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SOME INCLUSION RELATIONSHIPS FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH A FAMILY OF INTEGRAL OPERATORS

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Abstract. We define a family of integral operators using multiplier transformation on the space of normalized meromorphic functions and introduce several new subclasses using this operator. We investigate various inclusion relations for these subclasses and some interesting applications involving a certain class of the integral operator are also considered.

1. Introduction, Definitions And Preliminaries

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the punctured open unit disk

$$\mathcal{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 <| z |< 1\} = \mathcal{U} \setminus \{0\}.$$

A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}^*(\gamma)$ of meromorphic starlike functions of order γ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left(\frac{zf^{'}(z)}{f(z)}\right) < -\gamma \qquad (z \in \mathcal{U};\ 0 \leq \gamma < 1). \tag{2}$$

A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MC}(\gamma)$ of meromorphic convex functions of order γ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) < -\gamma \qquad (z \in \mathcal{U};\ 0 \leq \gamma < 1). \tag{3}$$

We now define a subclass of meromorphic close-to-convex functions of order δ type γ as follows. A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{C}^{'}(\delta, \gamma)$ if there

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exists a function $g(z) \in \mathcal{MC}(\gamma)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) < -\delta \qquad (z \in \mathcal{U};\ 0 \le \gamma < 1;\ 0 \le \delta < 1). \tag{4}$$

Similarly, a function $f \in \mathcal{M}$ is said to be in the class $\mathcal{K}(\delta, \gamma)$ if there exists a function $g(z) \in \mathcal{MC}(\gamma)$ such that

$$\operatorname{Re}\left(\frac{z(zf'(z))'}{g(z)}\right) < -\delta \qquad (z \in \mathcal{U};\ 0 \le \gamma < 1;\ 0 \le \delta < 1). \tag{5}$$

In recent years, several families of integral operators and differential operators were introduced using Hadamard product (or convolution). For example, we choose to mention the Ruscheweyh derivative [**11**], the Carlson-Shaffer operator [**1**], the Dziok-Srivastava operator [**2**], the Noor integral operator [**9**] and so on (see[**3, 5, 8, 10**]). Motivated by the work of N. E. Cho and K. I. Noor [**7**], we introduce a family of integral operators defined on the space meromorphic functions in the class $\mathcal{M}$. By using these integral operators, we define several subclasses of meromorphic functions and investigate various inclusion relationships and integral preserving properties for the meromorphic function classes introduced here.

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\mathbb{Z}_0^- = 0, -1, -2, \ldots$; $j = 1, \ldots, s$), we define the function $\phi(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s; z)$ by

$$\phi(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s; z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1}\,(\alpha_2)_{k+1}\,\ldots\,(\alpha_q)_{k+1}}{(\beta_1)_{k+1}\,(\beta_2)_{k+1}\,\ldots(\beta_s)_{k+1}} \frac{z^k}{(k+1)!}$$

$$(q \le s+1;\ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathcal{U}),$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & \text{if } k \in \mathbb{N}_0 = \{1, 2, , \ldots\}. \end{cases}$$

Now we introduce the following operator $I_\mu^p(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s)$: $\mathcal{M} \to \mathcal{M}$ as follows:
Let $F_{\mu,\,p}(z) = \frac{1}{z} + \sum\limits_{k=0}^{\infty} \left(\frac{k+\mu+1}{\mu}\right)^p z^k$, $p \in \mathbb{N}_0$, $\mu \neq 0$ and let $F_{\mu,\,p}^{-1}(z)$ be defined such that

$$F_{\mu,\,p}(z) * F_{\mu,\,p}^{-1}(z) = \phi(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s; z).$$

Then

$$I_\mu^p(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s)f = F_{\mu,\,p}^{-1}(z) * f(z). \tag{6}$$

From (6) it can be easily seen that

$$\begin{aligned} &I_\mu^p(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s)f \\ &\quad = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\mu}{k+\mu+1}\right)^p \frac{(\alpha_1)_{k+1}\,(\alpha_2)_{k+1}\,\ldots\,(\alpha_q)_{k+1}}{(\beta_1)_{k+1}\,(\beta_2)_{k+1}\,\ldots(\beta_s)_{k+1}} \frac{z^k}{(k+1)!}. \end{aligned} \tag{7}$$

For convenience, we shall henceforth denote

$$I_\mu^p(\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_s)f = I_\mu^p(\alpha_1, \beta_1)f. \tag{8}$$

For the choice of the parameters $p = 0$, $q = 2$, $s = 1$, the operator $I_\mu^p(\alpha_1, \beta_1)f$ is reduced to an operator introduced by N. E. Cho and K. I. Noor in [**7**] and when $p = 0$, $q = 2$, $s = 1$, $\alpha_1 = \lambda, \alpha_2 = 1$, $\beta_1 = (n+1)$, the operator $I_\mu^p(\alpha_1, \beta_1)f$ is reduced to an operator recently introduced by S.-M. Yuan et. al. in [**12**].

It can be easily verified from the above definition of the operator $I_\mu^p(\alpha_1, \beta_1)f$ that

$$z(I_\mu^{p+1}(\alpha_1, \beta_1)f(z))' = \mu I_\mu^p(\alpha_1, \beta_1)f(z) - (\mu+1)I_\mu^{p+1}(\alpha_1, \beta_1)f(z) \tag{9}$$

and

$$z(I_\mu^p(\alpha_1, \beta_1)f(z))' = \alpha_1 I_\mu^p(\alpha_1+1, \beta_1)f(z) - (\alpha_1+1)I_\mu^p(\alpha_1, \beta_1)f(z). \tag{10}$$

By using the operator $I_\mu^p(\alpha_1, \beta_1)f$, we now introduce the following subclasses of meromorphic functions:

$$\mathcal{MS}_\mu^p(\alpha_1, \beta_1, \gamma) := \big\{f : f \in \mathcal{M} \text{ and } I_\mu^p(\alpha_1, \beta_1)f \in \mathcal{MS}^*(\gamma)\big\},$$

$$\mathcal{MC}_\mu^p(\alpha_1, \beta_1, \gamma) := \big\{f : f \in \mathcal{M} \text{ and } I_\mu^p(\alpha_1, \beta_1)f \in \mathcal{MC}(\gamma)\big\},$$

$$\mathcal{QC}_\mu^p(\alpha_1, \beta_1, \gamma, \delta) := \big\{f : f \in \mathcal{M} \text{ and } I_\mu^p(\alpha_1, \beta_1)f \in \mathcal{C}'(\delta, \gamma)\big\}$$

and

$$\mathcal{QK}_\mu^p(\alpha_1, \beta_1, \gamma, \delta) := \big\{f : f \in \mathcal{M} \text{ and } I_\mu^p(\alpha_1, \beta_1)f \in \mathcal{K}(\delta, \gamma)\big\}.$$

We note that

$$f(z) \in \mathcal{MC}_\mu^p(\alpha_1, \beta_1, \gamma) \quad \Longleftrightarrow \quad -zf'(z) \in \mathcal{MS}_\mu^p(\alpha_1, \beta_1, \gamma) \tag{11}$$

and a similar relationship exists between the classes $\mathcal{QC}_\mu^p(\alpha_1, \beta_1, \gamma, \delta)$ and $\mathcal{QK}_\mu^p(\alpha_1, \beta_1, \gamma, \delta)$.

In order to establish our main results, we need the following lemma which is popularly known as the Miller-Mocanu Lemma.

Lemma 1.1 ([**6**]). *Let $u = u_1 + \mathrm{i}\,u_2$, $v = v_1 + \mathrm{i}\,v_2$ and let $\psi(u, v)$ be a complex function, $\psi : \mathbb{D} \to \mathbb{C}$, $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$. Suppose that ψ satisfies the following conditions*

(i) *$\psi(u, v)$ is continuous in $\mathbb{D}$;*
(ii) *$(1, 0) \in \mathbb{D}$ and $\mathrm{Re}\{\psi(1, 0)\} > 0$;*
(iii) *$\mathrm{Re}\{\psi(\mathrm{i}\,u_2, v_1)\} \le 0$ for all $(\mathrm{i}\,u_2, v_1) \in \mathbb{D}$ and such that $v_1 \le -\frac{(1+u_2^2)}{2}$.*

Let $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 \cdots$ be analytic in E, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathcal{U}$. If $\mathrm{Re}\{\psi(p(z), zp'(z))\} > 0 \quad (z \in \mathcal{U})$, then $\mathrm{Re}(p(z)) > 0$ for $z \in \mathcal{U}$.

2. Main Results

In this section, we give several inclusion relationships for meromorphic function classes, which are associated with the integral operator $I_\mu^p(\alpha_1, \beta_1)f$.

Theorem 2.1. *Let $\alpha_1,\, \mu > 0$ and $0 \le \gamma < 1$. Then*

$$\mathcal{MS}_\mu^p(\alpha_1+1, \beta_1,\, \gamma) \subset \mathcal{MS}_\mu^p(\alpha_1, \beta_1,\, \gamma) \subset \mathcal{MS}_\mu^{p+1}(\alpha_1, \beta_1,\, \gamma).$$

Proof. To prove the first part of Theorem 2.1, let $f \in \mathcal{MS}_\mu^p(\alpha_1+1, \beta_1,\, \gamma)$ and set

$$\frac{z(I_\mu^p(\alpha_1, \beta_1)f(z))^{'}}{I_\mu^p(\alpha_1, \beta_1)f(z)} + \gamma = -(1-\gamma)p(z) \tag{12}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and $p(z) \neq 0$ for all $z \in \mathcal{U}$.

Then by applying the identity (10), we obtain

$$\begin{aligned} \alpha_1 \frac{I_\mu^p(\alpha_1+1, \beta_1)f(z)}{I_\mu^p(\alpha_1, \beta_1)f(z)} &= \frac{z(I_\mu^p(\alpha_1, \beta_1)f(z))^{'}}{I_\mu^p(\alpha_1, \beta_1)f(z)} + (\alpha_1+1) \\ &= -(1-\gamma)p(z) - \gamma + (\alpha_1+1). \end{aligned} \tag{13}$$

By logarithmically differentiating both sides of the equation (13), we get

$$\begin{aligned} \frac{z(I_\mu^p(\alpha_1+1, \beta_1)f(z))^{'}}{I_\mu^p(\alpha_1+1, \beta_1)f(z)} &= \frac{z(I_\mu^p(\alpha_1, \beta_1)f(z))^{'}}{I_\mu^p(\alpha_1, \beta_1)f(z)} + \frac{(1-\gamma)zp^{'}(z)}{(1-\gamma)p(z)+\gamma-(\alpha_1+1)} \\ &= -\gamma - (1-\gamma)p(z) + \frac{(1-\gamma)zp^{'}(z)}{(1-\gamma)p(z)+\gamma-(\alpha_1+1)}. \end{aligned}$$

Now we form the equation $\psi(u, v)$ by choosing $u = p(z) = u_1 + \mathrm{i}\, u_2$, $v = zp^{'}(z) = v_1 + \mathrm{i}\, v_2$,

$$\psi(u, v) = (1-\gamma)u - \frac{(1-\gamma)v}{(1-\gamma)u+\gamma-(\alpha_1+1)}. \tag{14}$$

Then clearly, $\psi(u, v)$ is continuous in

$$\mathbb{D} = \left(\mathbb{C} \setminus \left\{\frac{\alpha_1+1-\gamma}{1-\gamma}\right\}\right) \times \mathbb{C}$$

and $(1, 0) \in \mathbb{D}$ with $\operatorname{Re}\big(\psi(1, 0)\big) > 0$.

Moreover, for all $(\mathrm{i}\, u_2, v_1) \in \mathbb{D}$ such that

$$v_1 \le -\frac{1}{2}(1+u_2^2),$$

we have

$$\begin{aligned} \operatorname{Re}\psi(\mathrm{i}\, u_2, v_1) = \operatorname{Re}\left\{\frac{-(1-\gamma)v_1}{(1-\gamma)\,\mathrm{i}\, u_2+\gamma-(\alpha_1+1)}\right\} &= \frac{(1-\gamma)(\alpha_1+1-\gamma)v_1}{(\gamma-\alpha_1-1)^2+(1-\gamma)^2u_2^2} \\ &\le -\frac{(1-\gamma)(1+u_2^2)(\alpha_1+1-\gamma)}{2[(\gamma-1-\alpha_1)^2+(1-\gamma)^2u_2^2]} < 0. \end{aligned}$$

Therefore $\psi(u,v)$ satisfies the hypothesis of the Miller-Mocanu Lemma. This shows that if $\operatorname{Re}\psi(p(z),zp^{'}(z)) > 0$ $(z \in \mathcal{U})$, then $\operatorname{Re}(p(z)) > 0$ $(z \in \mathcal{U})$, that is if $f(z) \in \mathcal{MS}_{\mu}^{p}(\alpha_1+1,\beta_1,\gamma)$ then $f(z) \in \mathcal{MS}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma)$.

To prove the second inclusion relationship asserted by Theorem 2.1, let $f \in \mathcal{MS}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma)$ and put

$$-(1-\gamma)s(z) = \gamma + \frac{z(I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z))^{'}}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)} \tag{15}$$

where the function $s(z)$ is analytic in $\mathcal{U}$ with $s(0)=1$. Then using the arguments to those detailed above with (9), it follows that $\mathcal{MS}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma) \subset \mathcal{MS}_{\mu}^{p+1}(\alpha_1,\beta_1,\,\gamma)$, which completes the proof of the Theorem 2.1. □

Theorem 2.2. *Let $\alpha_1,\mu > 0$ and $0 \le \gamma < 1$. Then*

$$\mathcal{MC}_{\mu}^{p}(\alpha_1+1,\beta_1,\,\gamma) \subset \mathcal{MC}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma) \subset \mathcal{MS}_{\mu}^{p+1}(\alpha_1,\beta_1,\,\gamma).$$

Proof. We observe that

$$\begin{aligned}
f(z) \in \mathcal{MC}_{\mu}^{p}(\alpha_1+1,\beta_1,\,\gamma) &\Longleftrightarrow I_{\mu}^{p}(\alpha_1+1,\beta_1)f(z) \in \mathcal{MC}(\gamma)\\
&\Longleftrightarrow -z(I_{\mu}^{p}(\alpha_1+1,\beta_1)f(z))' \in \mathcal{MS}^{*}(\gamma)\\
&\Longleftrightarrow I_{\mu}^{p}(\alpha_1+1,\beta_1)(-zf'(z)) \in \mathcal{MS}^{*}(\gamma)\\
&\Longleftrightarrow -zf'(z) \in \mathcal{MS}_{\mu}^{p}(\alpha_1+1,\beta_1,\,\gamma)\\
&\Longrightarrow -zf^{'}(z) \in \mathcal{MS}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma)\\
&\Longleftrightarrow I_{\mu}^{p}(\alpha_1,\beta_1)(-zf(z))' \in \mathcal{MS}^{*}(\gamma)\\
&\Longleftrightarrow -z(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))' \in \mathcal{MS}^{*}(\gamma)\\
&\Longleftrightarrow I_{\mu}^{p}(\alpha_1,\beta_1) \in \mathcal{MC}(\gamma)\\
&\Longleftrightarrow\ f(z) \in \mathcal{MC}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma)
\end{aligned}$$

and

$$\begin{aligned}
f(z) \in \mathcal{MC}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma) &\Longleftrightarrow -zf'(z) \in \mathcal{MS}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma)\\
&\Longrightarrow -zf^{'}(z) \in \mathcal{MS}_{\mu}^{p+1}(\alpha_1,\beta_1,\,\gamma)\\
&\Longleftrightarrow -z(I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z))' \in \mathcal{MS}^{*}(\gamma)\\
&\Longleftrightarrow f(z) \in \mathcal{MC}_{\mu}^{p+1}(\alpha_1,\beta_1,\gamma)
\end{aligned}$$

which evidently proves Theorem 2.2. □

Theorem 2.3. *Let $\alpha_1,\,\mu > 0$ and $0 \le \gamma,\,\delta < 1$. Then*

$$\mathcal{QC}_{\mu}^{p}(\alpha_1+1,\beta_1,\,\gamma,\,\delta) \subset \mathcal{QC}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma,\,\delta) \subset \mathcal{QC}_{\mu}^{p+1}(\alpha_1,\beta_1,\,\gamma,\,\delta).$$

Proof. We begin proving that

$$\mathcal{QC}_{\mu}^{p}(\alpha_1+1,\beta_1,\,\gamma,\,\delta) \subset \mathcal{QC}_{\mu}^{p}(\alpha_1,\beta_1,\,\gamma,\,\delta).$$

Let $f(z) \in \mathcal{QC}_\mu^p(\alpha_1+1, \beta_1, \gamma, \delta)$. Then, in view of the definition of the function class $\mathcal{QC}_\mu^p(\alpha_1+1, \beta_1, \gamma, \delta)$, there exists a function $q \in \mathcal{MC}(\gamma)$ such that

$$\operatorname{Re}\left(\frac{z\left(I_\mu^p(\alpha_1+1, \beta_1) f(z)\right)'}{q(z)}\right) < -\delta.$$

Choose the function $g(z)$ such that $q(z) = I_\mu^p(\alpha_1+1, \beta_1) g(z)$, then

$$g \in \mathcal{MC}_\mu^p(\alpha_1+1, \beta_1, \gamma) \quad \text{and} \quad \operatorname{Re}\left(\frac{z\left(I_\mu^p(\alpha_1+1, \beta_1) f(z)\right)'}{I_\mu^p(\alpha_1+1, \beta_1) g(z)}\right) < -\delta. \tag{16}$$

We next put

$$\frac{z\left(I_\mu^p(\alpha_1, \beta_1) f(z)\right)'}{I_\mu^p(\alpha_1, \beta_1) g(z)} + \delta = -(1-\delta) p(z), \tag{17}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and $p(z) \neq 0$ for all $z \in \mathcal{U}$. Thus, by using the identity (10), we have

(18)

$$\begin{aligned}\frac{z(I_\mu^p(\alpha_1+1, \beta_1) f(z))'}{I_\mu^p(\alpha_1+1, \beta_1) g(z)} &= \frac{I_\mu^p(\alpha_1+1, \beta_1)(z f'(z))}{I_\mu^p(\alpha_1+1, \beta_1) g(z)} \\ &= \frac{z[I_\mu^p(\alpha_1, \beta_1)(z f'(z))]' + (\alpha_1+1) I_\mu^p(\alpha_1, \beta_1)(z f'(z))}{z(I_\mu^p(\alpha_1, \beta_1) g(z))' + (\alpha_1+1) I_\mu^p(\alpha_1, \beta_1) g(z)} \\ &= \frac{\dfrac{z[I_\mu^p(\alpha_1,\beta_1)(z f'(z))]'}{I_\mu^p(\alpha_1,\beta_1) g(z)}}{\dfrac{z(I_\mu^p(\alpha_1,\beta_1) g(z))'}{I_\mu^p(\alpha_1,\beta_1) g(z)} + (\alpha_1+1)} + \frac{\dfrac{(\alpha_1+1) I_\mu^p(\alpha_1,\beta_1)(z f'(z))}{I_\mu^p(\alpha_1,\beta_1) g(z)}}{\dfrac{z(I_\mu^p(\alpha_1,\beta_1) g(z))'}{I_\mu^p(\alpha_1,\beta_1) g(z)} + (\alpha_1+1)}.\end{aligned}$$

Jack [**4**] showed that $g \in \mathcal{MC}(\gamma)$ implies that $g \in \mathcal{MS}^*(\sigma)$ where

$$\sigma = \frac{2\gamma - 1 + \sqrt{9 - 4\gamma + 4\gamma^2}}{4}.$$

Since $g(z) \in \mathcal{MC}_\mu^p(\alpha_1+1, \beta_1, \gamma)$ and $\mathcal{MC}_\mu^p(\alpha_1+1, \beta_1, \gamma) \subset \mathcal{MC}_\mu^p(\alpha_1, \beta_1, \gamma)$, for some σ we can set

$$\frac{z(I_\mu^p(\alpha_1, \beta_1) g(z))'}{I_\mu^p(\alpha_1, \beta_1) g(z)} + \sigma = -(1-\sigma) H(z)$$

where $H(z) = h_1(x, y) + \mathrm{i}\, h_2(x, y)$ and $\operatorname{Re}\big(H(z)\big) = h_1(x, y) > 0$, $(z \in \mathcal{U})$. Then

$$\frac{z(I_\mu^p(\alpha_1+1, \beta_1) f(z))'}{I_\mu^p(\alpha_1+1, \beta_1) g(z)} = \frac{\dfrac{z[I_\mu^p(\alpha_1, \beta_1)(z f'(z))]'}{I_\mu^p(\alpha_1, \beta_1) g(z)}}{-\sigma - (1-\sigma) H(z) + (\alpha_1+1)} - \frac{(\alpha_1+1)\big[\delta + (1-\delta) p(z)\big]}{-\sigma - (1-\sigma) H(z) + (\alpha_1+1)}. \tag{19}$$

We thus find from (17) that

$$z(I_\mu^p(\alpha_1,\beta_1)f(z))' = -I_\mu^p(\alpha_1,\beta_1)g(z)\big[\delta+(1-\delta)p(z)\big]. \tag{20}$$

Upon differentiating both sides of (20) with respect to z, we have

$$\begin{aligned}\frac{z\big[z(I_\mu^p(\alpha_1,\beta_1)f(z))'\big]'}{I_\mu^p(\alpha_1,\beta_1)g(z)} = &-(1-\delta)zp'(z)\\ &+\big[\sigma+(1-\sigma)H(z)\big]\big[\delta+(1-\delta)p(z)\big].\end{aligned} \tag{21}$$

By substituting (21) in (19), we obtain

$$\frac{z(I_\mu^p(\alpha_1+1,\beta_1)f(z))'}{I_\mu^p(\alpha_1+1,\beta_1)g(z)}+\delta = -\left\{(1-\delta)p(z)-\frac{(1-\delta)zp'(z)}{\sigma+(1-\sigma)H(z)-(\alpha_1+1)}\right\}.$$

We now choose $u = p(z) = u_1 + \mathrm{i}\,u_2$ and $v = zp'(z) = v_1 + \mathrm{i}\,v_2$, we define the function $\psi(u,v)$ by

$$\psi(u,v) = (1-\delta)u - \frac{(1-\delta)v}{\sigma+(1-\sigma)H(z)-(\alpha_1+1)} \tag{22}$$

where $(u,v)\in\mathbb{D} = (\mathbb{C}\setminus\mathbb{D}^*)\times\mathbb{C}$ and

$$\mathbb{D}^* = \left\{z : z\in\mathbb{C} \ \text{ and } \ \mathrm{Re}(H(z)) = h_1(z) \geq 1+\frac{\alpha_1}{1-\sigma}\right\}.$$

It is easy to see that $\psi(u,v)$ is continuous in $\mathbb{D}$ and $(1,0)\in\mathbb{D}$ with $\mathrm{Re}\,\big(\psi(1,0)\big) > 0$.

Moreover, for all $(\mathrm{i}\,u_2, v_1)\in\mathbb{D}$ such that

$$v_1 \leq -\frac{1}{2}(1+u_2^2),$$

we have

$$\begin{aligned}\mathrm{Re}\,\psi(\mathrm{i}\,u_2,v_1) &= \mathrm{Re}\left\{\frac{-(1-\delta)v_1}{(1-\sigma)H(z)+\sigma-(\alpha_1+1)}\right\}\\ &= \frac{(1-\delta)v_1\big[(\alpha_1+1)-(1-\sigma)h_1(x,y)-\sigma\big]}{\big[(1-\sigma)h_1(x,y)+\sigma-\alpha_1-1\big]^2+\big[(1-\sigma)h_2(x,y)\big]^2}\\ &\leq -\frac{(1-\delta)(1+u_2^2)\big[(\alpha_1+1)-(1-\sigma)h_1(x,y)-\sigma\big]}{2\big[(1-\sigma)h_1(x,y)+\sigma-\alpha_1-1\big]^2+2\big[(1-\sigma)h_2(x,y)\big]^2}\\ &< 0.\end{aligned}$$

Therefore $\psi(u,v)$ satisfies the hypothesis of the Miller-Mocanu Lemma. This shows that if $\mathrm{Re}\,\psi(p(z),zp'(z)) > 0$, $(z\in\mathcal{U})$, then $\mathrm{Re}(p(z)) > 0$, $(z\in\mathcal{U})$, that is if $f(z)\in\mathcal{QC}_\mu^p(\alpha_1+1,\beta_1,\gamma,\delta)$ then $f(z)\in\mathcal{QC}_\mu^p(\alpha_1,\beta_1,\gamma,\delta)$.

Using the arguments similar to those detailed above, we can prove the second part of the inclusion. We therefore choose to omit the details involved. □

Using arguments similar to those detailed in Theorem 2.2, we can prove

Theorem 2.4. *Let $\alpha_1,\mu > 0$ and $0\leq\gamma,\delta < 1$. Then*

$$\mathcal{QK}_\mu^p(\alpha_1+1,\beta_1,\gamma,\delta)\subset\mathcal{QK}_\mu^p(\alpha_1,\beta_1,\gamma,\delta)\subset\mathcal{QK}_\mu^{p+1}(\alpha_1,\beta_1,\gamma,\delta).$$

3. Inclusion Properties Involving the Operator L_c

In this section, we examine the closure properties involving the integral operator $L_c(f)$ defined by

$$L_c(f) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,\mathrm{d}t, \qquad (f \in \mathcal{M},\ c > 0). \tag{23}$$

In order to obtain the integral-preserving properties involving the integral $L_c(f)$, we need the following lemma which is popularly known as the Jack's Lemma.

Lemma 3.1 ([**4**])**.** *Let $w(z)$ be a nonconstant analytic function in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number and $k \geq 1$.*

Theorem 3.2. *Let $c, \alpha_1, \mu > 0$ and $0 \leq \gamma < 1$. If $f \in \mathcal{MS}_\mu^p(\alpha_1, \beta_1, \gamma)$, then $L_c(f) \in \mathcal{MS}_\mu^p(\alpha_1, \beta_1, \gamma)$.*

Proof. From definition of $L_c(f)$ and the linearity of operator $I_\mu^p(\alpha_1, \beta_1)f$ we have

$$z(I_\mu^p(\alpha_1, \beta_1)L_c(f))' = cI_\mu^p(\alpha_1, \beta_1)f(z) - (c+1)I_\mu^p(\alpha_1, \beta_1)L_c(f). \tag{24}$$

Suppose that $f(z) \in \mathcal{MS}_\mu^p(\alpha_1, \beta_1, \gamma)$ and let

$$\frac{z(I_\mu^p(\alpha_1, \beta_1)L_c f(z))'}{I_\mu^p(\alpha_1, \beta_1)L_c f(z)} = -\frac{1 + (1 - 2\gamma)w(z)}{1 - w(z)}, \tag{25}$$

where $w(0) = 1$. Then by applying (24) in (25), we have

$$\frac{I_\mu^p(\alpha_1, \beta_1)f(z)}{I_\mu^p(\alpha_1, \beta_1)L_c f(z)} = \frac{c - (c + 2 - 2\gamma)w(z)}{c[1 - w(z)]},$$

which upon logarithmic differentiation yields

$$\begin{aligned}\frac{z(I_\mu^p(\alpha_1, \beta_1)f(z))'}{I_\mu^p(\alpha_1, \beta_1)f(z)} = &-\frac{1 + (1 - 2\gamma)w(z)}{1 - w(z)} + \frac{zw'(z)}{1 - w(z)} \\ &- \frac{(c + 2 - 2\gamma)zw'(z)}{c - (c + 2 - 2\gamma)w(z)}.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{z(I_\mu^p(\alpha_1, \beta_1)f(z))'}{I_\mu^p(\alpha_1, \beta_1)f(z)} + \gamma = &\frac{(\gamma - 1)[1 + w(z)]}{1 - w(z)} + \frac{zw'(z)}{1 - w(z)} \\ &- \frac{(c + 2 - 2\gamma)zw'(z)}{c - (c + 2 - 2\gamma)w(z)}.\end{aligned} \tag{26}$$

Now, assuming that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (z_0 \in \mathcal{U}),$$

and applying Jack's Lemma 3.1, we have

$$z_0 w'(z_0) = kw(z_0) \quad (k \geq 1). \tag{27}$$

If we set $w(z_0) = \mathrm{e}^{\mathrm{i}\,\theta}$, $(\theta \in \mathbb{R})$ in (26) and observe that

$$\mathrm{Re}\left(\frac{(\gamma-1)\big[1+w(z_0)\big]}{1-w(z_0)}\right) = 0.$$

then we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{z_0(I^p_\mu(\alpha_1,\beta_1)f(z_0))'}{I^p_\mu(\alpha_1,\beta_1)f(z_0)}+\gamma\right) &= \mathrm{Re}\left(\frac{z_0w'(z_0)}{1-w(z_0)} - \frac{(c+2-2\gamma)z_0w'(z_0)}{c-(c+2-2\gamma)w(z_0)}\right)\\ &= \mathrm{Re}\left(-\frac{2(1-\gamma)k\,\mathrm{e}^{\mathrm{i}\,\theta}}{(1-\mathrm{e}^{\mathrm{i}\,\theta})\big[c-(c+2-2\gamma)\,\mathrm{e}^{\mathrm{i}\,\theta}\big]}\right)\\ &= \frac{2k(1-\gamma)(c+1-\gamma)}{c^2-2c(c+2-2\gamma)\cos\theta+(c+2-2\gamma)^2}\\ &\geq 0,\end{aligned}$$

which obviously contradicts the hypothesis $f(z) \in \mathcal{MS}^p_\mu(\alpha_1,\beta_1,\gamma)$. Consequently, we can deduce that $|w(z)| < 1$ $(z \in \mathcal{U})$ which in view of (25) proves the integral-preserving property asserted by Theorem 3.2. □

Theorem 3.3. *Let c, α_1, $\mu > 0$ and $0 \leq \gamma < 1$. If $f \in \mathcal{MC}^p_\mu(\alpha_1,\beta_1,\gamma)$, then $L_c(f) \in \mathcal{MC}^p_\mu(\alpha_1,\beta_1,\gamma)$.*

Proof. We observe that

$$\begin{aligned}f(z) \in \mathcal{MC}^p_\mu(\alpha_1,\beta_1,\gamma) &\Longleftrightarrow -zf'(z) \in \mathcal{MS}^p_\mu(\alpha_1,\beta_1,\gamma)\\ &\Longrightarrow L_c\big(-zf'(z)\big) \in \mathcal{MS}^p_\mu(\alpha_1,\beta_1,\gamma)\\ &\Longleftrightarrow -z(L_cf(z))' \in \mathcal{MS}^p_\mu(\alpha_1,\beta_1,\gamma)\\ &\Longleftrightarrow L_cf(z) \in \mathcal{MC}^p_\mu(\alpha_1,\beta_1,\gamma).\end{aligned}$$

which completes the proof of the Theorem 3.3. □

Next, we derive an inclusion property which is obtained by using (24) and the same techniques as in the proof of the Theorem 2.3.

Theorem 3.4. *Let c, $\alpha_1,\mu > 0$ and $0 \leq \gamma < 1$. If $f \in \mathcal{QC}^p_\mu(\alpha_1,\beta_1,\gamma,\delta)$ then so is $L_c(f)$.*

Finally, we obtain Theorem 3.5 below by using (24) and the same techniques as in the proof of the Theorem 3.3.

Theorem 3.5. *Let c, α_1, $\mu > 0$ and $0 \leq \gamma < 1$. If $f \in \mathcal{QK}^p_\mu(\alpha_1,\beta_1,\gamma,\delta)$ then so is $L_c(f)$.*

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