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ABELIAN MODULES

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Abstract. In this note, we introduce abelian modules as a generalization of abelian rings. Let R be an arbitrary ring with identity. A module M is called *abelian* if, for any $m \in M$ and any $a \in R$, any idempotent $e \in R$, $mae = mea$. We prove that every reduced module, every symmetric module, every semicommutative module and every Armendariz module is abelian. For an abelian ring R, we show that the module M_R is abelian iff $M[x]_{R[x]}$ is abelian. We produce an example to show that $M[x, \alpha]$ need not be abelian for an abelian module M and an endomorphism α of the ring R. We also prove that if the module M is abelian, then M is p.p.-module iff $M[x]$ is p.p.-module, M is Baer module iff $M[x]$ is Baer module, M is p.q.-Baer module iff $M[x]$ is p.q.-Baer module.

1. Introduction

Throughout this paper R denotes an associative ring with identity 1, and modules will be unitary right R-modules.

Recall that a ring R is *reduced* if it has no nonzero nilpotent elements. A module M is called *reduced* if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Let e be an idempotent in R. Lee and Zhou extending the notions of Baer rings, quasi-Baer rings and p.p.-rings to modules: A module M is called *Baer* if, for any subset X of M, $r_R(X) = eR$, and M is called *quasi-Baer* if, for any submodule $X \subseteq M$, $r_R(X) = eR$, and M is called *p.p.-module* if, for any $m \in M$, $r_R(m) = eR$ (see, namely [**5**]). In this note we call M is a *p.q.-Baer* if, for any $m \in M$, $r_R(mR) = eR$. We write $R[x]$, $R[[x]]$, $R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively.

In [**5**], Lee and Zhou introduced those notions and the following notations. For a module M, we consider

$$M[x] = \left\{ \sum_{i=0}^{s} m_i x^i \ : \ s \geq 0, m_i \in M \right\},$$

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$$M[[x]] = \left\{ \sum_{i=0}^{\infty} m_i x^i \ : m_i \in M \right\},$$
$$M[x, x^{-1}] = \left\{ \sum_{i=-s}^{t} m_i x^i \ : \ s \geq 0, \ t \geq 0, \ m_i \in M \right\},$$
$$M[[x, x^{-1}]] = \left\{ \sum_{i=-s}^{\infty} m_i x^i \ : \ s \geq 0, \ m_i \in M \right\}.$$

Each of these is an abelian group under an obvious addition operation. Moreover $M[x]$ becomes a module over $R[x]$ for

$$m(x) = \sum_{i=0}^{s} m_i x^i \in M[x], \qquad f(x) = \sum_{i=0}^{t} a_i x^i \in R[x]$$

such that

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i a_j \right) x^k$$

The modules $M[x]$ and $M[[x]]$ are called the *polynomial extension* and the *power series extension of* M respectively. With a similar scalar product, $M[x, x^{-1}]$ (resp. $M[[x, x^{-1}]]$) becomes a module over $R[x, x^{-1}]$ (resp. $R[[x, x^{-1}]]$). The modules $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ are called the *Laurent polynomial extension* and the *Laurent power series extension of* M, respectively.

The module M is called *Armendariz* if the following condition 1. is satisfied, and a module M is called *Armendariz of power series type* if the following condition 2. is satisfied:

1. For any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x]$, $m(x)f(x) = 0$ implies $m_i a_j = 0$ for all i and j.
2. For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x]]$, $m(x)f(x) = 0$ implies $m_i a_j = 0$ for all i and j.

The ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. A module M_R is called *semicommutative* if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$. Buhphang and Rege in [**2**] studied basic properties of semicommutative modules. Agayev and Harmanci continued further investigations for semicommutative rings and modules in [**1**] and focused on the semicommutativity of subrings of matrix rings.

2. Abelian Modules

In this section the notion of an abelian module is introduced as a generalization of abelian rings to modules. We prove that many results of abelian rings can be extended to modules for this general settings.

The ring R is called *abelian* if every idempotent is central, that is $ae = ea$ for any $e^2 = e$, $a \in R$.

Definition 2.1. A module M is called *abelian* if, for any $m \in M$ and any $a \in R$, any idempotent $e \in R$, $mae = mea$.

Lemma 2.2.
1. *R is an abelian ring if and only if every R-module is abelian.*
2. *R is an abelian ring if and only if R_R is an abelian module.*

Proof. Clear. □

Example 2.3 shows that it is not the case that every R-module is non-abelian if R is non-abelian ring.

Example 2.3. There are abelian modules M_R over a non-abelian rings R.

Proof. Let F be any field. Consider the upper triangle 2×2 matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and the module $M_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. It is easy to check for any $m \in M$ and $a, b \in R$ $mab = mba$. Hence M is an abelian right R-module. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in R$. Then e is an idempotent element in R. For $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$, we have $ae = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ and $ea = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Hence the idempotent e is not central. Thus R is not an abelian ring. □

Proposition 2.4. *The class of abelian modules is closed under submodules, direct products and homomorphic images. Therefore abelian modules are closed under direct sums.*

Proof. Clear from definitions. □

Corollary 2.5. *A ring R is abelian if and only if every flat module M_R is abelian.*

Proof. It is clear from Proposition 2.4. □

Recall that a module M is called *cogenerated* by R if it is embedded in a direct product of copies of R. A module M is *faithful* if the only $a \in R$ such that $Ma = 0$ is $a = 0$. Proposition 2.6 is clear from Proposition 2.4.

Proposition 2.6. *The following conditions are equivalent:*
1. *R is an abelian ring.*
2. *Every cogenerated R-module is abelian.*
3. *Every submodule of a free R-module is abelian.*
4. *There exists a faithful abelian R-module.*

Lemma 2.7. *If the module M is semicommutative, then M is abelian. The converse holds if M is a p.p.-module.*

Proof. Let e be an idempotent in R and $m \in M$, $a \in R$. Since e is idempotent and M is semicommutative, we have $me(1_R - e) = 0$ implies that $meR(1_R - e) = 0$. For any $a \in R$ we have $mea(1_R - e) = 0$, that is, $mea = meae$. On the other hand, $m(1_R - e)e = 0$ implies that $m(1_R - e)Re = 0$. Then $m(1_R - e)ae = 0$ and so $mae = meae$. Hence $mea = mae$. Thus M is abelian. Suppose now M is an abelian and p.p.-module. Let $m \in M$ and $a \in R$ with $ma = 0$. Then $a \in r(m) = eR$ for some $e^2 = e \in R$. So $me = 0$ and $a = ea$. Hence $meR = 0$. By the assumption $mRe = 0$. Multiplying from the right by a, we have $mRea = 0$. Since $a = ea$, $mRa = 0$. Thus M is semicommutative. □

Lemma 2.8. *If the module M is Armendariz, then M is abelian. The converse holds if M is a p.p.-module.*

Proof. Let $m_1(x) = me - mer(1-e)x$, $m_2(x) = m(1-e) - m(1-e)rex \in M[x]$ and $f_1(x) = 1 - e + er(1-e)x$, $f_2(x) = e + (1-e)rex \in R[x]$, where e is an idempotent in R, $m \in M$ and $r \in R$. Then $m_1(x)f_1(x) = 0$ and $m_2(x)f_2(x) = 0$. Since M is Armendariz, $mer(1-e) = 0$ and $m(1-e)re = 0$. Then

$$mer = mere = mre.$$

Suppose now M is an abelian and p.p.-module. For any idempotent $e \in R$, any $a \in R$ and $m \in M$, we have

$mea = mae$. From Lemma 2.7, M is semicommutative, that is, $ma = 0$ implies $mRa = 0$ for any $m \in M$ and $a \in R$. Let $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$. Assume $m(x)f(x) = 0$. Then we have

$$m_0 a_0 = 0 \tag{1}$$

$$m_0 a_1 + m_1 a_0 = 0 \tag{2}$$

$$m_0 a_2 + m_1 a_1 + m_2 a_0 = 0 \tag{3}$$

$$\cdots$$

By hypothesis there exists an idempotent $e_0 \in R$ such that $r(m_0) = e_0 R$. Then (1) implies $e_0 a_0 = a_0$. Multiplying (2) by e_0 from the right, we have

$$0 = m_0 a_1 e_0 + m_1 a_0 e_0 = m_0 e_0 a_1 + m_1 e_0 a_0 = 0 + m_1 a_0.$$

It follows that $m_1 a_0 = 0$. By (2) $m_0 a_1 = 0$. Let $r(m_1) = e_1 R$. So $e_0 a_1 = a_1$ and $e_1 a_0 = a_0$. Multiplying (3) by $e_0 e_1$ from the right and using

$$m_0 R e_0 = 0 \qquad \text{and} \qquad m_1 R e_1 = 0 \qquad \text{and} \qquad m_2 a_0 e_0 e_1 = m_2 a_0$$

we have

$$m_2 a_0 = 0.$$

Then (3) becomes $m_0 a_2 + m_1 a_1 = 0$.
Multiplying this equation by e_0 from right and using

$$m_0 a_2 e_0 = m_0 e_0 a_2 = 0 \qquad \text{and} \qquad m_1 a_1 e_0 = m_1 e_0 a_1 = m_1 a_1$$

we have

$$m_1 a_1 = 0.$$

From (3) $m_0a_2 = 0$. Continuing in this way, we may conclude that $m_ia_j = 0$ for all $1 \le i \le s$ and $1 \le j \le t$. Hence M is Armendariz. This completes the proof. $\square$

Corollary 2.9. *If M is an Armendariz module of power series type, then M is abelian. The converse is true if M is a p.p.-module.*

Proof. Similar to the proof of Lemma 2.8. $\square$

The following example shows that, the converse of the first part of Lemma 2.7 and Lemma 2.8 may not be true in general.

Example 2.10. There exists an abelian module that is neither Armendariz nor semicommutative.

Proof. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^{2\times 2}$ the 2×2 full matrix ring over $\mathbb{Z}$,

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : a \equiv d \mod 2, \ \ b \equiv c \equiv 0 \mod 2 \right\}$$

and consider M to be the right R-module R_R. Since 0 and 1 are only idempotents in R, M_R is an abelian module. For $\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \in M$ and $\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \in R$, we have $\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0$, but $\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \neq 0$. So, M is not semicommutative. On the other hand, let

$$m(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x \in M[x],$$

$$f(x) = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x \in R[x].$$

Then $m(x)f(x) = 0$, but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$. Therefore M is not an Armendariz module. $\square$

Lemma 2.11. *If M is a reduced module, then M is abelian. The converse holds if M is a p.p.-module.*

Proof. Let M be reduced. Since any reduced module is semicommutative and by Lemma 2.7, any semicommutative module is abelian, M is abelian. Conversely, let M be an abelian and p.p.-module. Suppose $ma = 0$ for $m \in M$ and $a \in R$. If $x \in mR \cap Ma$, then there exist $m_1 \in M$ and $r_1 \in R$ such that $x = mr_1 = m_1a$. Since M is a p.p.-module, $ma = 0$ implies that $a \in r_R(m) = eR$ for some idempotent $e^2 = e \in R$. Then $a = ea$ and $xe = mr_1e = m_1ae$. Since M is abelian and $me = 0$, $mr_1e = mer_1 = m_1ae = m_1ea = m_1a = 0$. Hence $mR \cap Ma = 0$, that is, M is reduced. $\square$

Example 2.12 shows that there exists a p.q.-Baer module M but it is not a p.p.-module, and M is an abelian module but it is not reduced. So the converse statement of Theorem 2.11 need not be true in general.

Example 2.12. There exists an abelian p.q.-Baer module M that it is neither a reduced nor p.p.-module.

Proof. We consider the ring R and module M as in Example 2.10, that is,

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \ : \ a \equiv d,\ b \equiv 0 \text{ and } c \equiv 0 \mod 2 \right\}$$

In [**3**, Example 2 (1)], it is proven that M is a p.q.-Baer but not p.p.-module. In Example 2.10, it is proven that M is an abelian module, but not semicommutative. Since every reduced module is semicommutative, M can not be a reduced module. □

In [**6**] the module M is called *symmetric* if, $mab = 0$ implies $mba = 0$, for any $m \in M$ and $a, b \in R$.

Lemma 2.13. *If M is a symmetric module, then M is abelian. The converse holds if M is a p.p.-module.*

Proof. Assume that M is a symmetric module. Let $m \in M$ and $e^2 = e, a \in R$. Then $me(1-e)a = 0$. Being M symmetric implies $mea(1-e) = 0$. Hence $mea = meae$. Similarly $m(1-e)ea = 0$ implies $m(1-e)ae = 0$ and so $mae = meae$. It follows that $mae = mea$.

Conversely, suppose that M is a p.p.-module and abelian. Let $m \in M$, $a, b \in R$ and $mab = 0$. Since M is a p.p.-module, $b \in r_R(ma) = eR$ for an idempotent $e \in R$. Then $b = eb$ and $mae = 0$. By Lemma 2.7 we have $mRae = 0$, in particular, $mbae = 0$. By hypothesis $mba = meba = mbae = 0$. Hence M is symmetric. □

Theorem 2.14. *Let M be a p.p.-module. Then the following statements are equivalent.*

1. *M is reduced.*
2. *M is symmetric.*
3. *M is semicommutative.*
4. *M is Armendariz.*
5. *M is Armendariz of power series type.*
6. *M is abelian.*

Proof. 1. $\Longleftrightarrow$ 6. From Lemma 2.11.
2. $\Longleftrightarrow$ 6. Clear from Lemma 2.13.
3. $\Longleftrightarrow$ 6. From Lemma 2.7.
4. $\Longleftrightarrow$ 6. Clear from Lemma 2.8.
5. $\Longleftrightarrow$ 6. From Corollary 2.9. □

Lemma 2.15. *Let M be an abelian and p.p.-module. Then $r_R(m) = r_R(mR)$, for any $m \in M$.*

Proof. We always have $r_R(mR) \subset r_R(m)$. Conversely, every abelian p.p.-module is semicommutative, so $ma = 0$ implies that $mRa = 0$. Hence $r_R(m) \subset r_R(mR)$. Therefore $r_R(m) = r_R(mR)$. □

Corollary 2.16. *Let M be an abelian and p.p.-module. Then M is a p.q.-Baer module.*

Proof. Let M be an abelian and p.p.-module. By Lemma 2.15, we have $r_R(m) = r_R(mR) = eR$ for any $m \in M$ and an idempotent $e \in R$. Therefore M is a p.q.-Baer module. $\square$

Remark 2.17. Let S be a subring of a ring R with $1_R \in S$ and $M_S \subseteq L_R$. If L_R is abelian, then M_S is also abelian.

Theorem 2.18. *Let R be an abelian ring. Then we have the following:*
1. *M_R is abelian if and only if $M[x]_{R[x]}$ is abelian.*
2. *M_R is abelian if and only if $M[[x]]_{R[[x]]}$ is abelian.*

Proof. 1. If $M[x]_{R[x]}$ is abelian, by Remark 2.17, M_R is abelian.

Conversely, suppose that M_R is an abelian module. If R is abelian, by [**4**, Lemma 8(1)] idempotent elements of $R[x]$ belong to the ring R. So let $m(x) \in M[x]$, $f(x) \in R[x]$ and $e(x) = e(x)^2 = e^2 = e \in R$. Since R is abelian, by [**4**, Lemma 8], $R[x]$ is abelian, hence $f(x)e(x) = e(x)f(x)$. Therefore $m(x)f(x)e(x) = m(x)e(x)f(x)$, that is, $M[x]_{R[x]}$ is abelian.

2. If R is abelian, by [**4**, Lemma 8] idempotent elements of $R[[x]]$ belong to the ring R. The rest is similar to the proof of 1. $\square$

Let α be a ring homomorphism from R to R with $\alpha(1) = 1$. $R[x;\alpha]$ will denote the skew polynomial ring over R, hence $R[x;\alpha]$ is the ring with carrier $R[x]$ and multiplication $xa = \alpha(a)x$ for all $a \in R$. Let

$$M[x;\alpha] = \left\{ \sum_{i=0}^{s} m_i x^i \;:\; s \geq 0, \quad m_i \in M \right\}.$$

Then $M[x;\alpha]$ is an abelian group under an obvious addition operation. Moreover $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: For $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{i=0}^{t} a_i x^i \in R[x;\alpha]$

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i \alpha^i(a_j) \right) x^k.$$

Recall that a module M is said to be $\alpha-$*reduced* in [**5**] if, for any $m \in M$ and any $a \in R$,
1. $ma = 0$ implies $mR \cap Ma = 0$
2. $ma = 0$ if and only if $m\alpha(a) = 0$.

The module M is reduced if it is $\mathbf{1}-$reduced, where $\mathbf{1}$ is the identity endomorphism of R. In [**5**, Theorem 1.6], it is proven that if M is α-reduced, then $M[x;\alpha]$ is reduced and by Lemma 2.11, $M[x;\alpha]$ is abelian. One may suspects that if M_R is abelian, then $M[x,\alpha]_{R[x,\alpha]}$ is abelian also. But this is not the case.

Example 2.19. There exist abelian modules M_R such that $M[x,\alpha]_{R[x,\alpha]}$ need not be abelian.

Proof. Let F be any field, $R = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} : a, b, u, v \in F \right\}$,
$\alpha : R \to R$ defined by

$$\alpha \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} = \begin{pmatrix} u & v & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} \in R$$

and consider M to be the right R-module R_R. Since R is commutative, R and M are abelian. We claim $M[x;\alpha]$ is not an abelian module. Let e_{ij} denote the 4×4 matrix units having alone 1 as its (i,j)-entry and all other entries 0. Consider $e = e_{11}+e_{22}$ and $f = e_{33}+e_{44} \in R$ and $e(x) = e+fx \in R[x;\alpha]$. Then $e(x)^2 = e(x)$, $ef = fe = 0$, $e^2 = e$, $f^2 = f$, $\alpha(e) = f$, $\alpha(f) = e$. An easy calculation reveals that $e(x)e_{12} = e_{12} + e_{34}x$, but $e_{12}e(x) = e_{12}$. Hence $M[x,\alpha]_{R[x,\alpha]}$ is not abelian. $\square$

We end this paper with some observations concerning Baer, p.q.-Baer and p.p.-modules. We show that if a module M is abelian, there is a strong connection between Baer, p.q.-Baer, p.p.-modules and polynomial extension, power series extension, Laurent polynomial extension and Laurent power series extension of M, respectively.

Theorem 2.20. *Let M be an abelian module. Then we have:*

1. *M is a p.p.-module if and only if $M[x]$ is a p.p.-module.*
2. *M is a Baer module if and only if $M[x]$ is a Baer module.*
3. *M is a p.q.-Baer module if and only if $M[x]$ is a p.q.-Baer module.*
4. *M is a p.p.-module if and only if $M[x,x^{-1}]$ is a p.p.-module.*
5. *M is a Baer module if and only if $M[x,x^{-1}]$ is a Baer module.*
6. *M is a Baer module if and only if $M[[x]]$ is a Baer module.*
7. *M is a Baer module if and only if $M[[x,x^{-1}]]$ is a Baer module.*

Proof. 1. "$\Leftarrow$": Assume that $M[x]$ is a p.p.-module. Let $m \in M$. By the assumption there exists an idempotent element $e(x) = e_0 + e_1x + \ldots + e_nx^n \in R[x]$ such that $r_{R[x]}(m) = e(x)R[x]$. Then $e_0^2 = e_0$ and so $e_0R \subset r_R(m)$. Now let $a \in r_R(m)$. Since $r_R(m) \subset r_{R[x]}(m)$, $ma = 0$ implies that $a = e(x)a$ and so $a = e_0a$. Hence $r_R(m) \subset e_0R$, that is, $r_R(m) = e_0R$. Therefore M is a p.p.-module.
"$\Rightarrow$": Let $m(x) = m_0 + m_1x + \ldots + m_tx^t \in M[x]$. We claim that

$$r_{R[x]}(m(x)) = eR[x],$$

where $e = e_0e_1\ldots e_t$, $e_i^2 = e_i$ and $r_R(m_i) = e_iR$, $i = 0,1,\ldots,t$. For if, since M is abelian,

$$m(x)e = m_0e_0e_1\ldots e_t + m_1e_1e_0e_2\ldots e_tx + \ldots + m_te_te_0e_1\ldots e_{t-1}x^t = 0.$$

Then $eR[x] \subseteq r_{R[x]}(m(x))$. Let $f(x) = a_0 + a_1x + \ldots + a_nx^n \in r_{R[x]}(m(x))$. Then $m(x)f(x) = 0$. Since M is an abelian and p.p.-module, by Lemma 2.8, M

is Armendariz. So, $m_i a_j = 0$ and this implies $a_j \in r_R(m_i) = e_i R$. Then $a_j = e_i a_j$ for any i. Therefore $f(x) = ef(x) \in eR[x]$. This completes the proof.

2. " $\Leftarrow$ ": Let $M[x]$ be a Baer module and X be a subset of M. Since $M[x]$ is Baer, then there exists $e(x)^2 = e(x) = e_0 + e_1x + \ldots + e_nx^n \in R[x]$ such that $r_{R[x]}(X) = e(x)R[x]$. We claim that $r_R(X) = e_0R$. If $a \in r_R(X)$, then $a = e(x)a$ and so $a = e_0a$. Hence $r_R(X) \subset e_0R$. On the other hand, since $Xe(x) = 0$, we have $Xe_0 = 0$, that is, $e_0R \subset r_R(X)$. Then M is a Baer module.
" $\Rightarrow$ ": Since M is Baer, M is a p.p.-module. By Lemma 2.8, M is Armendariz. Then from [**5**, Theorem 2.5.1(a)], $M[x]$ is Baer.

3. " $\Leftarrow$ ": Let $M[x]$ be a p.q.-Baer module and $m \in M$. Then $r_{R[x]}(mR[x]) = e(x)R[x]$, where $(e(x))^2 = e(x) \in R[x]$ and so, we may find $e_0^2 = e_0 \in R$ (e_0 is the constant term of $e(x)$). Since $mR[x]e(x) = 0$, $mR[x]e_0 = 0$ and $mRe_0 = 0$. So, $e_0R \subset r_R(mR)$. Let $r \in r_R(mR) = r_R(mR[x]) \subset r_{R[x]}(mR[x]) = e(x)R[x]$. Then $e(x)r = r$. This implies $e_0r = r$ and so $r \in e_0R$. Therefore $r_R(mR[x]) = e_0R$, i.e. M is a p.q.-Baer module.
" $\Rightarrow$ ": Let M be a p.q.-Baer module and $m(x) = m_0 + m_1x + \ldots + m_tx^t \in M[x]$. Claim:

$$r_{R[x]}(m(x)R[x]) = e(x)R[x],$$

where $e(x) = e_0e_1 \ldots e_t$, $r_R(m_iR) = e_iR$.
Since M is abelian, $m(x)f(x)e_0 \ldots e_t = 0$. Then $e(x)R[x]R[x](m(x)R[x])$. Let

$$f(x) = a_0 + a_1x + \ldots + a_nx^n \in r_{R[x]}(m(x)R[x]).$$

Then $m(x)R[x]f(x) = 0$ and so, $m(x)Rf(x) = 0$. From the last equality we get $m_0Ra_0 = 0$. Hence $a_0 \in r_R(m_0R) = e_0R$ and so, $a_0 = e_0a_0$. Since $m(x)Rf(x) = 0$, for any $r \in R$,

$$m_0ra_1 + m_1ra_0 = 0.$$

Multiplying from the right by e_0, we get

$$m_0ra_1e_0 + m_1ra_0e_0 = m_1ra_0e_0 = m_1ra_0 = 0.$$

This implies $m_1Ra_0 = 0$ and $m_0Ra_1 = 0$. Then $a_0 \in r_R(m_1R) = e_1R$ and $a_1 \in r_R(m_0R) = e_0R$. So, $a_0 = e_1a_0$ and $a_1 = e_0a_1$. Again, since $m(x)Rf(x) = 0$, for any $r \in R$, $m_0ra_2 + m_1ra_1 + m_2ra_0 = 0$. Multiplying this equality from right by e_0e_1 and using previous results, we get $m_2ra_0 = 0$. Then $a_0 \in r_R(m_2R) = e_2R$. So $a_0 = e_2a_0$. Continuing this process we get $a_i = e_ja_i$ for any i, j. This implies $f(x) = e_0e_1 \ldots e_tf(x)$. So, $M[x]$ is a p.q.-Baer module.

4. Since every abelian and p.p.-module is Armendariz by Lemma 2.8, the proof follows from [**5**, Theorem 2.11 (2)(a)].

5. Since every Baer module is a p.p.-module, the proof follows from [**5**, Theorem 2.5 (2)(a)].

6. Since, by Corollary 2.9, every abelian and Baer module is Armendariz of power series type, the proof follows from [**5**, Theorem 2.5 (2)(a)].

7. By Corollary 2.9, every abelian and Baer module is Armendariz of power series type, it follows from [**5**, Theorem 2.5 (2)(b)]. □

Proposition 2.21. *Let M be an abelian module. If for any countable subset X of M, $r_R(X) = eR$, where $e^2 = e \in R$, then $M[[x]]$ and $M[[x, x^{-1}]]$ are p.p.-modules.*

Proof. Let $m \in M$. Since $\{m\}$ is a countable set, $r_R(m) = eR$. Then from Theorem 2.14, M is Armendariz of power series type. By [**5**, Theorem 2.11.(1)(c)] and [**5**, Theorem 2.11.(2)(c)], $M[[x]]$ and $M[[x, x^{-1}]]$ are p.p.-modules. □

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