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CONTINUITY WITH RESPECT TO DATA AND PARAMETERS OF WEAK SOLUTIONS TO A STEFAN-LIKE PROBLEM

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ABSTRACT. We study a reaction-diffusion system with moving boundary describing a prototypical fast reaction-diffusion scenario arising in the chemical corrosion of concrete-based materials. We prove the continuity with respect to data and parameters of weak solutions to the resulting moving-boundary system of partial differential equations.

1. INTRODUCTION

Recently we have established the existence and uniqueness of weak solutions to a two-phase reaction-diffusion system with a free boundary where an aggressive fast reaction is concentrated; see [**12, 13**] for these results and [**9**] for a larger picture of the chemical corrosion issue motivating this work – the concrete carbonation problem. Details about the chemo-physical problem, its civil engineering importance as well as some aspects of what mathematics can say concerning the prediction of the speed of the involved deterioration mechanism are reported in [**10**]. Within this framework, we focus on the continuity with respect to data and parameters of weak solutions to the mathematical model in question. It is worth mentioning that relatively general results on continuous dependence of solutions of scalar Stefan-like problems were proved in the past by several authors (see, for instance, [**3, 6, 2, 1**] and [**17**]). Particularly, we mention the contributions by Mohamed [**14**] and Pawell [**16**] who study the continuous dependence problem for (scalar) moving-boundary descriptions of some non-corrosive chemical reactions taking place in concrete. Since here we deal with a non-linearly coupled system of semi-linear parabolic PDEs in two moving *a priori* unknown phases, whose motion is driven by a non-equilibrium moving-boundary condition of kinetic type, none of these formulations seem to be applicable. The working framework we have chosen to prove the stability estimate is that one prepared in [**13**].

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This note is organized in the following fashion: In Section 2, we present the moving-boundary system and shortly comment on the underlying physics. Preliminary technical information (like function spaces used, our concept of weak formulations, review of known basic estimates, a local existence and uniqueness result for weak solutions) is detailed in Section 3. We state the main result (that is Theorem 4.1) in Section 4 and prove it in Section 5.

2. The moving-boundary problem

We investigate the moving-boundary problem of finding the vector of concentrations $(\bar{u}_1,\dots,\bar{u}_6)^t$ and the interface position $s(t)$ which satisfy for all $t\in S_T :=]0,T[$ $(0<T<\infty$ fixed) the equations

$$(2.1)\quad \begin{cases} (\phi\phi_w\bar{u}_i)_{,t} + (-D_i\nu_{i2}\phi\phi_w\bar{u}_{i,x})_x = f_{i,Henry}, x\in]0,s(t)[, \quad i\in\{1,2\},\\ (\phi\phi_w\bar{u}_3)_{,t} + (-D_3\phi\phi_w\bar{u}_{3,x})_x = f_{Diss}, x\in]s(t),L[\\ (\phi\phi_w\bar{u}_4)_{,t} = f_{Prec}+f_{Reac\Gamma}, \quad x=s(t)\in\Gamma(t),\\ (\phi\bar{u}_5)_{,t} + (-D_5\phi\bar{u}_{5,x})_x = 0, x\in]0,s(t)[,\\ (\phi\bar{u}_6)_{,t} + (-D_6\phi\bar{u}_{6,x})_x = 0, x\in]s(t),L[,\end{cases}$$

$$(2.2)\quad \phi\phi_w\nu_{i2}\bar{u}_i(x,0) = \hat{u}_{i0}(x),\ i\in\mathcal{I}:=\mathcal{I}_1\cup\mathcal{I}_2,\quad x\in\Omega(0),$$

$$(2.3)\quad \phi\phi_w\bar{u}_4(x,0) = \hat{u}_{40}(x), \qquad x\in\Omega(0),$$

$$(2.4)\quad \phi\phi_w\nu_{i2}\bar{u}_i(0,t) = \lambda_i(t), \qquad i\in\mathcal{I}_1 := \{1,2,5\},$$

$$(2.5)\quad \bar{u}_5(s(t),t) = \bar{u}_6(s(t),t),$$

$$(2.6)\quad \bar{u}_{i,x}(L,t) = 0, \qquad i\in\mathcal{I}_2 := \{3,6\},$$

$$(2.7)\quad \begin{cases} [j_1\cdot n]_{\Gamma(t)} = -\tilde{\eta}_\Gamma(s(t),t) + s'(t)[\phi\phi_w\bar{u}_1]_{\Gamma(t)},\\ [j_i\cdot n]_{\Gamma(t)} = \tilde{\eta}_\Gamma(s(t),t)\delta_{5i} + s'(t)[\phi\phi_w\nu_{i2}\bar{u}_i]_{\Gamma(t)}, \qquad i\in\{2,5,6\},\\ [j_3\cdot n]_{\Gamma(t)} = -\tilde{\eta}_\Gamma(s(t),t) + s'(t)[\phi\phi_w\bar{u}_3]_{\Gamma(t)},\end{cases}$$

and

$$(2.8)\quad s'(t) = \alpha\frac{\tilde{\eta}_\Gamma(s(t),t)}{\phi\phi_w\bar{u}_3(s(t),t)} =: \tilde{\psi}_\Gamma(s(t),t), s(0)=s_0>0.$$

In (2.7), n is the outer normal to the interface $\Gamma(t)$, while $[A]_{\Gamma(t)}$ denotes the jump in the quantity A across $\Gamma(t)$. For fixing ideas, we assume that the only relevant chemistry intervening here is the so called carbonation reaction (details are given in [**4, 9**] and references cited therein), that is

$$(2.9)\quad \mathrm{CO_2(g\to aq)} + \mathrm{Ca(OH)_2(s\to aq)} \to \mathrm{CaCO_3(aq\to s)} + \mathrm{H_2O}.$$

In this framework, $\bar{u}_1$ and $\bar{u}_2$ denote the aqueous and respectively gaseous CO_2 concentrations, $\bar{u}_3$ is the concentration of dissolved $Ca(OH)_2$, $\bar{u}_4$ is the immobile rapidly precipitating species (here: $CaCO_3$(aq)), while $\bar{u}_5$ and $\bar{u}_6$ point out the moisture concentrations (produced via (2.9)) within $]0,s(t)[$ and $]s(t),L[$, respectively. The process can be briefly described as follows: Molecules of atmospheric CO_2 penetrate concrete structures via the air-filled parts of the pores (see Fig. 1), dissolve in pore water where they meet a lot of aqueous $Ca(OH)_2$ ready to react via (2.9). There is chemical evidence [**4**] showing that (2.9) is sufficiently fast so that the two spatial supports of the reactants (CO_2(aq) and $Ca(OH)_2$(aq)) are separated by a sharp interface positioned at $x = s(t)$.

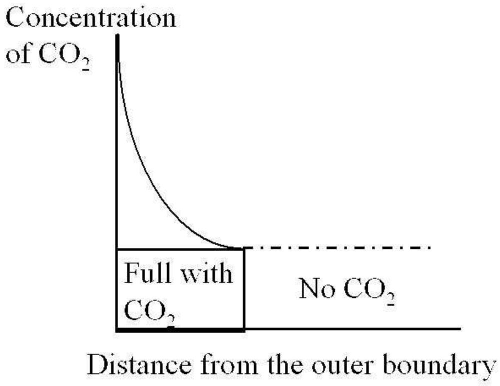

Figure 1. Complete separation of reactants in the carbonation process. The task is to predict the depth at which CO_2 is able to penetrate until a given time $t \in S_T$.

Remark 2.1. The complete segregation of the reactants and the fact that for this reaction-diffusion scenario the associated Thiele modulus is much larger than unity motivates us to apply a moving-boundary strategy in order to predict the penetration of front (here – a sharp interface) of CO_2 in concrete. Conceptually similar reaction-diffusion problems with fast reaction and relatively slow transport arise, for instance, in geochemistry [**15**].

Furthermore, $\nu_{12} = \nu_{32} := 1$, $\nu_{22} := \frac{\phi_a}{\phi_w}$, $\nu_{52} = \nu_{62} := \frac{1}{\phi_w}$, $\nu_{i\ell} := 1$ $(i \in \mathcal{I}, \ell \in \mathcal{I} - \{2\})$, δ_{ij} $(i, j \in \mathcal{I})$ is Kronecker's symbol, $j_i := -D_i \nu_{i\ell} \phi \phi_w \bar{u}_i$ $(i, \ell \in \mathcal{I}_1 \cup \mathcal{I}_2)$ are the corresponding effective diffusive fluxes and $\alpha > 0$. The parameters D_i, L and s_0 are assumed to be constant and strictly positive; the boundary data λ_i are prescribed in agreement with the environmental conditions to which $\Omega =]0, L[$ – a part of a concrete sample – is exposed. The interior boundary conditions (2.7) are derived using an argument based on the pillbox lemma; see [**7**]. Following [**18**] (and subsequent papers, e.g., [**5**]), equation (2.8) represents a non-equilibrium type of free boundary condition that is called *kinetic* condition. For a derivation via the first principles of (2.8) for this particular reaction-diffusion setting, we refer the reader to [**10**, Section 2.3.1].

The initial conditions $\hat{u}_{i0} > 0$ are determined by the chemistry of the cement. The hardened mixture of aggregate, cement and water determines numerical ranges for the porosity $\phi > 0$ and also for the water and air fractions, $\phi_w > 0$ and $\phi_a > 0$. In this paper, we set ϕ, ϕ_a and ϕ_w to be constant. The productions terms $f_{i,Henry}$, f_{Diss}, f_{Prec} and $f_{Reac\Gamma}$ are sources or sinks by Henry-like interfacial transfer mechanisms (see [**8**] for a related application of Henry's law), dissolution, precipitation, and carbonation reactions. We assume

$$
\begin{cases} f_{i,Henry} := (-1)^i P_i(\phi\phi_w \bar{u}_1 - Q_i \phi\phi_a \bar{u}_2) \\ \qquad\qquad (P_i > 0, Q_i > 0), \quad i \in \{1,2\}, \\ f_{Diss} := -S_{3,diss}(\phi\phi_w \bar{u}_3 - u_{3,eq}), \\ \qquad\qquad S_{3,diss} > 0,\ f_{Prec} := 0,\ f_{Reac\Gamma} := \tilde{\eta}_\Gamma. \end{cases} \tag{2.10}
$$

In (2.10), $\tilde{\eta}_\Gamma(s(t),t)$ denotes the interface-concentrated reaction rate. It is defined in the following fashion: Let $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_6)^t$ be the vector of concentrations and M_Λ the set of parameters $\Lambda := (\Lambda_1, \ldots, \Lambda_m)^t$ chosen to describe the reaction rate. We assume that M_Λ is a non-empty compact subset of $\mathbb{R}^m_+$. We introduce the function

$$
\begin{aligned} &\bar{\eta}_\Gamma : \mathbb{R}^6 \times M_\Lambda \to \mathbb{R}_+ \\ &\text{by} \ \ \bar{\eta}_\Gamma(\bar{u}(x,t),\Lambda) := k\phi\phi_w \bar{u}_1^p(x,t))\bar{u}_3^q(x,t), \quad x = s(t). \end{aligned} \tag{2.11}
$$

In (2.11), $m := 3$ and $\Lambda := \{p, q, k\phi\phi_w\} \in \mathbb{R}^3_+$. We define the reaction rate $\tilde{\eta}_\Gamma(s(t),t)$ by

$$
\tilde{\eta}_\Gamma(s(t),t) := \bar{\eta}_\Gamma(\bar{u}(s(t),t),\Lambda), \tag{2.12}
$$

where $\bar{\eta}_\Gamma$ is given by (2.11) and represents the classical power-law ansatz. Note that some mass-balance equations act in $]0, s(t)[$, while other act in $]s(t), L[$ or at $\Gamma(t)$. All of the three space regions are varying in time and they are *a priori* unknown. The system (2.1)–(2.12) forms the *sharp-interface carbonation model*.

Remark 2.2.

(i) The local existence and uniqueness of weak solutions to the sharp-interface carbonation model was reported in [**13**, Theorem 3.3], while the global solvability was addressed in [**13**, Theorem 3.7]. In this paper, we show the continuity of the weak solution to (2.1)–(2.12) with respect to initial data, boundary data and model parameters. The importance of our result is twofold: (1) On one side, we complete the well-posedness study of (2.1)–(2.12), which has been started in [**13**]. (2) On the other side, we prepare a theoretical framework for numerically testing the stability with respect to model parameters. Note that for the carbonation problem many important material parameters are typically unknown. Our stability estimates suggest that there is a little place of "playing games" with the most critical parameters, i.e. those entering (2.8), e.g. It is worth mentioning that unsuitable choices of reaction rates (and hence, of velocities) may produce the blow up in concentration near the interface position (like in [**11**], e.g.).

(ii) The strategy of the proof is the following: We subtract the weak formulation written in terms of two different solutions compared within the same time interval S_T. In order to obtain the desired result, we make use of a lot of *a priori* knowledge of the solution behavior. In particular, we essentially rely on positivity and L^∞ bounds for all involved concentrations (cf. [**13**, Theorem 4.2]) as well as energy estimates (cf. [**13**, Lemma 4.3]) the weak solutions to (2.1)–(2.12). The result is obtained by conveniently applying

Gronwall's inequality in combination with an interpolation inequality as well as with some particular algebraic inequalities tailored to deal with the special non-linearities induced by Landau-like transformations.

3. Technical preliminaries

3.1. Fixing the moving boundary

We take advantage of the $1D$ geometry and immobilize the moving boundary via the fixed-domain transformations (also called Landau's transformations)

$$(x,t) \in [0, s(t)] \times \bar{S}_T \longmapsto (y,t) \in [a,b] \times \bar{S}_T, \quad y = \frac{x}{s(t)}, \quad i \in \mathcal{I}_1, \tag{3.1}$$

$$(x,t) \in [s(t), L] \times \bar{S}_T \longmapsto (y,t) \in [a,b] \times \bar{S}_T, \quad y = a + \frac{x - s(t)}{L - s(t)}, \quad i \in \mathcal{I}_2, \tag{3.2}$$

where $t \in S_T$ is arbitrarily fixed. We introduce the notation $u_i(y,t) := \hat{u}_i(x,t) - \lambda_i(t)$ for all $y \in [a,b]$ and $t \in S_T$. Further, let $\hat{u}_i := \phi\phi_w \bar{u}_i, i \in \{1,3,4\}, \hat{u}_2 := \phi\phi_a \bar{u}_2, \hat{u}_i := \phi \bar{u}_i, i \in \{5,6\}$ and write down the original moving-boundary system (2.1)–(2.12) on fixed domains. As a result of this procedure, we obtain the transformed PDEs system (3.3)–(3.13). The model equations have the forms

$$(u_i + \lambda_i)_{,t} - \frac{(D_i u_{i,y})_{,y}}{s^2(t)} = f_i(u+\lambda) + y\frac{s'(t)}{s(t)} u_{i,y}, \qquad i \in \mathcal{I}_1, \tag{3.3}$$

$$(u_i + \lambda_i)_{,t} - \frac{(D_i u_{i,y})_{,y}}{(L - s(t))^2} = f_i(u+\lambda) + (2-y)\frac{s'(t)}{L - s(t)} u_{i,y}, \qquad i \in \mathcal{I}_2, \tag{3.4}$$

where u is the concentration vector $(u_1, u_2, u_3, u_5, u_6)^t$ and λ represents the boundary data $(\lambda_1,\lambda_2,\lambda_3,\lambda_5,\lambda_6)^t$. We make use of λ_3 and λ_6 only for notational simplicity ($\lambda_3 := \lambda_6 := 0$). The vectors of concentrations u_0 and λ are assumed to be compatible, i.e.

$$u_{0i}(0) = \lambda_i(0), \text{ and hence } \hat{u}_i(0) = 0 \text{ for } i \in \mathcal{I}_1. \tag{3.5}$$

Our initial boundary and interface conditions are now:

$$\begin{aligned} u_i(y,0) = u_{i0}(y), \qquad & i \in \mathcal{I}_1 \cup \mathcal{I}_2, \qquad u_i(a,t) = 0, \\ & i \in \mathcal{I}_1, \qquad u_{i,y}(b,t) = 0, \qquad i \in \mathcal{I}_2, \end{aligned} \tag{3.6}$$

$$\frac{-D_1}{s(t)} u_{1,y}(1) = \eta_\Gamma(1,t) + s'(t)(u_1(1) + \lambda_1), \tag{3.7}$$

$$\frac{-D_2}{s(t)} u_{2,y}(1) = s'(t)(u_2(1) + \lambda_2), \tag{3.8}$$

$$\frac{-D_3}{L - s(t)} u_{3,y}(1) = -\eta_\Gamma(1,t) + s'(t)(u_3(1) + \lambda_3), \tag{3.9}$$

$$\frac{-D_5}{s(t)} u_{5,y}(1) + \frac{D_6}{L - s(t)} u_{6,y}(1) = \eta_\Gamma(1,t), u_5(1) + \lambda_5 = u_6(1) + \lambda_6, \tag{3.10}$$

where $\eta_\Gamma(1,t)$ denotes the reaction rate that acts in the y-t plane. We also mention that $u_{i0}(y) = \hat{u}_{i0}(x) - \lambda_i(0)$, where $x = ys_0$, $y \in [0,1]$ for $i \in \mathcal{I}_1$, and $x =$

$s_0 + (y-1)(L-s_0)$, $y \in [1,2]$ for $i \in \mathcal{I}_2$. The vectors of concentrations u_0 and λ are assumed to be compatible, i.e.

$$u_{0i}(0) = \lambda_i(0), \text{ and hence, } \hat{u}_i(0) = 0 \text{ for } i \in \mathcal{I}_1. \tag{3.11}$$

The formulation is completed with two ordinary differential equations

$$s'(t) = \psi_\Gamma(1,t) \text{ and } v_4'(t) = f_4(v_4(t)) \text{ a.e. } t \in S_T, \tag{3.12}$$

where $v_4(t) := \hat{u}_4(s(t),t)$ for $t \in S_T$, for which we take

$$s(0) = s_0 > 0, \qquad v_4(0) = \hat{u}_{40}. \tag{3.13}$$

3.2. Function spaces. Weak formulation

The definition and properties of the function spaces used here can be found in [**19**], e.g. For each $i \in \mathcal{I}_1 \cup \mathcal{I}_2$, we denote $H_i := L^2(a,b)$ and set $[a,b] := [0,1]$ for $i \in \mathcal{I}_1$ and $[a,b] := [1,2]$ for $i \in \mathcal{I}_2$. Moreover, $\mathbb{H} := \prod_{i\in\mathcal{I}_1\cup\mathcal{I}_2} H_i$ and $\mathbb{V} := \prod_{i\in\mathcal{I}_1\cup\mathcal{I}_2} V_i$, where V_i are the Sobolev spaces $V_i := \{u \in H^1(a,b) : u_i(a) = 0\}, i \in \mathcal{I}_1$ and $V_i := H^1(a,b), i \in \mathcal{I}_2$. In addition, $|\cdot| := ||\cdot||_{L^2(a,b)}$ and $||\cdot|| := ||\cdot||_{H^1(a,b)}$. If $(X_i : i \in \mathcal{I})$ is a sequence of given sets X_i, then $X^{|\mathcal{I}_1\cup\mathcal{I}_2|}$ denotes the product $\prod_{i\in\mathcal{I}_1\cup\mathcal{I}_2} X_i := X_1 \times X_2 \times X_3 \times X_5 \times X_6$.

Let $\varphi := (\varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_6)^t \in \mathbb{V}$ be an arbitrary test function and take $t \in S_T$. The weak formulation of (3.3)–(3.13) reads as follows:

$$\begin{cases} a(s,u,\varphi) := \dfrac{1}{s}\displaystyle\sum_{i\in\mathcal{I}_1}(D_i u_{i,y}, \varphi_{i,y}) + \dfrac{1}{L-s}\displaystyle\sum_{i\in\mathcal{I}_2}(D_i u_{i,y}, \varphi_{i,y}), \\ b_f(u,s,\varphi) := s\displaystyle\sum_{i\in\mathcal{I}_1}(f_i(u), \varphi_i) + (L-s)\displaystyle\sum_{i\in\mathcal{I}_2}(f_i(u), \varphi_i), \\ e(s',u,\varphi) := \displaystyle\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2} g_i(s,s',u(1))\varphi_i(1), \\ h(s',u_{,y},\varphi) := s'\displaystyle\sum_{i\in\mathcal{I}_1}(y u_{i,y}, \varphi_i) + s'\displaystyle\sum_{i\in\mathcal{I}_2}((2-y)u_{i,y}, \varphi_i), \end{cases} \tag{3.14}$$

for any $u \in \mathbb{V}$ and $\lambda \in W^{1,2}(S_T)^{|\mathcal{I}_1\cup\mathcal{I}_2|}$. The term $a(\cdot)$ incorporates the diffusive part of the model, $b_f(\cdot)$ comprises volume productions, $e(\cdot)$ sums up reaction terms acting on $\Gamma(t)$ and $h(\cdot)$ is a non-local term due to fixing the domain. The interface terms $g_i(i \in \mathcal{I}_1 \cup \mathcal{I}_2)$ are given by

$$\begin{cases} g_1(s,s',u) := \eta_\Gamma(1,t) + s'(t)u_1(1), & g_2(s,s',u) := s'(t)u_2(1), \\ g_3(s,s',u) := \eta_\Gamma(1,t) - s'(t)u_3(1), & g_5(s,s',u) := \eta_\Gamma(1,t), \\ g_6(s,s',u) := 0, & \end{cases} \tag{3.15}$$

whereas the volume terms f_i $(i \in \mathcal{I})$ are defined as

$$\begin{cases} f_1(u) := P_1(Q_1 u_2 - u_1), & f_4(\hat{u}) := +\tilde{\eta}_\Gamma(s(t),t), \\ f_2(u) := -P_2(Q_2 u_2 - u_1), & f_5(u) := 0, \\ f_3(u) := S_{3,diss}(u_{3,eq} - u_3), & f_6(u) := 0. \end{cases} \tag{3.16}$$

The initial and boundary data as well as the model parameters are assumed to satisfy the following set of restrictions:

$$\lambda \in W^{1,2}(S_T)^{|\mathcal{I}_1 \cup \mathcal{I}_2|}, \qquad \lambda(t) \geq 0 \ \text{ a.e. } \ t \in \bar{S}_T, \tag{3.17}$$

$$u_{3,eq} \in L^\infty(S_T), \qquad u_{3,eq}(t) \geq 0 \ \text{ a.e. } \ t \in \bar{S}_T, \tag{3.18}$$

$$u_0 \in L^\infty(a,b)^{|\mathcal{I}_1 \cup \mathcal{I}_2|}, \qquad u_0(y) + \lambda(0) \geq 0 \ \text{ a.e. } \ y \in [a,b], \tag{3.19}$$

$$\hat{u}_{40} \in L^\infty(0,s_0), \qquad \hat{u}_4(x,0) > 0 \ \text{ a.e. } \ x \in [0,s_0], \tag{3.20}$$

$$s_0 > 0, \qquad L_0 < L < +\infty, \qquad s_0 < L_0, \tag{3.21}$$

$$\min\{S_{3,diss}, P_1, Q_1, P_2, Q_2, D_\ell(\ell \in \mathcal{I}_1 \cup \mathcal{I}_2)\} > 0. \tag{3.22}$$

We denote

$$m_0 := \min\{s_0, L - L_0\}, \qquad M_0 := \max\{L_0, L - s_0\}. \tag{3.23}$$

Set

$$\mathcal{K} := \prod_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} [0, k_i], \tag{3.24}$$

and, for fixed $\Lambda \in M_\Lambda$, we take

$$M_{\eta_\Gamma} := \max_{\bar{u} \in \mathcal{K}} \{\bar{\eta}_\Gamma(\bar{u}, \Lambda)\}. \tag{3.25}$$

In (3.24), we set

$$\begin{cases} k_i := \max\{u_{i0}(y) + \lambda_i(t), \lambda_i(t) : y \in [a,b], \ t \in \bar{S}_T\}, \quad i = 1,2,3,6, \\ k_4 := \max\{\hat{u}_{40}(x) + M_{\eta_\Gamma} T : x \in [0, s(t)], \ t \in \bar{S}_T\}, \\ k_5 := \max\{u_{50}(y) + \lambda_5(t), \lambda_6(t), \ \kappa : y \in [a,b], \ t \in \bar{S}_T\}, \\ k_6 := k_5, \end{cases} \tag{3.26}$$

where

$$\kappa := \frac{L_0}{D_5 - M_{\eta_\Gamma} L L_0} \left(M_{\eta_\Gamma} + \frac{L}{2} |\lambda_{5,t}|_\infty + 1 \right). \tag{3.27}$$

Definition 3.1 (Local Weak Solution; cf. [**10, 13**])**.** We call the triple (u, v_4, s) a local weak solution to the problem (3.3)–(3.13) if there is a $\delta \in]0,T]$ with $S_\delta :=]0,\delta[$ such that

$$s_0 < s(\delta) \leq L_0, \tag{3.28}$$

$$v_4 \in W^{1,4}(S_\delta), \quad s \in W^{1,4}(S_\delta), \tag{3.29}$$

$$u \in W_2^1(S_\delta; \mathbb{V}, \mathbb{H}) \cap [\bar{S}_\delta \mapsto L^\infty(a,b)]^{|\mathcal{I}_1 \cup \mathcal{I}_2|}, \tag{3.30}$$

For all $\varphi \in \mathbb{V}$ and a.e. $t \in S_\delta$ we have

$$\begin{cases} s \sum_{i \in \mathcal{I}_1} (u_{i,t}(t), \varphi_i) + (L - s) \sum_{i \in \mathcal{I}_2} (u_{i,t}(t), \varphi_i) + a(s, u, \varphi) + e(s', u + \lambda, \varphi) \\ = b_f(u + \lambda, s, \varphi) + h(s', u_{,y}, \varphi) - s \sum_{i \in \mathcal{I}_1} (\lambda_{i,t}(t), \varphi_i) - (L - s) \sum_{i \in \mathcal{I}_2} (\lambda_{i,t}(t), \varphi_i), \\ s'(t) = \eta_\Gamma(1,t), \qquad v_4'(t) = f_4(v_4(t)) \text{ a.e. } t \in S_\delta, \\ u(0) = u_0 \in \mathbb{H}, \qquad s(0) = s_0, \qquad v_4(0) = \hat{u}_{40}. \end{cases}$$

3.3. Assumptions of the model parameters and constitutive reaction-rate law

The only assumptions that are needed are the following:

(A) Fix $\Lambda \in M_\Lambda$. Let $\bar{\eta}_\Gamma(\bar{u},\Lambda) > 0$, if $\bar{u}_1 > 0$ and $\bar{u}_3 > 0$, and $\bar{\eta}_\Gamma(\bar{u},\Lambda) = 0$, otherwise. For any fixed $\bar{u}_1 \in \mathbb{R}$, $\bar{\eta}_\Gamma$ is bounded.

(B) The reaction rate $\bar{\eta}_\Gamma : \mathbb{R}^6 \times M_\Lambda \to \mathbb{R}_+$ is locally Lipschitz. This restricts the choice of p and q in (2.11).

(C1) $1 > k_3 \geq \max_{\bar{S}_T}\{|u_{3,eq}(t)| : t \in \bar{S}_T\}$; $D_5 - M_{\eta_\Gamma} L > 0$;

(C2) $P_1 Q_1 k_2 \leq P_1 k_1$; $\qquad P_2 k_1 \leq P_2 Q_2 k_2$.

Remark 3.2.

(i) We refer to reader to [**10**] to see a possible physical interpretation of the restrictions (A)–(C).

(ii) For our convenience, we define the constants $K_1 = K_3 := 0$, and K_2 and K_4 via (3.44) and (5.5), respectively.

By (A) and (B), we deduce that $\eta_\Gamma(0,\Lambda) = 0$ for all $\Lambda \in M_\Lambda$. For all $\bar{u} \in \mathbb{R}^6$ there is an ϵ-neighborhood $\mathcal{U}_\epsilon(\bar{u})$ and a positive constant $C_\eta = C_\eta(\Lambda, \lambda, \epsilon, T_{\text{fin}})$ such that the inequality

$$\bar{\eta}_\Gamma(\bar{u}(s(t),t),\Lambda) \leq C_\eta |\bar{u}(s(t),t)| \tag{3.31}$$

holds for all $t \in S_T$. (3.31) can be reformulated as

$$\eta_\Gamma(1,t) \leq C_\eta |u(1,t)| \quad \text{for all} \quad t \in S_T. \tag{3.32}$$

Note also that there exists a function $c_g = c_g(C_\eta)$ such that

$$|e(s', u(1), \varphi(1))| \leq c_g |u(1)||\varphi(1)| \quad \text{for all} \quad \varphi \in \mathbb{V} \tag{3.33}$$

and a constant $c_f = c_f(C_\eta, K_1) > 0$ such that

$$|b_f(u, s, \varphi)| \leq c_f \left(|u_{3,eq}|_\infty^2 + |u|^2 + |\varphi|^2\right) \quad \text{for all} \quad \varphi \in \mathbb{V}, \tag{3.34}$$

where $K_1 > 0$ is a constant depending on the material parameters entering f_i ($i \in \mathcal{I}$), i.e. P_1, P_2, Q_1, Q_2, and $S_{3,diss}$. The exact structure of c_g, c_f and K_1 is dictated by the definition of the production terms f_i and g_i ($i \in \mathcal{I}$), see (3.16) and (3.15). Since $\psi_\Gamma(1,t)$ has essentially the *same* structure as $\eta_\Gamma(1,t)$, it also satisfies (A) and (B).

3.4. Known results

In this section, we list a couple of known results (see [**10, 13**]) which will be extensively used in section 5.

Lemma 3.3 (Some Basic Estimates)**.** *Let* $c_\xi > 0$, $\xi > 0$, $\theta \in [\frac{1}{2}, 1[$ *and* $s \in W^{1,1}(S_\delta)$.

(i) *There exists the constant* $\hat{c} = \hat{c}(\theta) > 0$ *such that*

$$|u_i|_\infty \leq \hat{c} |u_i|^{1-\theta} ||u_i||^\theta \tag{3.35}$$

for all $u_i \in V_i$, *where* $i \in \mathcal{I}_1 \cup \mathcal{I}_2$.

(ii) *It holds*

$$|u_i|^{1-\theta}||u_i||^{\theta} \leq \xi||u_i|| + c_\xi|u_i| \tag{3.36}$$

for all $u_i \in V_i$, *where* $i \in \mathcal{I}_1 \cup \mathcal{I}_2$.

(iii) *Let* $\varphi \in \mathbb{V}$ *with* $\varphi = (\varphi_1, \ldots, \varphi_6)^t$, $t \in S_\delta$, $\hat{c}$ *as in* (i), *and* ξ, c_ξ *as in* (ii). *Then, for* $i \in \mathcal{I}_1$ *and* $j \in \mathcal{I}_2$, *we have the following inequalities:*

$$\frac{|s'(t)|}{s(t)}(y\varphi_{i,y}, \varphi_i) = \frac{1}{2}\frac{|s'(t)|}{s(t)}\{\varphi_i(1)^2 - |\varphi_i|^2\} \leq \frac{1}{2}\frac{|s'(t)|}{s(t)}\{\hat{c}^2|\varphi_i|^{2(1-\theta)}||\varphi_i||^{2\theta} - |\varphi_i|^2\};$$

$$\frac{|s'(t)|}{s(t)}|\varphi_i(1)|^2 \leq \frac{|s'(t)|}{s(t)}|\varphi_i|^2_\infty \leq \frac{\xi}{s^2(t)}||\varphi_i||^2 + c_\xi\hat{c}^{\frac{2}{1-\theta}} \times s(t)^{\frac{2\theta-1}{1-\theta}}|s'(t)|^{\frac{1}{1-\theta}}|\varphi_i|^2;$$

$$\begin{aligned}\frac{|\varphi_i(1)|^2}{s^2(t)} &\leq \frac{1}{s^2(t)}|\varphi_i|^2_\infty \leq \hat{c}^2 s(t)^{2\theta-2}|\varphi_i|^{2(1-\theta)}\left(s(t)^{-1}||\varphi_i||\right)^{2\theta} \\ &\leq \frac{\xi}{s^2(t)}||\varphi_i||^2 + c_\xi\hat{c}^{\frac{2}{1-\theta}}|s(t)|^{\frac{2(\theta-1)}{1-\theta}}|\varphi_i|^2; \\ \frac{|\varphi_i(1)|^2}{s(t)} &\leq \frac{\xi}{s^2(t)}||\varphi_i||^2 + c_\xi\hat{c}^{\frac{2}{1-\theta}}|s(t)|^{\frac{2\theta-1}{1-\theta}}|\varphi_i|^2\end{aligned}$$

$$\frac{|s'(t)|}{L-s(t)}((2-y)\varphi_{j,y}, \varphi_j) = \frac{1}{2}\frac{|s'(t)|}{L-s(t)}|\varphi_j(1)|^2 + \frac{1}{2}\frac{|s'(t)|}{L-s(t)}|\varphi_j|^2.$$

Theorem 3.4 (Positivity and L^∞-Estimates)**.** *Let the triple* (u, v_4, s) *as in Definition 3.1 satisfy the assumptions* (A)–(C2). *Then the following statements hold:*

(i) (Positivity) $u(t) + \lambda(t) \geq 0$ *in* $\mathbb{V}$ *for all* $t \in S_\delta$.

(ii) (L^∞-estimates) *Let* $\ell \in \mathcal{I}_1 \cup \mathcal{I}_2$ *be arbitrarily fixed. There exists a constant* $k_\ell > 0$ *(see* (3.26)*) such that* $u_\ell(t) + \lambda_\ell(t) \leq k_\ell$ *in* V_ℓ *(*$\ell \in \mathcal{I} - \{4, 5\}$*) for all* $t \in S_\delta$. *In addition, there exists a constant* $k_5 > 0$ *such that* $u_5(t) \leq k_5 y$ *a.e.* $y \in [0, 1]$ *and all* $t \in S_\delta$.

(iii) (Localization of the interface)

$s_0 \leq s(t) \leq s_0 + \delta M_{\eta\Gamma}$ *for all* $t \in S_\delta$, *where* $M_{\eta\Gamma}$ *is given in* (3.26).

(iv) (Positivity and boundedness of v_4 at $\Gamma(t)$)

$$0 < \hat{u}_{40} \leq v_4(t) \leq \hat{u}_{40} + \delta M_{\eta\Gamma} \text{ for all } t \in S_\delta.$$

Lemma 3.5 (Energy Estimates)**.** *Assume that* (A)–(C2) *hold and let the triple* (u, v_4, s) *be as in Definition 3.1. The following statements hold a.e. in* S_δ*:*

$$|u(t) + \lambda(t)|^2 \leq \alpha(t) \exp\left(\int_0^t \beta(\tau)\mathrm{d}\tau\right); \tag{3.37}$$

$$|u(t) + \lambda(t)|^2 \leq \alpha(t) + \int_0^t \beta(s)\alpha(s) \exp\left(\int_s^t \beta(\tau)\mathrm{d}\tau\right)\mathrm{d}s; \tag{3.38}$$

$$\int_0^t ||u(\tau) + \lambda(\tau)||^2\mathrm{d}\tau \leq d_0^{-1}\alpha(t) \exp\left(\int_{t_0}^t \beta(\tau)\mathrm{d}\tau\right), \tag{3.39}$$

where

$$d_0 := \min\left\{\min_{i\in\mathcal{I}_1}\frac{s_0 D_i}{L^2 m_0},\ \min_{i\in\mathcal{I}_2}\frac{(L-L_0)D_i}{(L-s_0)^2 m_0}\right\},\ m_0 \text{ as in } (3.23). \tag{3.40}$$

The factors $a(t)$, $\alpha(t)$ *and* $\beta(t)$ *are given by*

$$a(t) := \frac{(s'(t))^2}{2} + \frac{(L-s(t))^2 K_2}{2}, \tag{3.41}$$

$$\alpha(t) := |\varphi(0)|^2 + \frac{2}{m_0}\int_0^t a(\tau)\mathrm{d}\tau, \tag{3.42}$$

$$\beta(t) := \left[\frac{s'(t)}{2} + K_2\left(2 + \frac{D_3}{L-s(t)} + \frac{s'(t)}{2}\right)^2\right]\frac{1}{m_0}, \tag{3.43}$$

whereas

$$K_2 := 1 + (S_{3,diss}|u_{3,eq}|_\infty)^2 + \frac{LP_1Q_1}{2} + c_\xi \hat{c}^4. \tag{3.44}$$

Furthermore, we have

$$u \in L^2(S_\delta, \mathbb{V}), u_{,t} \in L^2(S_\delta, \mathbb{V}^*), u \in C(\bar{S}_\delta, \mathbb{H}). \tag{3.45}$$

Theorem 3.6 (Local Existence and Uniqueness)**.** *Assume the hypotheses* (A)–(C2) *and let the conditions* (3.17)–(3.2) *be satisfied. Then the following assertions hold:*

(a) *There exists a* $\delta \in]0, T[$ *such that the problem* (3.3)–(3.13) *admits a unique local solution on* S_δ *in the sense of Definition 3.1;*
(b) $0 \le u_i(y,t) + \lambda_i(t) \le k_i$ *a.e.* $y \in [a,b]$ $(i \in \mathcal{I}_1 \cup \mathcal{I}_2)$ *for all* $t \in S_\delta$*. Moreover,* $0 \le \hat{u}_4(x,t) \le k_4$ *a.e.* $x \in [0, s(t)]$ *for all* $t \in S_\delta$*;*
(c) $v_4, s \in W^{1,\infty}(S_\delta)$.

4. Main result

Select $i \in \{1,2\}$ and let $(u^{(i)}, v_4^{(i)}, s_i)$ be two weak solutions on S_δ in the sense of Definition 3.1. They correspond to the sets of data

$$\mathcal{D}_i := (u_0^{(i)}, \lambda^{(i)}, \Xi^{(i)}, \Upsilon^{(i)}, \Lambda^{(i)})^t,$$

where $u_0^{(i)}$, $\lambda^{(i)}$, $\Xi^{(i)}$, $\Upsilon^{(i)}$, and $\Lambda^{(i)}$ denote the respective initial data, boundary data, and the model parameters describing diffusion, dissolution mechanisms and carbonation reaction, respectively.

In this context, we have $\Xi^{(i)} := (D_\ell^{(i)}\ (\ell \in \mathcal{I}_1 \cup \mathcal{I}_2),\ P_k^{(i)}(k \in \{1,2\}), Q_k^{(i)}(k \in \{1,2\}),\ S_{3,diss}^{(i)})^t \subset M_\Xi$ and $\Upsilon^{(i)} = (u_{3,eq}^{(i)}) \subset M_\Upsilon$, $i \in \{1,2\}$. Here M_Ξ and M_Υ are compact subsets of $\mathbb{R}_+^{10}$ and $L^2(S_\delta)$.

Set $\Delta u := u^{(2)} - u^{(1)}$, $\Delta v_4 := v_4^{(2)} - v_4^{(1)}$, $\Delta s := s_2 - s_1$, $\Delta\lambda := \lambda^{(2)} - \lambda^{(1)}$, $\Delta u_0 := u_0^{(2)} - u_0^{(1)}$, $\Delta\Xi := \Xi^{(2)} - \Xi^{(1)}$, $\Delta\Upsilon := \Upsilon^{(2)} - \Upsilon^{(1)}$, $\Delta\Lambda := \Lambda^{(2)} - \Lambda^{(1)}$, and $\Delta\eta_\Gamma := \tilde{\eta}_\Gamma^{(2)} - \tilde{\eta}_\Gamma^{(1)} := \bar{\eta}_\Gamma^{(2)}(\bar{u}^{(2)}, \Lambda^{(2)}) - \bar{\eta}_\Gamma^{(1)}(\bar{u}^{(1)}, \Lambda^{(1)})$. The Lipschitz condition of η_Γ reads: There exists a constant $c_L = c_L(\mathcal{D}_1, \mathcal{D}_2) > 0$ such that the inequality $|\Delta\eta_\Gamma| \le c_L(|\Delta u| + |\Delta\Lambda|)$ holds locally pointwise, see (B).

Having these notations available, we can state now the main result of the paper.

Theorem 4.1. *Let* $(u^{(i)}, v_4^{(i)}, s_i)(i \in \{1,2\})$ *be two local weak solutions on* S_δ *in the sense of Definition 3.1 satisfying the assumptions of Theorem 3.6. Let* $(u_0^{(i)}, \lambda^{(i)}, \Lambda^{(i)})$ *be the vector of initial, boundary and reaction data. Then the function* $\mathbb{H} \times W^{1,2}(S_\delta)^{|\mathcal{I}_1 \cup \mathcal{I}_2|} \times M_\Xi \times M_\Upsilon \times M_\Lambda \to W_2^1(S_\delta, \mathbb{V}, \mathbb{H}) \times W^{1,4}(S_\delta)^2$, *which maps* $(u_0, \lambda, \Xi, \Upsilon, \Lambda)^t$ *into* $(u, v_4, s)^t$, *is Lipschitz in the following sense: There exists a constant* $c = c(\delta, s_0, \hat{u}_{40}, L, k_i, c_L, \delta) > 0$ $(i \in \mathcal{I}_1 \cup \mathcal{I}_2)$ *such that*

$$\begin{aligned}
&\|\Delta u\|^2_{W_2^1(S_\delta,\mathbb{V},\mathbb{H})\cap L^\infty(S_\delta,\mathbb{H})} + \|\Delta v_4\|^2_{W^{1,4}(S_\delta)\cap L^\infty(S_\delta)} + ||\Delta s||^2_{W^{1,4}(S_\delta)\cap L^\infty(S_\delta)}\\
&\quad \le c\left(\|\Delta u_0\|^2_{\mathbb{H}\cap L^\infty([a,b]^{|\mathcal{I}_1\cup\mathcal{I}_2|})} + \|\Delta\lambda\|^2_{(W^{1,2}(S_\delta)\cap L^\infty(S_\delta))^{|\mathcal{I}_1\cup\mathcal{I}_2|}}\right)\\
&\qquad + c\left(\max_{M_\Xi}|\Delta\Xi|^2 + ||\Delta\Upsilon||^2_{M_\Upsilon\cap L^\infty(S_\delta)} + \max_{M_\Lambda}|\Delta\Lambda|^2\right).
\end{aligned}\tag{4.1}$$

We prove Theorem 4.1 in Section 5. A direct consequence of this result is the stability of the moving boundary as stated in the next result.

Corollary 4.2 (Stability of the Interface)**.** *Assume that the hypotheses of Theorem 4.1 are satisfied. Then the function* $\mathbb{H} \times W^{1,2}(S_\delta)^{|\mathcal{I}_1\cup\mathcal{I}_2|} \times M_\Xi \times M_\Lambda \to W^{1,4}(S_\delta)$, *which maps the data* $(u_0, \lambda, \Xi, \Upsilon, \Lambda)^t$ *into the position of the interface* s, *is Lipschitz in the following sense: There exists a constant* $c = c(\delta, s_0, \hat{u}_{40}, L, k_i, c_L, \delta) > 0$ *such that*

$$\begin{aligned}
\|\Delta s\|^2_{W^{1,4}(S_\delta)\cap L^\infty(S_\delta)} &\le c\left(\|\Delta u_0\|^2_{\mathbb{H}\cap L^\infty([a,b]^{|\mathcal{I}_1\cup\mathcal{I}_2|})} + \|\Delta\lambda\|^2_{(W^{1,2}(S_\delta)\cap L^\infty(S_\delta))^{|\mathcal{I}_1\cup\mathcal{I}_2|}}\right)\\
&\quad + c\left(\max_{M_\Xi}|\Delta\Xi|^2 + ||\Delta\Upsilon||^2_{M_\Upsilon\cap L^\infty(S_\delta)} + \max_{M_\Lambda}|\Delta\Lambda|^2\right).
\end{aligned}\tag{4.2}$$

Putting together the statements of Theorem 4.1 with those of [**13**, Theorem 3.3 and Theorem 3.4], the well-posedness of the moving boundary system described in Section 1 is shown.

5. Proof of Theorem 4.1

Let $(u^{(i)}, v_4^{(i)}, s_i)(i \in \{1,2\})$ be two weak solutions on S_δ (in the sense of Definition 3.1), which satisfy the assumptions of Theorem 3.6. We want to show that the function $\mathbb{H} \times W^{1,2}(S_\delta)^{|\mathcal{I}_1\cup\mathcal{I}_2|} \times M_\Xi \times M_\Upsilon \times M_\Lambda \to W_2^1(S_\delta, \mathbb{V}, \mathbb{H}) \times W^{1,4}(S_\delta)^2$ that maps $(u_0, \lambda, \Xi, \Upsilon, \Lambda)^t$ into $(u, v_4, s)^t$ is Lipschitz continuous in the sense of (4.1). By (3.2), the positions $s_i(t)$, $i = 1, 2$ of the interfaces $\Gamma_i(t)$ $(i \in \{1,2\})$ satisfy the geometrical restriction

$$0 < s_{i0} := s_i(0) \le s_i(t) \le L_{i0} < L \text{ for } i \in \{1,2\} \text{ and } t \in S_\delta.$$

Denoting $s_0 := \max\{s_{10}, s_{20}\}$ and $L_0 := \min\{L_{10}, L_{20}\}$, the common space domain traveled by the interfaces $\Gamma_i(t)$ is $\Omega :=]s_0, L_0[$. Within this frame we only discuss the case $s_0 < L_0$. Set

$$L_* := \min_{t\in \bar{S}_\delta}\{\min\{s_i(t), L - s_i(t)\} : i = 1, 2\}, \tag{5.1}$$

$$D_0 := \min\{D_j^{(i)} : j \in \mathcal{I}_1 \cup \mathcal{I}_2, i \in \{1, 2\}\} > 0. \tag{5.2}$$

We subtract the weak formulation (3.31) for the solution $(u^{(1)}, v_4^{(1)}, s_1)$ from the weak formulation written in terms of $(u^{(2)}, v_4^{(2)}, s_2)$. Choosing $w = (u^{(2)} - u^{(1)})^t + (\lambda^{(2)} - \lambda^{(1)})^t \in \mathbb{V}$ (i.e. $w_j = u_j^{(2)} - u_j^{(1)} + \lambda_j^{(2)} - \lambda_j^{(1)} \in V_j$ for each $j \in \mathcal{I}_1 \cup \mathcal{I}_2$) as test function, we obtain

$$\begin{aligned} & s_2 \sum_{i\in\mathcal{I}_1} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w_i(t)|^2 + (L - s_2)) \sum_{i\in\mathcal{I}_2} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w_i(t)|^2 \\ & + \frac{1}{s_2}\sum_{i\in\mathcal{I}_1} \|\sqrt{D_i}^{(2)} w_i\|^2 + \frac{1}{(L - s_2)}\sum_{i\in\mathcal{I}_2} \|\sqrt{D_i}^{(2)} w_i\|^2 \le \sum_{\ell=1}^{5} J_\ell, \end{aligned} \tag{5.3}$$

where the terms J_ℓ $(\ell \in \{1, \dots, 5\})$ are defined by

$$\begin{aligned} J_1 &:= \Delta s \sum_{i\in\mathcal{I}_1}(u_{i,t}^{(1)}, w_i) - \Delta s \sum_{i\in\mathcal{I}_2}(u_{i,t}^{(1)}, w_i) \\ J_2 &:= \frac{\Delta s}{s_1 s_2}\sum_{i\in\mathcal{I}_1}(D_i^{(1)} u_{i,y}^{(1)}, w_{i,y}) - \frac{\Delta s}{(L - s_1)(L - s_2)}\sum_{i\in\mathcal{I}_2}(D_i^{(1)} u_{i,y}^{(1)}, w_{i,y}) \\ &\quad + \frac{|\Delta D|}{s_1}\sum_{i\in\mathcal{I}_1}(u_{i,y}^{(1)}, w_{i,y}) + \frac{|\Delta D|}{L - s_1}\sum_{i\in\mathcal{I}_2}(u_{i,y}^{(1)}, w_{i,y}), \\ J_3 &:= s_2\left[P_1^{(2)}(Q_1^{(2)} u_2^{(2)} - u_1^{(2)}, w_1) - P_2^{(2)}(Q_2^{(2)} u_2^{(2)} - u_1^{(2)}, w_2)\right] \\ &\quad - s_1\left[P_1^{(1)}(Q_1^{(1)} u_2^{(1)} - u_1^{(1)}, w_1) - P_2^{(1)}(Q_2^{(1)} u_2^{(1)} - u_1^{(1)}, w_2)\right] \\ &\quad + (L - s_2)S_{3,diss}^{(2)}(u_{3,eq}^{(2)} - u_3^{(2)}, w_3) - (L - s_1)S_{3,diss}^{(1)}(u_{3,eq}^{(1)} - u_3^{(1)}, w_3) \\ J_4 &:= \left[\eta_\Gamma^{(2)} + s_2' u_1^{(2)}(1)\right] w_1(1) - s_2' u_2^{(2)}(1) w_2(1) \\ &\quad + \left[\eta_\Gamma^{(2)} - s_2' u_3^{(2)}(1)\right] w_3(1) - \eta_\Gamma^{(2)} w_5(1) \\ &\quad - \left[\eta_\Gamma^{(1)} + s_1' u_1^{(1)}(1)\right] w_1(1) + s_1' u_2^{(1)}(1) w_2(1) \\ &\quad - \left[\eta_\Gamma^{(1)} - s_1' u_3^{(1)}(1)\right] w_3(1) + \eta_\Gamma^{(1)} w_5(1) \\ &\quad + \frac{1}{s_2}\sum_{i\in\mathcal{I}_1} D_i^{(2)}|w_i(1)|^2 + \frac{1}{L - s_2}\sum_{i\in\mathcal{I}_2} D_i^{(2)}|w_i(1)|^2 \\ J_5 &:= s_2'\sum_{i\in\mathcal{I}_1}(y u_{i,y}^{(2)}, w_i) + s_2'\sum_{i\in\mathcal{I}_2}((2 - y) u_{i,y}^{(2)}, w_i) \\ &\quad - s_1'\sum_{i\in\mathcal{I}_1}(y u_{i,y}^{(1)}, w_i) - s_1'\sum_{i\in\mathcal{I}_2}((2 - y) u_{i,y}^{(1)}, w_i). \end{aligned} \tag{5.4}$$

To simplify the writing of the estimates, we employ the constant K_4, which is given by

$$\begin{aligned}K_4 := \; & 1+ c_\xi c_{\bar\xi}\left(\hat c\bar k\right)^{\frac{2}{1-\theta}} + \bar k^2 + \bar k^4\hat c^4 + 2c_\xi \bar k^2 + \max\left\{1, \frac{L}{2}\right\}\\ & + c_\xi + (\hat c^2\tilde c)^{\frac{1}{1-\theta}} + c_\xi \sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\left(D_i^{(1)}\right)^2 + \left[(k_1+k_2)P_1^{(2)}Q_1^{(2)}\right]^2\\ & + (LQ_1^{(2)}k_2)^2 + (P_1^{(2)}k_2)^2 + 2\left(P_1^{(2)}Q_1^{(2)}\right)^2 .\end{aligned} \tag{5.5}$$

Note that K_4 is finite and depends on k_ℓ ($\ell \in \mathcal{I}_1\cup\mathcal{I}_2$), c_ξ, $\tilde c$, $c_{\bar\xi}$, θ, and δ. To estimate the above terms $|J_\ell|$ ($\ell \in \{1,\dots,5\}$) we use all of the estimates that we have already possed, that is positivity, maximum, and energy estimates. We obtain

$$\begin{aligned}|J_1| &\le \frac{|\Delta s|^2}{2}|w_{1,t}|^2 + \frac{|w|^2}{2}.\\ |J_2| &\le 2\xi\sum_{i\in\mathcal{I}_1}\frac{\|w_i\|^2}{s_2^2} + 2\xi\sum_{i\in\mathcal{I}_2}\frac{\|w_i\|^2}{(L-s_2)^2}\\ &\quad + K_4\left(\frac{1}{s_1^2}+\frac{1}{(L-s_1)^2}\right)\|u_1^{(1)}\|^2|\Delta s|^2\\ &\quad + K_4\left[\left(\frac{s_2}{s_1}\right)^2 + \left(\frac{L-s_2}{L-s_1}\right)^2\right]\|u_1^{(1)}\|^2|\Delta D|^2\end{aligned} \tag{5.6}$$

$$|J_3| \le \frac{3}{2}|\Delta s|^2 + \frac{L}{2}\left(|\Delta S_{3,diss}|^2 + |\Delta u_{3,eq}|_\infty^2\right) + |\Delta P|^2 + |\Delta Q|^2 + K_4|w|^2. \tag{5.7}$$

Since $M_{\eta_\Gamma} < \infty$, then there exists a constant $\tilde c \in \mathbb{R}_+^*$ such that

$$\tilde c > 1 + 3C_\eta + 4k_2^2 + k_3^2 + 2M_{\eta_\Gamma} + \frac{L-L_0+s_0}{s_0(L-L_0)}\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2} D_i. \tag{5.8}$$

Using (5.8), we obtain

$$\begin{aligned}|J_4| &\le |\Delta\Lambda|^2 + \frac{3}{2}|\Delta s|^2 + \tilde c|w(1)|^2\\ &\le |\Delta\Lambda|^2 + \frac{3}{2}|\Delta s|^2 + \tilde c\hat c^2 s_2^{2\theta}\sum_{i\in\mathcal{I}_1}\frac{\|w_i\|^{2\theta}}{s_2^{2\theta}}|w_i|^{2(1-\theta)}\\ &\quad + \tilde c\hat c^2(L-s_2)^{2\theta}\sum_{i\in\mathcal{I}_2}\frac{\|w_i\|^{2\theta}}{(L-s_2)^{2\theta}}|w_i|^{2(1-\theta)}\\ &\le \xi\sum_{i\in\mathcal{I}_1}\frac{\|w_i\|^2}{s_2^2} + \xi\sum_{i\in\mathcal{I}_2}\frac{\|w_i\|^2}{(L-s_2)^2} + |\Delta\Lambda|^2 + \frac{3}{2}|\Delta s|^2\\ &\quad + K_4\left[s_2^{\frac{2\theta}{1-\theta}} + (L-s_2)^{\frac{2\theta}{1-\theta}}\right]|w|^2.\end{aligned} \tag{5.9}$$

Furthermore, it holds

$$\begin{aligned}
J_5 &= h(s_2', u_{,y}^{(2)}, w) - h(s_1', u_{,y}^{(1)}, w)\\
&= s_2\frac{s_2'}{s_2}\sum_{i\in\mathcal{I}_1}(yu_{i,y}^{(2)}, w_i) - s_1\frac{s_1'}{s_1}\sum_{i\in\mathcal{I}_1}(yu_{i,y}^{(1)}, w_i)\\
&\quad + (L-s_2)\frac{s_2'}{L-s_2}\sum_{i\in\mathcal{I}_2}((2-y)u_{i,y}^{(2)}, w_i) - (L-s_1)\frac{s_2'}{L-s_2}\sum_{i\in\mathcal{I}_2}((2-y)u_{i,y}^{(2)}, w_i)\\
&= J_{51} + J_{52}.
\end{aligned}$$

Using again Lemma 3.3, we establish upper bounds for these terms in the following fashion:

$$\begin{aligned}
J_{51} &\le L\frac{s_2'}{s_2}\sum_{i\in\mathcal{I}_1}|(yw_{i,y}, w_i)| + L|\left(\frac{s_2'}{s_2} - \frac{s_1'}{s_1}\right)|\sum_{i\in\mathcal{I}_1}|(yu_{i,y}^{(1)}, w_i)|,\\
J_{52} &\le L\frac{s_2'}{L-s_2}\sum_{i\in\mathcal{I}_2}|((2-y)w_{i,y}, w_i)|\\
&\quad + L\left|\left(\frac{s_2'}{L-s_2} - \frac{s_1'}{L-s_1}\right)\right|\sum_{i\in\mathcal{I}_2}|((2-y)u_{i,y}^{(1)}, w_i)|.
\end{aligned}$$

It holds

$$\begin{aligned}
\frac{1}{L}|J_{51}| &\le \frac{s_2'}{s_2}\sum_{i\in\mathcal{I}_1}|(yw_{i,y}, w_i)| + \frac{|\Delta s'|}{s_2}\sum_{i\in\mathcal{I}_1}|(yu_{i,y}^{(1)}, w_i)|\\
&\quad + \frac{s_1'}{s_1 s_2}|\Delta s|\sum_{i\in\mathcal{I}_1}|(yu_{i,y}^{(1)}, w_i)|.
\end{aligned}\tag{5.10}$$

Firstly, we see that

$$\frac{s_2'}{s_2}\sum_{i\in\mathcal{I}_1}|(yw_{i,y}, w_i)| \le \xi\sum_{i\in\mathcal{I}_1}\frac{||w_i||^2}{s_2^2} + c_\xi\hat{c}^{\frac{2}{1-\theta}}\left(\frac{s_2'}{2}\right)^{\frac{1}{1-\theta}} s_2^{\frac{2\theta-1}{1-\theta}}\sum_{i\in\mathcal{I}_1}|w_i|^2.$$

Furthermore, for each $i \in \mathcal{I}_1$ we use the relation $|(yu_{i,y}^{(1)}, w_i)| \le |u_i^{(1)}(1)w_i(1)| + |(yw_{i,y}, u_i^{(1)})| + |(u_i^{(1)}, w_i)|$ to split the last two sums in (5.10) as follows:

$$\begin{aligned}
\frac{1}{L}|J_{51}| &\le \xi\sum_{i\in\mathcal{I}_1}\frac{\|w_i\|^2}{s_2^2} + c_\xi\hat{c}^{\frac{2}{1-\theta}}\left(\frac{s_2'}{2}\right)^{\frac{1}{1-\theta}} s_2^{\frac{2\theta-1}{1-\theta}}\sum_{i\in\mathcal{I}_1}|w_i|^2\\
&\quad + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} := \frac{|\Delta s'|}{s_2}\sum_{i\in\mathcal{I}_1}|u_i^{(1)}(1)w_i(1)| &\le \sum_{i\in\mathcal{I}_1}|\Delta s'|\frac{\hat{c}\bar{k}}{s_2^{1-\theta}}\frac{\|w_i\|^\theta}{s_2^\theta}|w_i|^{1-\theta}\\
&\le 2\bar{\xi}|\Delta s'|^2 + \xi c_{\bar{\xi}}\sum_{i\in\mathcal{I}_1}\frac{\|w_i\|^2}{s_2^2} + c_\xi c_{\bar{\xi}}\frac{(\hat{c}\bar{k})^{\frac{2}{1-\theta}}}{s_2^2}\sum_{i\in\mathcal{I}_1}|w_i|^2\\
&\quad (\text{with } \bar{k} \ge 2\max k_i \quad (i\in\mathcal{I})),
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &:= \frac{|\Delta s'|}{s_2} \sum_{i\in\mathcal{I}_1} |(y w_{i,y}, u_i^{(1)})| \le \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|}{s_2} \bar{k}|\Delta s'| \\
&\le 2c_\xi \bar{k}^2 |\Delta s'|^2 + \xi \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|^2}{s_2^2}, \\
\mathrm{III} &:= \frac{|\Delta s'|}{s_2} \sum_{i\in\mathcal{I}_1} |(u_i^{(1)}, w_i)| \le 2|\Delta s'|^2 + \frac{\bar{k}^2}{2s_2^2} \sum_{i\in\mathcal{I}_1} |w_i|^2, \\
\mathrm{IV} &:= \frac{s_1'}{s_1 s_2} |\Delta s| \sum_{i\in\mathcal{I}_1} |u_i^{(1)}(1) w_i(1)| \\
&\le 2\bar{\xi}|\Delta s|^2 + \xi c_{\bar{\xi}} \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|^2}{s_2^2} + \frac{1}{s_2^2} \left(\bar{k}\hat{c}\frac{s_1'}{s_1}\right)^{\frac{2}{1-\theta}} \sum_{i\in\mathcal{I}_1} |w_i|^2, \\
\mathrm{V} &:= \frac{s_1'}{s_1 s_2} |\Delta s| \sum_{i\in\mathcal{I}_1} |(y w_{i,y}, u_i^{(1)})| \le 2c_\xi \bar{k}^2 |\Delta s|^2 \left(\frac{s_1'}{s_1}\right)^2 + \xi \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|^2}{s_2^2}, \\
\mathrm{VI} &:= \frac{s_1'}{s_1 s_2} |\Delta s| \sum_{i\in\mathcal{I}_1} |(u_i^{(1)}, w_i)| \le 2|\Delta s|^2 + \frac{s_1'^2 \bar{k}^2}{s_1^2 s_2^2} \sum_{i\in\mathcal{I}_1} |w_i|^2.
\end{aligned}$$

These inequalities yield an upper bound on $|J_{51}|$. It holds

$$\begin{aligned}
\frac{1}{L}|J_{51}| &\le \xi(3 + 2c_{\bar{\xi}}) \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|^2}{s_2^2} + |\Delta s|^2 \left[2(1+\bar{\xi}) + K_4 \left(\frac{s_1'}{s_1}\right)^2\right] \\
&\quad + 2|\Delta s'|^2 (1 + \bar{\xi} + K_4) \\
&\quad + K_4 \left[\frac{1}{s_2^2} + \frac{(s_1')^2}{s_1^2 s_2^2} + \frac{(s_2')^2}{4} + \frac{(s_1')^4}{s_1^4 s_2^2}\right] \sum_{i\in\mathcal{I}_1} |w_i|^2.
\end{aligned} \tag{5.11}$$

Using the inequality

$$|((2-y)u_{i,y}^{(1)}, w_i)| \le |u_i^{(1)}(1) w_i(1)| + |((2-y)w_{i,y}, u_i^{(1)})| + |(u_i^{(1)}, w_i)|, \qquad i \in \mathcal{I}_2,$$

we find that

$$\begin{aligned}
\frac{1}{L}|J_{52}| &\le \xi(3 + 2c_{\bar{\xi}}) \sum_{i\in\mathcal{I}_2} \frac{\|w_i\|^2}{(L-s_2)^2} \\
&\quad + |\Delta s|^2 \left[1 + \bar{\xi} + K_4 \left(\frac{s_1'}{L - s_1}\right)^2\right] \\
&\quad + |\Delta s'|^2 (1 + \bar{\xi} + K_4) + K_4 \left[\frac{1}{(L-s_2)^2} + \frac{(s_1')^2}{(L-s_1)^2 (L-s_2)^2}\right. \\
&\qquad \left. + \frac{(s_2')^2}{4} + \frac{(s_1')^4}{(L-s_1)^4 (L-s_2)^2}\right] \sum_{i\in\mathcal{I}_2} |w_i|^2.
\end{aligned} \tag{5.12}$$

By (5.11) and (5.12), it yields
(5.13)

$$
\begin{aligned}
|J_5| \le{} & \xi L(3+2c_{\bar{\xi}}) \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|^2}{s_2^2} + \xi L(3+2c_{\bar{\xi}}) \sum_{i\in\mathcal{I}_2} \frac{\|w_i\|^2}{(L-s_2)^2} \\
& + L\left[3(1+\bar{\xi}) + K_4\left(\frac{s_1'}{s_1}\right)^2 + K_4\left(\frac{s_1'}{L-s_1}\right)^2\right]|\Delta s|^2 \\
& + 3L(1+\bar{\xi}+K_4)|\Delta s'|^2 \\
& + LK_4\left[\frac{1}{(s_2)^2} + \frac{(s_1')^2}{(s_1)^2(s_2)^2} + \frac{(s_2')^2}{4} + \frac{(s_1')^4}{(s_1)^4(s_2)^2} + \frac{1}{(L-s_2)^2}\right. \\
& \qquad \left. + \frac{(s_1')^2}{(L-s_1)^2(L-s_2)^2} + \frac{(s_2')^2}{4} + \frac{(s_1')^4}{(L-s_1)^4(L-s_2)^2}\right]|w|^2.
\end{aligned}
$$

Simple algebraic manipulations show that we can bound the sum $\sum_{\ell=1}^5 |J_\ell|$ by

$$
\begin{aligned}
& \xi(3+3L+2c_{\bar{\xi}}) \sum_{i\in\mathcal{I}_1} \frac{\|w_i\|^2}{s_2^2} + \xi(3+3L+2c_{\bar{\xi}}) \sum_{i\in\mathcal{I}_2} \frac{\|w_i\|^2}{(L-s_2)^2} + K_4|\Delta\Xi|^2 \\
& + K_4|\Delta\Lambda|^2 + K_4\left[\left(\frac{s_2}{s_1}\right)^2 + \left(\frac{L-s_2}{L-s_1}\right)^2\right]\|u_1^{(1)}\|^2|\Delta D|^2 \\
& + |\Delta s^2|\left[3+3L+3L\bar{\xi} + \frac{|w_{1,t}|^2}{2} + K_4\left(\frac{1}{s_1^2} + \frac{1}{(L-s_1)^2}\right)\|u_1^{(1)}\|^2\right. \qquad (5.14) \\
& \qquad \left. + LK_4\left(\frac{s_1'}{s_1}\right)^2 + LK_4\left(\frac{s_1'}{L-s_1}\right)^2\right] + |\Delta s'|^2 3L(1+\bar{\xi}+K_4) \\
& + |w|^2\left[\frac{1}{2} + K_4\left(\chi_2(t) + s_2^{\frac{2\theta}{1-\theta}} + (L-s_2)^{\frac{2\theta}{1-\theta}}\right)\right],
\end{aligned}
$$

where the expression of $\chi_2(t)$ is given by

$$
\begin{aligned}
\chi_2(t) := L & \left[\frac{1}{(s_2)^2} + \frac{(s_1')^2}{(s_1)^2(s_2)^2} + \frac{(s_2')^2}{4} + \frac{(s_1')^4}{(s_1)^4(s_2)^2} + \frac{1}{(L-s_2)^2}\right. \\
& \left. + \frac{(s_1')^2}{(L-s_1)^2(L-s_2)^2} + \frac{(s_2')^2}{4} + \frac{(s_1')^4}{(L-s_1)^4(L-s_2)^2}\right].
\end{aligned}
\tag{5.15}
$$

We select $\bar{\xi} > 0$ and $\xi > 0$ such that the first two sums in (5.14) can be neglected when they are compared with the diffusive part from the left-hand side of (5.3). On this way, we obtain $\frac{D_0}{L_*^2} - \psi(\xi,\bar{\xi}) > 0$, where $\psi(\xi,\bar{\xi}) := \xi(3+3L+2c_{\bar{\xi}})$, and also

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 + \left(\frac{D_0}{L_*^2} - \psi(\xi,\bar{\xi})\right)\|w(t)\|^2 \le a(t) + b(t)|w(t)|^2, \tag{5.16}
$$

where the expressions of $a(t)$ and $b(t)$ ($t \in S_\delta$) are given by

$$a(t) := K_4|\Delta\Lambda|^2 + K_4|\Delta\xi|^2 + a_{11}(t)|\Delta s|^2 + a_{12}(t)|\Delta s'|^2 + a_{13}(t)|\Delta D_i|^2$$

$$b(t) := \frac{1}{2} + K_4\left(\chi_2(t) + s_2^{\frac{2\theta}{1-\theta}} + (L - s_2)^{\frac{2\theta}{1-\theta}}\right).$$

We do not need here to list the exact expressions of $a_{1k}(t)$ ($k \in \{1,2,3\}$). They can be easily obtained when comparing the right-hand side of (5.16) to the estimate on $\sum_{\ell=1}^{5}|J_\ell|$. Here, we only need to know that $\int_{S_\delta} a_{1k}(\tau)\mathrm{d}\tau < \infty$ ($k \in \{1,2,3\}$). The latter inequality follows via the energy estimates. Additionally, we note that for any $t_0 \in S_\delta$ we have

$$|a_{11}(t)|\Delta s(t)|^2 + a_{12}(t)|\Delta s'(t)|^2 \leq a_{11}(t)(t-t_0)\int_{t_0}^{t}|\Delta\eta(\tau)|^2\mathrm{d}\tau + a_{12}(t)|\Delta\eta(t)|^2.$$

Now, denoting by $\tilde{a}(t)$ the sum

$$\tilde{a}(t) := K_4|\Delta\Lambda|^2 + K_4|\Delta\Xi|^2 + a_{13}(t)|\Delta D|^2,$$

we re-write (5.16) in the form

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 + \left(\frac{D_0}{L_*^2} - \psi(\xi,\bar{\xi})\right)\|w(t)\|^2 &\leq \tilde{a}(t) + a_{11}(t)\delta\int_0^t|\Delta\eta(\tau)|^2\mathrm{d}\tau \\ &\quad + a_{12}(t)|\Delta\eta(t)|^2 + b(t)|w(t)|^2.\end{aligned} \tag{5.17}$$

Let the functions $\alpha, \beta : S_\delta \to \mathbb{R}_+$ be defined by

$$\alpha(t) := 2\int_0^t a(\tau)\mathrm{d}\tau \qquad \text{and} \qquad \beta(t) := 2b(t).$$

Here

$$a(t) = \tilde{a}(t) + a_{11}(t)\delta\int_0^t|\Delta\eta(\tau)|^2\mathrm{d}\tau + a_{12}(t)|\Delta\eta(t)|^2.$$

Note that α is strictly increasing on S_δ. By (5.16) or (5.17), and Gronwall's inequality, we infer that

$$|w(t)|^2 \leq \left(|w(0)|^2 + \alpha(t)\right)\exp\left(\int_0^t \beta(\tau)\mathrm{d}\tau\right) \quad \text{a.e. } t \in S_\delta. \tag{5.18}$$

Owing to (5.16) and (5.18), and reasoning in the standard way (see, e.g. the proof of Claim 3.3.27 in [**10**]), we derive the desired upper bound on $\int_{S_\delta}\|w(\tau)\|^2\mathrm{d}\tau$. The conclusion of the Theorem follows in a straightforward manner.

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