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CERTAIN CLASSES OF p-VALENT FUNCTIONS ASSOCIATED WITH WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

G. MURUGUSUNDARAMOORTHY

Abstract. The Wright's generalized hypergeometric function is used here to introduce a new class of p-valent functions $\mathcal{WT}_p(\lambda, \alpha, \beta)$ defined in the open unit disc and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity of functions belonging to the class $\mathcal{WT}_p(\lambda, \alpha, \beta)$.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n, \qquad p < k; \quad p, k \in \mathbb{N} = \{1, 2, 3, \dots\} \tag{1.1}$$

which are analytic in the open disc $U = \{z : z \in \mathcal{C}; \quad |z| < 1\}$. For functions $f \in \mathcal{A}(p)$ given by (1.1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n, \qquad p \in \mathbb{N} = \{1, 2, 3, \dots\}$$

we define the Hadamard product (or convolution) of f and g by

$$f(z) * g(z) = (f * g)(z) = z^p + \sum_{n=k}^{\infty} a_n b_n z^n, \qquad z \in U. \tag{1.2}$$

For positive real parameters $\alpha_1, A_1 \dots, \alpha_l, A_l$ and $\beta_1, B_1 \dots, \beta_m, B_m$ $(l, m \in \mathbb{N} = 1, 2, 3, \dots)$ such that

$$1 + \sum_{n=k}^{m} B_n - \sum_{n=k}^{l} A_n \geq 0, \qquad z \in U,$$

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the Wright's generalized hypergeometric function [**11**]

$$
\begin{aligned}
&{}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z] \\
&\qquad\qquad\qquad = {}_l\Psi_m[(\alpha_j, A_j)_{1,l}(\beta_j, B_j)_{1,m}; z]
\end{aligned}
$$

is defined by

$$
\begin{aligned}
&{}_l\Psi_m[(\alpha_j, A_j)_{1,l}(\beta_t, B_t)_{1,m}; z] \\
&\qquad = \sum_{n=k}^{\infty} \left(\prod_{j=0}^{l} \Gamma(\alpha_j + nA_j \right) \left(\prod_{j=0}^{m} \Gamma(\beta_j + nB_j \right)^{-1} \frac{z^n}{n!}, \qquad z \in U.
\end{aligned}
$$

If $A_j = 1 (j = 1, 2, \ldots, l)$ and $B_j = 1 (j = 1, 2, \ldots, m)$, we have the relationship:

$$
\begin{aligned}
\Omega_l \Psi_m[(\alpha_j, 1)_{1,l}(\beta_j, 1)_{1,m}; z] &\equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \\
&= \sum_{n=k}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \qquad (1.3)
\end{aligned}
$$

$(l \leq m+1; \ \ l, m \in N_0 = N \cup \{0\}; \ z \in U)$ is the generalized hypergeometric function (see for details [**2**]) where $(\alpha)_n$ is the Pochhammer symbol and

$$
\Omega = \left(\prod_{j=0}^{l} \Gamma(\alpha_j) \right)^{-1} \left(\prod_{j=0}^{m} \Gamma(\beta_j) \right). \qquad (1.4)
$$

By using the generalized hypergeometric function Dziok and Srivastava [**2**] introduced the linear operator recently. In [**3**] Dziok and Raina extended the linear operator by using Wright's generalized hypergeometric function. First we define a function

$$
{}_l\phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z] = \Omega z^p {}_l\Psi_m[(\alpha_j, A_j)_{1,l}(\beta_j, B_j)_{1,m}; z].
$$

Let $\mathbf{\Theta}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}] : \mathcal{A}(p) \to \mathcal{A}(p)$ be a linear operator defined by

$$
\mathbf{\Theta}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}] f(z) := z^p \ {}_l\phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z] * f(z)
$$

We observe that, for $f(z)$ of the form (1.1), we have

$$
\mathbf{\Theta}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}] f(z) = z^p + \sum_{n=k}^{\infty} \sigma_n \ a_n z^n \qquad (1.5)
$$

where σ_n is defined by

$$
\sigma_n = \frac{\Omega\Gamma(\alpha_1 + A_1(n-p)) \ldots \Gamma(\alpha_l + A_l(n-p))}{(n-p)!\Gamma(\beta_1 + B_1(n-p)) \ldots \Gamma(\beta_m + B_m(n-p))}. \qquad (1.6)
$$

For convenience, we write

$$\Theta[\alpha_1]f(z) = \Theta[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z) \tag{1.7}$$

Indeed, by setting $A_j = 1(j = 1, \ldots, l)$, $B_j = 1(j = 1, \ldots, m)$ and $p = 1$ the linear operator $\Theta[\alpha_1]$, leads immediately to the Dziok-Srivastava operator [**2**] which contains, as its further special cases, such other linear operators of Geometric Function Theory as the Hohlov operator, the Carlson-Shaffer operator [**1**], the Ruscheweyh derivative operator [**6**], the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator [**8**]. See also [**2**] and [**3**] in which comprehensive details of various other operators are given.

Motivated by the earlier works of [**2, 4, 5, 7, 9, 10**] we introduce a new subclass of p-valent functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we let $\mathcal{W}_p(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}(p)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{\mathcal{J}_\lambda(z) - 1}{\mathcal{J}_\lambda(z) + (1 - 2\alpha)} \right| < \beta \qquad (z \in U) \tag{1.8}$$

where

$$\mathcal{J}_\lambda(z) = (1 - \lambda)\frac{\Theta[\alpha_1]f(z)}{z^p} + \lambda \frac{(\Theta[\alpha_1]f(z))'}{pz^{p-1}}, \tag{1.9}$$

$\Theta[\alpha_1]f(z)$ is given by (1.7). Further let $\mathcal{WT}_p(\lambda, \alpha, \beta) = \mathcal{W}_p(\lambda, \alpha, \beta) \cap T(p)$, where

$$T(p) := \left\{ f \in \mathcal{A}(p) : f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n, \quad a_n \geq 0; \quad z \in U \right\}. \tag{1.10}$$

The purpose of the present paper is to investigate the coefficient estimates, extreme points, distortion theorems and the radii of convexity and starlikeness of the class $\mathcal{WT}_p(\lambda, \alpha, \beta)$.

2. Coefficient Bounds

In this section we obtain coefficient estimates and extreme points of the class $\mathcal{WT}_p(\lambda, \alpha, \beta)$.

Theorem 2.1. *Let the function f be defined by* (1.10). *Then $f \in \mathcal{WT}_p(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=k}^{\infty} (p + n\lambda - p\lambda)(1 + \beta)\sigma_n a_n \leq 2p\beta(1 - \alpha). \tag{2.1}$$

Proof. Suppose f satisfies (2.1). Then for $z \in U$ we have

$$
\begin{aligned}
&|\mathcal{J}_\lambda(z) - 1| - \beta\,|\mathcal{J}_\lambda(z) + (1-2\alpha)| \\
&\qquad = \left| -\sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}(1+\beta)\sigma_n a_n z^{n-p} \right| \\
&\qquad\quad - \beta \left| 2(1-\alpha) - \sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}\sigma_n a_n z^{n-p} \right| \\
&\qquad \le \sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}\sigma_n a_n - 2\beta(1-\alpha) + \sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}\beta\sigma_n a_n \\
&\qquad = \sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}[1+\beta]\sigma_n a_n - 2\beta(1-\alpha) \le 0.
\end{aligned}
$$

Hence, by maximum modulus theorem and (1.8), $f \in \mathcal{WT}_p(\lambda,\alpha,\beta)$. To prove the converse assume that

$$
\left| \frac{\mathcal{J}_\lambda(z) - 1}{\mathcal{J}_\lambda(z) + (1-2\alpha)} \right| = \left| \frac{-\sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}\sigma_n a_n z^{n-p}}{2(1-\alpha) - \sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}\sigma_n a_n z^{n-p}} \right| \le \beta, \qquad z \in U.
$$

Thus

$$
\operatorname{Re}\left\{ \frac{\sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p} a_n \sigma_n z^{n-p}}{2(1-\alpha) - \sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)}{p}\sigma_n a_n z^{n-p}} \right\} < \beta, \tag{2.2}
$$

since $\operatorname{Re}(z) \le |z|$ for all z. Choose values of z on the real axis such that $\mathcal{J}_\lambda(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain the desired inequality (2.1). □

Corollary 2.1. *If $f(z)$ of the form* (1.10) *is in $\mathcal{WT}_p(\lambda,\alpha,\beta)$, then*

$$
a_n \le \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n}, \qquad n = k, k+1, \dots, \tag{2.3}
$$

with the equality only for the function

$$
f(z) = z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} z^n, \qquad n = k, k+1, \dots, . \tag{2.4}
$$

Theorem 2.2 (Extreme Points)**.** *Let*

$$
\begin{aligned}
f_p(z) &= z^p \qquad \text{and} \\
f_n(z) &= z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} z^n, \qquad n = k, k+1, \dots .
\end{aligned} \tag{2.5}
$$

Then $f(z)$ is in the class $\mathcal{WT}_p(\lambda,\alpha,\beta)$ if and only if it can be expressed in the form

$$
f(z) = \mu_p z^p + \sum_{n=k}^{\infty} \mu_n f_n(z), \tag{2.6}
$$

where $\mu_n \ge 0$ and $\mu_p + \sum_{n=k}^{\infty} \mu_n = 1$.

Proof. Suppose $f(z)$ can be written as in (2.6). Then

$$\begin{aligned} f(z) &= \mu_p z^p - \sum_{n=k}^{\infty} \mu_n \left[z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} z^n \right] \\ &= z^p - \sum_{n=k}^{\infty} \mu_n \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} z^n. \end{aligned}$$

Now,

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n}{2p\beta(1-\alpha)} \mu_n \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} \\ &= \sum_{n=k}^{\infty} \mu_n = 1 - \mu_p \leq 1. \end{aligned}$$

Thus $f \in \mathcal{WT}_p(\lambda,\alpha,\beta)$. Conversely, let us have $f \in \mathcal{WT}_p(\lambda,\alpha,\beta)$. Then by using (2.3), we set

$$\mu_n = \frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n a_n}{2p\beta(1-\alpha)}, \qquad n \geq k$$

and $\mu_p = 1 - \sum_{n=k}^{\infty} \mu_n$. Then we have (2.6) and hence this completes the proof of Theorem 2.2. $\square$

3. Distortion Bounds

In this section we obtain distortion bounds for the class $\mathcal{WT}_p(\lambda,\alpha,\beta)$.

Theorem 3.1. *Let f be in the class $\mathcal{WT}_p(\lambda,\alpha,\beta)$, $|z| = r < 1$ and $c_n = (p+n\lambda-p\lambda)\sigma_n$. If the sequence $\{c_k\}$ is nondecreassing for $n > k$, then*

$$\begin{aligned} r^p - \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^k &\leq |f(z)| \\ &\leq r^p + \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^k \end{aligned} \tag{3.1}$$

$$\begin{aligned} pr^{p-1} - \frac{2pk\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^{k-1} &\leq |f'(z)| \\ &\leq pr^{p-1} + \frac{2pk\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^{k-1}. \end{aligned} \tag{3.2}$$

The bounds in (3.1) *and* (3.2) *are sharp since the equalities are attained by the function*

$$f(z) = z^p - \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} z^k. \tag{3.3}$$

Proof. In the view of Theorem 2.1, we have

$$\sum_{n=k}^{\infty} a_n \leq \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} \tag{3.4}$$

Using (1.10) and (3.4), we obtain

$$\begin{aligned} |z|^p - |z|^k \sum_{n=k}^{\infty} a_n \le |f(z)| \le \ & |z|^p + |z|^k \sum_{n=k}^{\infty} a_n \\ r^p - r^k \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} \le |f(z)| \le \ & r^p + r^k \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k}. \end{aligned} \tag{3.5}$$

Hence (3.1) follows from (3.5). Also,

$$|f'(z)| \le pr^{p-1} + r^{k-1} \sum_{n=k}^{\infty} na_n \le pr^{p-1} + r^{k-1} \frac{2pk\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k}.$$

Similarly, we can prove the left hand inequality given in (3.2) which completes the proof of the theorem. □

4. Radius of Starlikeness and Convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{WT}_p(\lambda, \alpha, \beta)$ are given in this section.

Theorem 4.1. *Let the function $f(z)$ defined by (1.10) belong to the class $\mathcal{WT}_p(\lambda, \alpha, \beta)$. Then $f(z)$ is p-valently close-to-convex of order δ $(0 \le \delta < p)$ in the disc $|z| < r_1$, where*

$$r_1 := \inf_{n \ge k} \left[\frac{(p-\delta)(p+n\lambda-p\lambda)[1+\beta]\ \sigma_n}{2pn\beta(1-\alpha)} \right]^{\frac{1}{n-p}}. \tag{4.1}$$

Proof. The function $f \in T(p)$ is close-to-convex of order δ, if

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \delta. \tag{4.2}$$

For the left-hand side of (4.2) we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{n=k}^{\infty} na_n |z|^{n-p}.$$

The last expression is less than $p - \delta$ if

$$\sum_{n=k}^{\infty} \frac{n}{p-\delta} a_n |z|^{n-p} < 1.$$

Using the fact that $f \in \mathcal{WT}_p(\lambda, \alpha, \beta)$ if and only if

$$\sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n a_n}{2p\beta(1-\alpha)} \le 1,$$

we can say (4.2) is true if

$$\frac{n}{p-\delta} |z|^{n-p} \le \frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n}{2p\beta(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n-p} = \left[\frac{(p-\delta)(p+n\lambda-p\lambda)[1+\beta]\ \sigma_n}{2pn\beta(1-\alpha)}\right]$$

which completes the proof. □

Theorem 4.2. *Let $f \in \mathcal{WT}_p(\lambda,\alpha,\beta)$. Then*

(1) *f is p-valently starlike of order δ $(0 \le \delta < p)$ in the disc $|z| < r_2$; that is,* $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, *$(|z| < r_2)$ where*

$$r_2 = \inf_{n\ge k}\left\{\frac{(p-\delta)(p+n\lambda-p\lambda)[1+\beta]\ \sigma_n}{2p\beta(1-\alpha)(k+p-\delta)}\right\}^{\frac{1}{n}}.$$

(2) *f is p-valently convex of order δ $(0 \le \delta < p)$ in the disc $|z| < r_3$, that is* $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, *$(|z| < r_3)$ where*

$$r_3 = \inf_{n\ge p+1}\left\{\frac{(p-\delta)(p+n\lambda-p\lambda)[1+\beta]\sigma_n}{2n\beta(1-\alpha)(n-\delta)}\right\}^{\frac{1}{n}}.$$

Proof. (1) The function $f \in T(p)$ is p-valently starlike of order δ, if

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \delta. \tag{4.3}$$

For the left hand side of (4.3) we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{n=k}^{\infty}(n-p)a_n\ |z|^n}{1-\sum_{n=k}^{\infty} a_n\ |z|^n}.$$

The last expression is less than $p-\delta$ if

$$\sum_{n=k}^{\infty}\frac{n-\delta}{p-\delta}a_n|z|^n < 1.$$

Using the fact that $f \in \mathcal{WT}_p(\lambda,\alpha,\beta)$ if and only if

$$\sum_{n=k}^{\infty}\frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n a_n}{2p\beta(1-\alpha)} < 1,$$

we can say (4.3) is true if

$$\frac{n-\delta}{p-\delta}|z|^n < \frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n}{2p\beta(1-\alpha)}.$$

Or, equivalently,

$$|z|^n < \frac{(p-\delta)(p+n\lambda-p\lambda)[1+\beta]\sigma_n}{2p\beta(1-\alpha)(n-\delta)}$$

which yields the starlikeness of the family.

(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar to the proof of (1). □

Remark. In view of the relationship (1.3) the linear operator (1.5) and by setting $A_j = 1$ $(j = 1, \ldots, l)$ and $B_j = 1 (j = 1, \ldots, m)$ and specific choices of parameters l, m, α_1, β_1 the various results presented in this paper would provide interesting extensions and generalizations of p-valent function classes. The details involved in the derivations of such specializations of the results presented here are fairly straightforward.

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G. Murugusundaramoorthy, School of Science and Humanities, VIT University, Vellore-632014, India, *e-mail*: `gms@vit.ac.in`