi = 1\} = d(\chi_7(k)) = q^2(q-1)) \qquad \text{and}$$

$$\min\{c(\chi)(1) : \ker\chi = 1\} = c(\chi_7(k))(1) = q^3.$$

B. The quasi-permutation representations of maximal parabolic subgroup P of $G_2(3^n)$ are constructed by the same method in Theorem 3.1. In this case, by [**8**, Table III], we have

$$\ker\theta_{11}\bigcap\ker\chi_6(k) = 1.$$

This helps us to calculate

$$\min\{d(\chi) : \ker\chi = 1\} = q(q-1)(q+1) \qquad \text{and}$$

$$\min\{c(\chi)(1) : \ker\chi = 1\} = q^2(q+1).$$

For the both parts, it is elementary to see that $\lim\limits_{q\to\infty}\dfrac{c(P)}{r(P)} = 1$. Therefore the result follows. □

In the following theorem, we construct $r(G)$ and $c(G)$ of another parabolic subgroup Q of $G_2(q)$.

Theorem 3.3.

A. *Let Q be a maximal parabolic subgroup of $G_2(p^n)$, $p \neq 3$, then*

(1) $r(Q) = q(q^2-1)|\Gamma(\chi_7(k))|$,

(2) $c(Q) = q^3|\Gamma(\chi_7(k))|$,

(3) $\lim\limits_{q\to\infty}\dfrac{c(Q)}{r(Q)} = 1.$

B. *Let Q be a maximal parabolic subgroup of $G_2(3^n)$, then*

(1) $r(Q) = q(q-1)(q+2)$,

(2) $c(Q) = q^2(q+1)$,

(3) $\lim\limits_{q\to\infty}\dfrac{c(Q)}{r(Q)} = 1.$

Proof. **A)** As we have mentioned before, in order to calculate $r(Q)$ and $c(Q)$ we need to determine $d(\chi)$ and $c(\chi)(1)$ for all characters that are faithful or $\bigcap_\chi \ker\chi = 1$.

Now, in this case, since the degrees of faithful characters are minimal, so we consider just the faithful characters and by Corollary 2.3, Lemmas 2.4, 2.5 and [**9**, Table III-2] for the maximal parabolic subgroup Q of $G_2(2^n)$ we have

$$d(\chi_7(k)) = |\Gamma(\chi_7(k))|\chi_7(k)(1) \geq q(q^2-1) \;\text{ and }\; c(\chi_7)(k)(1) \geq q^3,$$

$$d(\theta_2) = |\Gamma(\theta_2)|\theta_2(1) \geq q(q-1)(q^2-1) \;\text{ and }\; c(\theta_2(1) \geq q^3(q-1),$$

$$d(\Sigma_{l=0}^2\theta_2(k,l)) = |\Gamma(\Sigma_{l=0}^2\theta_2(k,l))|(\Sigma_{l=0}^2\theta_2(k,l))(1) \geq q(q-1)(q^2-1) \;\text{ and }$$
$$c(\Sigma_{l=0}^2\theta_2(k,l)(1) \geq q^4(q-1),$$

$$d(\Sigma_{x\in X}\theta_3(x)) = |\Gamma(\Sigma_{x\in X}\theta_3(x))|(\Sigma_{x\in X}\theta_3(x))(1) = q^2(q-1)(q^2-1) \;\text{ and }$$
$$c(\Sigma_{x\in X}\theta_3(x))(1) = q^4(q-1)$$

The values are set out in Table IV.

For the character $\chi_7(k)$, $k \in R_0$ as $|R_0| = q-1$, so $|\Gamma(\chi_7(k))| \leq q-1$, where $\Gamma(\chi_7(k)) = \Gamma(Q(\chi_7(k)) : Q)$. Therefore we have

$$q(q^2-1) \leq d(\chi_7(k)) \leq q(q-1)(q^2-1).$$

Now, by Table IV we have

$$\min\{d(\chi) : \ker\chi = 1\} = d(\chi_7(k)) = mq(q^2-1) \qquad \text{and}$$
$$\min\{c(\chi)(1) : \ker\chi = 1\} = c(\chi_7(k))(1) = mq^3, \qquad \text{where} \ \ m = |\Gamma(\chi_7(k))|.$$

Table IV

χ	$d(\chi)$	$c(\chi)(1)$
$\chi_7(k)$	$\geq q(q^2-1)$	$\geq q^3$
$\chi_8(k)$	$\geq q(q-1)(q^2-1)$	$\geq q^3(q-1)$
$\Sigma_{l=0}^2 \theta_2(k,l)$	$\geq q(q-1)(q^2-1)$	$\geq q^3(q-1)$
$\Sigma_{x\in X}\theta_3(x)$	$= q^2(q-1)(q^2-1)$	$= q^4(q-1)$

For the maximal parabolic subgroup Q of $G_2(p^n)$, $p \neq 3$, by the same method and [**1**, Table A.6], Table V is constructed.

Table V

χ	$d(\chi)$	$c(\chi)(1)$
${}_Q\chi_7(k)$	$\geq q(q^2-1)$	$\geq q^3$
$\Sigma_{l=0}^2 \ {}_Q\theta_2(k,l)$	$\geq q(q-1)(q^2-1)$	$\geq q^3(q-1)$
$\Sigma_{x\in F_q^*} \ {}_Q\theta_3(x)$	$\geq q(q-1)^2(q^2-1)$	$\geq q^4(q-1)^2$
$\Sigma_{x\in F_q} \ {}_Q\theta_4(x)$	$= q^2(q-1)(q^2-1)$	$= q^4(q-1)$
${}_Q\theta_5(k) +_Q \theta_6(k)$	$\geq q(q-1)(q^2-1)$	$\geq q^3(q-1)$

For the character ${}_Q\chi_7(k)$, $k \in R_0$ as $|R_0| = q-1$, so $|\Gamma({}_Q\chi_7(k))| \leq q-1$, where $\Gamma({}_Q\chi_7(k)) = \Gamma(Q({}_Q\chi_7(k)) : Q)$. Therefore we have

$$q(q^2-1) \leq d(\chi_7(k)) \leq q(q-1)(q^2-1).$$

Now, by Table V we have

$$\min\{d(\chi) : \ker\chi = 1\} = d({}_Q\chi_7(k)) = mq(q^2-1) \qquad \text{and}$$
$$\min\{c(\chi)(1) : \ker\chi = 1\} = c({}_Q\chi_7(k))(1) = mq^3, \qquad \text{where} \ \ m = |\Gamma({}_Q\chi_7(k))|.$$

B. The quasi-permutation representations of maximal parabolic subgroup Q of $G_2(3^n)$ are constructed by the same method as in Theorem 3.1. In this case, by Table III of [**8**], we have

$$\ker\theta_{11} \bigcap \ker\chi_6(k) = 1.$$

This helps us to obtain

$$\min\{d(\chi) : \ker\chi = 1\} = q(q-1)(q+2) \qquad \text{and}$$
$$\min\{c(\chi)(1) : \ker\chi = 1\} = q^2(q+1).$$

It is obviously that also in this case $\lim_{q\to\infty} \dfrac{c(Q)}{r(Q)} = 1$. Therefore the result follows. □

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