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# EXISTENCE THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. This paper concernes with the study of existence theorems for a general class of functional differential equations of the form

$$u'(t) = f(t, u \circ \gamma(t, \cdot)).$$

The obtained results generalize the retarded functional differential equations [**5, 6, 8**] and cover singular functional differential equations [**1, 2, 4, 7, 9, 12**].

## 1. Introduction

Let $(E, |\cdot|_E)$ be a Banach space. For a fixed $r > 0$, we define $\mathcal{C} = C([-r, 0]; E)$ to be the Banach space of continuous $E$-valued functions on $J := [-r, 0]$ with the usual supremum norm $\|\varphi\| = \sup_{\theta\in[-r,0]} |\varphi(\theta)|_E$.

For a continuous function $u : \mathbb{R} \to E$ and any $t \in \mathbb{R}$, we denote by $u_t$ the element of $\mathcal{C}$, defined by

$$u_t(\theta) = u(t + \theta), \qquad\qquad \theta \in J.$$

For each $(\sigma, a) \in \mathbb{R} \times \mathbb{R}^*_+$, we consider

$$\begin{aligned}\Gamma_{\sigma,a} = \{\, \gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a] \text{ continuous functions such that}\\ \text{for all } \theta \in [-r, 0],\ s \in [0, a], \quad \gamma(\sigma, \theta) = \sigma + \theta \text{ and } \gamma(\sigma + s, 0) = \sigma + s \,\}.\end{aligned}$$

It is clear that if $u \in C([\sigma - r, \sigma + a]; E)$ and $\gamma \in \Gamma_{\sigma,a}$, then $u \circ \gamma(t, \cdot) \in \mathcal{C}$ and $t \mapsto u \circ \gamma(t, \cdot)$ is a continuous function for $t \in [\sigma, \sigma + a]$, where $u \circ \gamma(t, \cdot)(\theta) := u(\gamma(t, \theta))$ for all $\theta \in J$, in particular, if $\gamma(t, \theta) = t + \theta$, then $u \circ \gamma(t, \cdot) = u_t \in \mathcal{C}$ and $t \mapsto u_t$ is continuous for $t \in [\sigma, \sigma + a]$.

Now we introduce a general class of functional differential equations

$$u'(t) = f(t, u \circ \gamma(t, \cdot)) \tag{1.1}$$

where $f$ is a continuous function from $[\sigma, \sigma + a] \times \mathcal{C}$ into $E$.

If $\gamma(t, \theta) = t+\theta$, then the equation $R(f, \gamma)$ coincides with the classical retarded functional differential equation $u'(t) = f(t, u_t)$ (see, for example [**5, 6, 8**]).

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If $\gamma(t,\theta) = \rho(\rho^{-1}(t)+\theta)$ where $\rho : [\sigma - r, \sigma + b] \to [\sigma - r, \sigma + a]$, $(b > 0)$ is defined by

$$\rho(\tau) = \begin{cases} \sigma + \displaystyle\int_\sigma^\tau \frac{\mathrm{d}s}{\psi(s)} & \text{if} \quad \tau \in [\sigma, \sigma + b] \\ \tau & \text{if} \quad \tau \in [\sigma - r, \sigma], \end{cases}$$

$\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$, then the equation $R(f,\gamma)$ coincides with the following initial value problem for the singular functional differential equation (see [**7**]):

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in (\sigma, \sigma + b] \\ x_\sigma = \varphi, \end{cases}$$

where $g : [\sigma, \sigma + b] \times \mathcal{C} \to E$ is completely continuous and $f(t,\phi) := g(\rho^{-1}(t), \phi)$.

Also, in the Section 5, we shall study the general form

$$\begin{cases} \psi(\tau)x^{(n)}(\tau) = g(\tau, x_\tau, x'_\tau, \ldots, x_\tau^{(n-1)}), & \tau \in (\sigma, \sigma + b], \quad (b > 0) \\ x_\sigma = \varphi, \end{cases}$$

or the second order delay equation of the form

$$\begin{cases} \psi(\tau)x''(\tau) = g(\tau, x(\tau), x(\tau - r_1), x'(\tau), x'(\tau - r_2)), & \tau \in (\sigma, \sigma + b], \quad (b > 0) \\ x_\sigma = \varphi, \quad x'_\sigma = \varphi' \quad \text{on } [-r, 0], \end{cases}$$

where $r = \max(r_1, r_2)$, (see [**12**]).

## 2. Preliminaries

Let $D$ be a subset of $\mathbb{R} \times \mathcal{C}$ and let $f$ be a continuous function from $D$ into $E$. In the sequel, we give $(\sigma, a) \in \mathbb{R} \times \mathbb{R}_+^*$ and $\gamma \in \Gamma_{\sigma,a}$. We say that the relation

$$u'(t) = f(t, u \circ \gamma(t, \cdot)), \qquad ((t, u \circ \gamma(t, \cdot)) \in D),$$

is a functional differential equation on $D$ and will denote this equation by $R(f,\gamma)$.

**Definition 2.1.** A function $u$ is said to be a solution of the equation $R(f,\gamma)$, if there exists a real $A$ such that $0 < A \leq a$ and $u \in C([\sigma - r, \sigma + A); E)$, $(t, u \circ \gamma(t, \cdot)) \in D$ and $u$ satisfies the equation $R(f,\gamma)$ for $t \in [\sigma, \sigma + A)$. Then, we say that $u$ is a solution of $R(f,\gamma)$ on $[\sigma, \sigma + A)$

For $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, we say $u := u(\sigma, \varphi)$ is a solution of equation $R(f,\gamma)$ through $(\sigma, \varphi)$, if there is $A$ such that $0 < A \leq a$ and $u(\sigma, \varphi)$ is a solution of $R(f,\gamma)$ on $[\sigma - r, \sigma + A)$ and $u_\sigma(\sigma, \varphi) = \varphi$.

Let $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, we consider the function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t - \sigma) & \text{if} \quad t \in [\sigma - r, \sigma] \\ \varphi(0) & \text{if} \quad t \geq \sigma. \end{cases}$$

We have $\widetilde{\varphi} \in C([\sigma - r, +\infty); E)$, $\widetilde{\varphi}_\sigma = \varphi$ and $\widetilde{\varphi}(t + \sigma) = \varphi(0)$ for $t \geq 0$.

It is easy to see that the following result is immediate.

**Lemma 2.1.** *Suppose that $f \in C(D;E)$, $\varphi \in \mathcal{C}$ and $0 < A \leq a$. Then, there are equivalent statements:*

i) *$u$ is solution of $R(f,\gamma)$ on $[\sigma - r, \sigma + A)$ through $(\sigma, \varphi)$,*

ii) *$u \in C([\sigma - r, \sigma + A); E)$, $(t, u \circ \gamma(t, \cdot)) \in D$ for all $t \in [\sigma, \sigma + A)$ and*

$$\begin{cases} u_\sigma = \varphi \\ u(t) = \varphi(0) + \displaystyle\int_\sigma^t f(s, u \circ \gamma(s, \cdot))\mathrm{d}s, \qquad t \in [\sigma, \sigma + A); \end{cases}$$

iii) *there exists $y \in C([-r, A); E)$ such that $(\sigma + t, y_t + \widetilde{\varphi} \circ \gamma(\sigma + t, \cdot)) \in D$ for all $t \in [0, A)$ and*

$$\begin{cases} y_0 = 0 \\ y(t) = \displaystyle\int_0^t f(\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))\mathrm{d}s, \qquad t \in [0, A). \end{cases}$$

For any real $\alpha, \beta > 0$, define

$$I_\alpha = [0, \alpha], \qquad \widehat{I_\alpha} = (0, \alpha], \qquad B_\beta = \{\psi \in \mathcal{C} : \|\psi\| \leq \beta\},$$
$$A(\alpha, \beta) = \{y \in C([-r, \alpha]; E) : y_0 = 0 \text{ and } y_t \in B_\beta, \ \ t \in I_\alpha\}$$

and

$$C^0(D, E) = \{f \in C(D, E) : f \text{ is bounded on } D\}.$$

We have $A(\alpha, \beta)$ is a closed bounded convex subset of $C([-r, \alpha]; E)$ and $C^0(D, E)$ is a Banach space with the norm $\|f\|_0 = \sup_{(t,\varphi) \in D} |f(t, \varphi)|_E$.

**Lemma 2.2.** *Suppose that $\Omega \subset \mathbb{R} \times \mathcal{C}$ is open, $W \subset \Omega$ is compact and $f^0 \in C(\Omega; E)$. Then, there exists a neighborhood $V \subset \Omega$ of $W$ such that $f^0 \in C^0(V; E)$, there exists a neighborhood $U \subset C^0(V; E)$ of $f^0$ and three positive constants $M$, $\alpha \leq a$ and $\beta$ such that*

$$|f(\sigma, \varphi)|_E < M \qquad \text{for all } (\sigma, \varphi) \in V \text{ and } f \in U,$$

*$(\sigma^0 + t, y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) \in V$ for any $(\sigma^0, \varphi^0) \in W$, $t \in I_\alpha$, $y \in A(\alpha, \beta)$ and $\gamma \in \Gamma_{\sigma^0, a}$.*

*Proof.* Since $f^0(W)$ is a compact subset of the Banach space $E$, it is bounded, and therefore exists $M > 0$ such that

$$\left|f^0(\sigma^0, \varphi^0)\right|_E < \frac{M}{3}$$

for all $(\sigma^0, \varphi^0) \in W$. However $f^0$ is continuous at $(\sigma^0, \varphi^0)$, and therefore for $0 < \varepsilon < \frac{M}{3}$, there exists $(\alpha', \beta') \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ (with $\alpha' \leq a$) such that for $(t, \psi) \in \big(\sigma^0 - \alpha', \sigma^0 + \alpha'\big) \times B(\varphi^0, \beta') \subset \Omega$ we have

$$\left|f^0(t, \psi)\right|_E \leq \left|f^0(t, \psi) - f^0(\sigma^0, \varphi^0)\right|_E + \left|f^0(\sigma^0, \varphi^0)\right|_E < \frac{2M}{3}.$$

Consider $\beta$ such that $0 < \beta < \beta'$ and $\gamma \in \Gamma_{\sigma',a}$. Since the function $s \in [\sigma^0, \sigma^0 + a] \mapsto \widetilde{\varphi^0} \circ \gamma(s, \cdot)$ is continuous at $\sigma^0$, then there exists $\alpha$ such that $0 < \alpha < \alpha'$ and

$$\left\| \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot) - \widetilde{\varphi^0} \circ \gamma(\sigma^0, \cdot) \right\| = \left\| \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0 \right\| < \beta' - \beta, \qquad t \in I_\alpha.$$

Define $V = \bigcup_{(\sigma^0,\varphi^0)\in W} \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right) \times B(\varphi^0, \beta')$. Then $W \subset V \subset \Omega$, $V$ is a neighborhood of $W$ and $f^0 \in C^0(V; E)$. Moreover $(\sigma^0 + t, y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) \in V$ for all $t \in I_\alpha$, $y \in A(\alpha, \beta)$, $\gamma \in \Gamma_{\sigma^0,a}$. Indeed $\sigma^0 + t \in \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right)$ and $y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot) \in B(\varphi^0, \beta')$ because

$$\left\| y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0 \right\| \le \|y_t\| + \left\| \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0 \right\| \le \beta'.$$

Define $U = \left\{ f \in C^0(V; E) : \left\| f - f^0 \right\|_0 < \frac{M}{3} \right\}$. Then $U$ is a neighborhood of $f^0$, $U \subset C^0(V; E)$ and for all $(\sigma, \varphi) \in V$, $f \in U$

$$|f(\sigma, \varphi)|_E \le \left| f(\sigma, \varphi) - f^0(\sigma, \varphi) \right|_E + \left| f^0(\sigma, \varphi) \right|_E < M.$$

$\square$

The next lemma will be used to apply fixed point theorems for existence of solutions of the equation $R(f, \gamma)$.

**Lemma 2.3.** *Suppose that $\Omega \subset \mathbb{R} \times \mathcal{C}$ is open, $W = \{(\sigma, \varphi)\} \subset \Omega$ and $f^0 \in C(\Omega; E)$ are given, the neighborhoods $V$, $U$ and the constants $M$, $\alpha$ and $\beta$ are the ones obtained from Lemma 2.2. Define an operator $T : U \times A(\alpha, \beta) \to C([-r, \alpha]; E)$ by*

$$T(f, y)(t) = \begin{cases} 0 & \text{if } t \in [-r, 0] \\ \displaystyle\int_0^t f((\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))\mathrm{d}s & \text{if } t \in I_\alpha. \end{cases}$$

*If $M\alpha \le \beta$, then $T : U \times A(\alpha, \beta) \to A(\alpha, \varphi)$ and $T$ is continuous on $U \times A(\alpha, \beta)$.*

*Proof.* It is clear that $T$ maps $U \times A(\alpha, \beta)$ into $C([-r, \alpha]; E)$ and by Lemma 2.2, for all $t, t' \in I_\alpha$

$$|T(f, y)(t) - T(f, y)(t')|_E \le M\,|t - t'| \qquad \text{and} \qquad |T(f, y)(t)|_E \le M\alpha.$$

It is easy to see that for all $t, t' \in [-r, \alpha]$

$$|T(f, y)(t) - T(f, y)(t')|_E \le M\,|t - t'| \qquad \text{and} \qquad |T(f, y)(t)|_E \le M\alpha.$$

Hence the family $\Im = \{T(f, y) : (f, y) \in U \times A(\alpha, \beta)\}$ is bounded and uniformly equicontinuous. Also, we have $(T(f, y))_0 = 0$ and $(T(f, y))_t \in B_\beta$ if $M\alpha \le \beta$, thus $T$ maps $U \times A(\alpha, \beta)$ into $A(\alpha, \beta)$.

It remains to show that $T$ is continuous on $U \times A(\alpha, \beta)$.

Let $(f^n, y^n)$ be a sequence in $U \times A(\alpha, \beta)$ that converges to a member $(f, y)$ of $U \times A(\alpha, \beta)$.

It is clear that for each $s \in I_\alpha$

$$\begin{aligned}\|y^n_s - y_s\| &= \sup_{\theta\in[-r,0]} |y^n(s+\theta) - y(s+\theta)|_E \le \|y^n - y\|_1 \\ &:= \sup_{t\in[-r,\alpha]} |y^n(t) - y(t)|_E\,.\end{aligned}$$

We have $(\sigma + s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))$, $(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) \in V$ because $s \in I_\alpha$ and $y^n, y \in A(\alpha,\beta)$. Since $(f^n)$ converges uniformly to $f$ in $V$, then the sequence $(f^n(\sigma+s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)))$ converges to $f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))$ in $E$, but $f^n$ and $f$ are bounded on $V$ (see Lemma 2.2) and by the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t f^n(\sigma+s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s \longrightarrow \int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s$$

in $E$. Hence for all $t \in I_\alpha$, $T(f^n, y^n)(t)$ converges to $T(f,y)(t)$ in $E$ and then $(T(f^n,y^n)(t))$ converges to $T(f,y)(t)$ in $E$ for all $t \in [-r,\alpha]$. This implies that the set $\{T(f^n,y^n)(t) : t \in [-r,\alpha]\}$ is relatively compact in $E$, but the family $\{T(f^n,y^n) : n \in \mathbb{N}\}$ is bounded and uniformly equicontinuous and therefore by the Ascoli theorem [**3, 11**], the family$\{T(f^n,y^n) : n \in \mathbb{N}\}$ is relatively compact in $C([-r,\alpha];E)$. We shall show that $T(f^n,y^n)$ converges to $T(f,y)$ in $C([-r,\alpha];E)$. Suppose, for the sake of contradiction, that there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$

$$\exists n > N :\ \|T(f^n,y^n) - T(f,y)\|_1 \ge \varepsilon.$$

Then for

$$N = n_0, \quad \exists n_1 > n_0 : \qquad \|T(f^{n_1},y^{n_1}) - T(f,y)\|_1 \ge \varepsilon$$

and for $k > 1$ and

$$N = n_{k-1}, \quad \exists n_k > n_{k-1} : \quad \|T(f^{n_k},y^{n_k}) - T(f,y)\|_1 \ge \varepsilon.$$

If necessary, passing, to a subsequence, we can assume that $(T(f^{n_k},y^{n_k}))$ converges to $z \in A(\alpha,\beta)$ such that $\|z - T(f,y)\|_1 \ge \varepsilon$. Since $(T(f^{n_k},y^{n_k}))$ converges to $z$ in $C([-r,\alpha];E\ )$, then $(T(f^{n_k},y^{n_k}))(t)$ converges to $z(t)$ in $E$ for each $t \in [-r,\alpha]$, but this sequence converges to $T(f,y)(t)$ in $E$, which is a contradiction. Therefore $T$ is continuous on $U \times A(\alpha,\beta)$. ☐

## 3. Local existence of solutions

In this section we shall show existence theorem of solutions to $R(f,\gamma)$ by using the results obtained in the section two.

**Definition 3.1.** Suppose that $\Omega$ is an open set in $\mathbb{R}\times\mathcal{C}$. A function $f \in C(\Omega;E)$ is said to have the condition $(l)$ if, for all $(\sigma,\varphi) \in \Omega$, there exists a neighborhood $V' \subset \Omega$ of $(\sigma,\varphi)$ and a positive constant $k$ such that for all bounded $I\times S_1 \subset V'$ with bounded $f(I\times S_1)$, then $\chi(f(I,S_1)) \le k\chi_0(S_1)$ where $\chi$ (resp. $\chi_0$) is the measure of noncompactness [**3, 11**] on $E$ (resp. $\mathcal{C}$).

**Theorem 3.1.** *Suppose that* $\Omega$ *is an open set in* $\mathbb{R}\times\mathcal{C}$ *and* $f \in C(\Omega; E)$. *If* $f$ *is compact or satisfying the condition* $(l)$ *(resp.* $f(t,\cdot)$ *is locally Lipschitz), then for all* $(\sigma,\varphi) \in \Omega$ *and* $\gamma \in \Gamma_{\sigma,a}$ *there exists a positive constant* $\alpha \leq a$ *and a solution (resp. a unique solution) of the equation* $R(f,\gamma)$ *on* $[\sigma - r, \sigma + \alpha]$ *through* $(\alpha, \varphi)$.

*Proof.* By notations of Lemmas 2.2 and 2.3 with $W = \{(\sigma,\varphi)\}$, the operator $T_1 = T(f,\cdot)$ maps $A(\alpha,\beta)$ into $A(\alpha,\beta)$ if $\alpha \leq \frac{\beta}{M}$ and $T_1$ is continuous.

*First case*. If $f$ is compact we shall show that $T_1$ is compact.

Let $B$ be a bounded subset of $A(\alpha,\beta)$ and $(z^n)$ a sequence of $T_1B$, then there exists a sequence $(y^n)$ of $B$ such that $z^n = T_1 y^n$.

The set $\{f(\sigma + s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) : n \in \mathbb{N},\ s \in I_\alpha\}$ is relatively compact because $f$ is completely continuous. By Mazur theorem [**3, 11**] its closed convex hull is compact. But, for all $t \in \widehat{I_\alpha}$, we have (see [**11**, p. 25])

$$\frac{1}{t}\int_0^t f(\sigma+s, y^n_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s \in \overline{\mathrm{Co}}\{f(\sigma+s, y^n_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \colon n \in \mathbb{N},\ s \in I_\alpha\}.$$

Then,

$$\{T_1 y^n(t) : n \in \mathbb{N}, s \in \widehat{I_\alpha}\} \subset \alpha\overline{\mathrm{Co}}\{f(\sigma+s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) : n \in \mathbb{N}, s \in I_\alpha\}$$

which is compact, hence $\{T_1 y^n(t) : n \in \mathbb{N}, s \in \widehat{I_\alpha}\}$ is relatively compact. However $\{T_1 y^n(t) : n \in \mathbb{N}\}$ is bounded and uniformly equicontinuous, then by the Ascoli theorem, $\{T_1 y^n(t) : n \in \mathbb{N}\}$ is relatively compact, thus $(z^n)$ has a subsequence that converges in $C([-r,\alpha]; E)$. By Schauder fixed-point theorem and Lemma 2.3, $R(f,\gamma)$ has a solution on $[\sigma - r, \sigma + \alpha]$ through $(\sigma,\varphi)$.

*Second case*. If $f$ satisfies the condition $(l)$.

Let $V = (\sigma - \alpha', \sigma + \alpha') \times B(\varphi, \beta')$ be the neighborhood obtained in the Lemma 2.2 and by the condition $(l)$, there exist $V' = (\sigma - \alpha'', \sigma + \alpha'') \times B(\varphi, \beta'')$ and $k > 0$ such that if $f(I \times S_1)$ is bounded for all bounded $I \times S_1 \subset V'$, then $\chi(f(I\times S_1)) \leq k\chi_0(S_1)$. Take $\alpha_2 = \min(\alpha',\alpha'')$, $\beta_2 = \min(\beta',\beta'')$ and $V_1 = V \cap V'$. Let $0 < \beta_1 < \beta_2$, then there exists $\alpha_1$ such that $0 < \alpha_1 < \alpha_2$ (see proof of Lemma 2.2) and for all $s \in I_{\alpha_1}$

$$\|\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) - \widetilde{\varphi}\circ\gamma(\sigma,\cdot)\| = \|\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) - \varphi\| < \beta_2 - \beta_1.$$

Then for every $s \in I_{\alpha_1}$ and every $y \in A(\alpha_1,\beta_1)$, $(\sigma + s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \in V_1$. Thus, if $\alpha_1 \leq \frac{\beta_1}{M}$, then $T_1$ maps $A(\alpha_1,\beta_1)$ into itself and $T_1$ is continuous. Moreover $A(\alpha_1,\beta_1)$ is closed, bounded convex subset of $C([-r,\alpha_1]; E)$. In order to apply Darboux fixed point theorem [**3, 11**], we shall show that there exists $\delta \in [0,1)$ such that $\chi^1_0(T_1 S) \leq \delta\chi^1_0(S)$ for all $S \subset A(\alpha_1,\beta_1)$ where $\chi^1_0$ is the measure of noncompactness on $C([-r,\alpha_1]; E)$.

Let $S \subset A(\alpha_1, \beta_1)$, then for each $t \in \widehat{I_{\alpha_1}}$

$$\begin{aligned}\chi(T_1S(t)) &= \chi\left(\left\{\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s:\ y\in S\right\}\right)\\ &= \chi\left(\left\{t\frac{1}{t}\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s:\ y\in S\right\}\right)\\ &\le \alpha_1\chi(\overline{\mathrm{Co}}\{f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):\ y\in S,\ s\in[0,t]\})\\ &\le \alpha_1\chi(\{f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):\ y\in S,\ s\in[0,t]\}).\end{aligned}$$

By definition of $V_1$, we have for each $t \in \widehat{I_{\alpha_1}}$

$$\{(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):\ y\in S,\ s\in[0,t]\}\subset V_1.$$

Take

$$I=\{\sigma+s:\ s\in[0,t]\}\subset[\sigma,\sigma+\alpha_1]\qquad\text{and}$$

$$S_1=\{y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,.):\ y\in S,\ s\in[0,t]\}.$$

Then, $I\times S_1$ and $f(I\times S_1)$ are bounded (because $f$ is bounded on $V$ see Lemma 2.2). Hence, for each $t\in\widehat{I_{\alpha_1}}$, $\chi(T_1S(t))\le k\alpha_1\chi_0(S_1)$ and

$$\begin{aligned}\chi_0(S_1) &= \chi_0(\{y_s:\ y\in S,\ s\in[0,t]\})+\chi_0(\{\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot):\ s\in[0,t]\})\\ &\le \chi_0(\{y_s: y\in S,\ s\in[0,t]\}).\end{aligned}$$

Since $\chi_0(\{\widetilde{\varphi}\circ\gamma(\sigma+s,.): s\in[0,t]\})=0$, the set $\{\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot): s\in[0,t]\}$ is relatively compact.

Thus, for each $t\in\widehat{I_{\alpha_1}}$

$$\chi(T_1S(t))\le k\alpha_1\chi_0(S_s)\le k\alpha_1\chi_0^1(S)\qquad\text{(see [}\mathbf{13}\text{])}$$

where $S_s=\{y_s: s\in[0,t],\ y\in S\}$.

But the family $\left\{t\mapsto\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\,\mathrm{d}s\right\}$ is uniformly bounded and equicontinuous, then by Ambrosetti theorem [**3, 11**], we obtain

$$\chi_0^1(T_1S)=\sup_{t\in[-r,\alpha_1]}\chi(T_1S(t))\le k\alpha_1\chi_0^1(S).$$

Take $\alpha_1<\min\left\{\frac{1}{k},\frac{\beta_1}{M}\right\}$ and $\delta=k\alpha_1\in[0,1)$.

*Third case*. If $f(t,\cdot)$ is locally Lipschitz.

Let $V=(\sigma-\alpha',\sigma+\alpha')\times B(\varphi,\beta')$ be the neighborhood obtained in the Lemma 2.2 and since $f(t,\cdot)$ is locally Lipschitz then there exists $V'=(\sigma-\alpha''$, $\sigma+\alpha'')\times B(\varphi,\beta'')$ such that $f(t,\cdot)$ is Lipschitz on $V'$. $T_1$ maps $A(\alpha_1,\beta_1)$ into $A(\alpha_1,\beta_1)$ if $\alpha_1\le\frac{\beta_1}{M}$ (see the second case of the existence of $\alpha_1,\beta_1$) and $T_1$ is a contraction strict if $\alpha_1 < \min\left\{\frac{\beta_1}{M}, \frac{1}{k}\right\}$. Indeed, for all $y, z \in A(\alpha_1, \beta_1)$

$$\begin{aligned}\|T_1 y - T_1 z\|_1 &= \sup_{t\in[-r,\alpha_1]} |T_1 y(t) - T_1 z(t)|_E \\ &\le \sup_{t\in\widehat{I_{\alpha_1}}} \int_0^t |f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \\ &\qquad - f(\sigma+s, z_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))|_E \mathrm{d}s \\ &\le k \sup_{t\in\widehat{I_{\alpha_1}}} \int_0^t \|y_s - z_s\|\,\mathrm{d}s,\end{aligned}$$

but for all $s \in (0, t] \subset \widehat{I_{\alpha_1}}$

$$\|y_s - z_s\| \le \sup_{\theta\in[-r,0]\cap[-s,0]} |y(s+\theta) - z(s+\theta)|_E \le \|y - z\|_1 .$$

Finally

$$\|T_1 y - T_1 z\|_1 \le k\alpha_1 \|y - z\|_1 \qquad \text{for all } y, z \in A(\alpha_1, \beta_1).$$

Thus, the equation $R(f,\gamma)$ has a unique solution on $[\sigma - r, \sigma + \alpha_1]$ through $(\sigma, \varphi)$. □

**Remark.** If $f = f_1 + f_2$ where $f_1$ is completely continuous and $f_2$ is locally Lipschitz, then the condition $(l)$ is verified.

## 4. Global existence solutions

**Definition 4.1.** Let $u$ (resp. $v$) be a solution of $R(f,\gamma)$ on $J_u = [\sigma - r, \sigma + A)$ (resp. $J_v = [\sigma - r, \sigma + B)$) where $0 < A, B \le a$. The solution $v$ is said to be a continuation of $u$ if $J_v \supset J_u$ and $v = u$ on $J_u$.

The solution $u$ is said to be noncontinuable if it has no proper continuation.

The following result of the existence of noncontinuable solutions follows from Zorn lemma [**14**].

**Proposition 4.1.** *If $u$ is a solution of the equation $R(f,\gamma)$ on $J_u$, then there exists a noncontinuable solution $\widehat{u}$ of $R(f,\gamma)$ on $J_{\widehat{u}}$ such that $\widehat{u}$ is a continuation of $u$.*

Theorem 3.1 gives a criterion of local existence for solutions to $R(f,\gamma)$, then we use the previous proposition to study the continuation of solutions to the equation $R(f,\gamma)$.

**Theorem 4.1.** *Suppose that $\Omega$ is an open subset of $\mathbb{R}\times\mathcal{C}$ and $f \in C(\Omega; E)$. If $f$ is compact or verifies the condition $(l)$ (resp. $f(t,\cdot)$ is locally Lipschitz), then for all $(\sigma,\varphi) \in \Omega$ and $\gamma \in \Gamma_{\sigma,a}$, there exists a noncontinuable solution (resp. a unique solution) of $R(f,\gamma)$ on $[\sigma - r, \sigma + \alpha)$ $(0 < \alpha \le a)$ through $(\sigma,\varphi)$.*

*Proof.* It remains to show unicity of a noncontinuable solution if $f(t,\cdot)$ is locally Lipschitz. Suppose that there exist two noncontinuable solutions $u : [\sigma - r,$

$\sigma+\alpha_u) \to E$ and $v : [\sigma-r, \sigma+\alpha_v) \to E$ of $R(f,\gamma)$ through $(\sigma,\varphi)$, then $u_\sigma = v_\sigma = \varphi$. If $\alpha_u < \alpha_v$, then $v$ is a continuation of $u$, which is a contradiction, so $\alpha_u = \alpha_v$.

By Lemma 2.2, we associated with $u$ (resp. $v$), $y$ (resp. $z$) and we will see $y = z$ on $J = (0,\alpha)$. Suppose that in $J$ there exists $t' > 0$ such that $y(t') \neq z(t')$. Define $t_0 = \inf\{t \in J : y(t) \neq z(t)\}$, by continuity of $y$ and $z$, then $y(t_0) = z(t_0)$ and $y_{t_0} = z_{t_0}$. Set $\varphi^0 = y_{t_0} + \widetilde{\varphi}\circ\gamma(\sigma+t_0,\cdot) = z_{t_0} + \widetilde{\varphi}\circ\gamma(\sigma+t_0,\cdot)$, then $(\sigma+t_0,\varphi^0) \in \Omega$ and there exist a neighborhood $V \subset \Omega$ of $(\sigma,\varphi)$ and a positive constant $k$ such that $f$ is $k$-Lipschitz on $V$. Let $\alpha' > 0$ such that for all $t \in J$, $0 < t - t_0 \leq \alpha'$, then $(\sigma+t, y_t + \widetilde{\varphi}\circ\gamma(\sigma+t,\cdot))$, $(\sigma+t, z_t + \widetilde{\varphi}\circ\gamma(\sigma+t,\cdot)) \in V$. However

$$\begin{aligned} y(t) - z(t) &= y(t) - y(t_0) + z(t_0) - z(t) \\ &= \int_{t_0}^{t} [f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) - f(\sigma+s, z_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))]\mathrm{d}s, \end{aligned}$$

and thus

$$|y(t) - z(t)|_E \leq \int_{t_0}^{t} k\,\|y_s - z_s\|\,\mathrm{d}s.$$

It follows from Gronwall inequality [**5**] that $y(t) = z(t)$ for all $t_0 \leq t \leq t_0 + \alpha'$, which is a contradiction to the definition of $t_0$. $\square$

Now, let $\sigma \in \mathbb{R}$ and $\Gamma_\sigma$ be a set of continuous functions $\gamma : [\sigma,+\infty) \times [-r,0] \to [\sigma-r,+\infty)$ such that $\gamma(\sigma,\theta) = \sigma+\theta$, $\gamma(\sigma+t,0) = \sigma+t$ and $\gamma([\sigma,\sigma+t]\times[-r,0]) = [\sigma-r,\sigma+t]$ for all $\theta \in [-r,0]$, $t \geq 0$.

The following theorem gives a global solutions of the equation $R(f,\gamma)$ on $[\sigma-r, +\infty)$.

**Theorem 4.2.** *Let $f : \mathbb{R}\times\mathcal{C} \to E$ be a continuous function. Suppose that $f$ is compact or verifies the condition $(l)$ (resp. $f(t,\cdot)$ is locally Lipschitz). Suppose further that there exists a continuous function $m : \mathbb{R} \to \mathbb{R}^+$ such that*

$$|f(t,\psi)|_E \leq m(t)h(\|\psi\|), \qquad (t,\psi) \in \mathbb{R}\times\mathcal{C},$$

*where $h$ is continuous nondecreasing on $\mathbb{R}^+$, positive on $\mathbb{R}^+_*$ and $\int_0^{+\infty} \frac{\mathrm{d}s}{h(s)} = +\infty$. Then, for all $(\sigma,\varphi) \in \mathbb{R}\times\mathcal{C}$ and $\gamma \in \Gamma_\sigma$, there exists a function (resp. a unique solution) $u \in C([\sigma-r,+\infty);E)$ which verifies the Cauchy problem:*

$$\begin{cases} u'(t) = f(t, u\circ\gamma(t,\cdot)), & t \geq 0 \\ u_\sigma = \varphi. \end{cases} \tag{4.1}$$

*Proof.* By Theorem 4.1, there exists a noncontinuable solution (resp. a unique solution) $u$ of problem (4.1) on $[\sigma-r,\beta)$ where $\beta > \sigma$. We shall show $\beta = +\infty$. Suppose that $\beta < +\infty$. By Lemma 2.1, we have for $t \in [\sigma,\beta)$

$$\begin{aligned} |u(t)|_E &\leq |\varphi(0)|_E + \int_\sigma^t |f(s, u\circ\gamma(s,\cdot))|_E\,\mathrm{d}s \\ &\leq \|\varphi\| + \int_\sigma^t m(s)h(\|u\circ\gamma(s,\cdot)\|)\mathrm{d}s. \end{aligned} \tag{$*$}$$

Consider the function $v$ given by

$$v(t) = \sup\left\{|u(s)|_E : \sigma - r \leq s \leq t\right\}, \qquad t \in [\sigma, \beta).$$

It is clear that

$$(**) \qquad v(t) \leq \|\varphi\| + \int_\sigma^t m(s)h(\|u \circ \gamma(s,\cdot)\|)\mathrm{d}s \leq \|\varphi\| + \int_\sigma^t m(s)h(v(s))\mathrm{d}s.$$

Denoting by $w(t)$ the right-hand side of the above inequality $(**)$, we obtain $w(\sigma) = \|\varphi\|$ and

$$v(t) \leq w(t), \qquad w'(t) = m(t)h(v(t)) \leq m(t)h(w(t)), \qquad t \in [\sigma, \beta).$$

Integrating over $[\sigma, t]$, we obtain

$$\int_\sigma^t \frac{w'(s)}{h(w(s))}\mathrm{d}s = \int_{w(\sigma)}^{w(t)} \frac{\mathrm{d}s}{h(s)} \leq \int_\sigma^t m(s)\mathrm{d}s < +\infty.$$

This inequality implies that there is a positive constant $c$ such that for all $t \in [\sigma, \beta)$, $w(t) \leq c$, then $v(t) \leq c$. This majoration implies $|u'(t)|_E$ is bounded, hence $u$ is uniformly continuous on $[\sigma - r, \beta)$, then there exists a unique continuous function $\overline{u} : [\sigma - r, \beta] \to E$ defined by

$$\overline{u}(t) = \begin{cases} u(t) & \text{if } t < \beta \\ \lim\limits_{s \to \beta} u(s) & \text{if } t = \beta. \end{cases}$$

Since $\gamma(s, \theta) \in [\sigma - r, s]$, then $\overline{u} \circ \gamma(s, \cdot) = u \circ \gamma(s, \cdot)$ and

$$\begin{aligned} \overline{u}(\beta) = \lim_{s \to \beta} u(s) &= \varphi(0) + \lim_{s \to \beta} \int_\sigma^s f(s', u \circ \gamma(s', \cdot))\mathrm{d}s' \\ &= \varphi(0) + \lim_{s \to \beta} \int_\sigma^s f(s', \overline{u} \circ \gamma(s', \cdot))\mathrm{d}s' \\ &= \varphi(0) + \int_\sigma^\beta f(s', \overline{u} \circ \gamma(s', \cdot))\mathrm{d}s'. \end{aligned}$$

This implies $\overline{u}$ is a solution to (4.1) on $[\sigma - r, \beta]$, which is a contradiction, and thus $\beta = +\infty$. □

**Remark.** By the fixed-point theorem for a strict contraction, we obtain the following result easily.

**Theorem 4.3.** *Let $\Gamma$ be the set of continuous functions $\gamma : \mathbb{R} \times [-r, 0] \to \mathbb{R}$ such that*

$$\gamma([\sigma, \sigma + T] \times [-r, 0]) \subset [\sigma - r, \sigma + T] \qquad \textit{for all } \ (\sigma, T) \in \mathbb{R} \times \mathbb{R}_+^*$$

*and $f : \mathbb{R} \times \mathcal{C} \to E$ be a continuous function such that $f(t, \cdot)$ is Lipschitz. Then, for all $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$ and $\gamma \in \Gamma$, there exists a unique function $u \in C([\sigma - r, +\infty); E) \cap C^1([\sigma, +\infty); E)$ which verifies the Cauchy problem:*

$$(4.2) \qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \geq 0 \\ u_\sigma = \varphi. \end{cases}$$

## 5. Applications

### Retarded functional differential equations

Let $(\sigma, a) \in \mathbb{R} \times \mathbb{R}_+^*$. Consider $\gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a]$ defined by $\gamma(t, \theta) = t + \theta$. Then $\gamma \in \Gamma_{\sigma,a}$ and the equation $R(f, \gamma)$ coincides with the classical retarded functional differential equations $u'(t) = f(t, u_t)$ (see, for example [**5, 6, 8**]).

### Singular functional differential equations

Singular functional differential equations have been studied by many authors, for instance, Baxely [**1**], Bobisud and O'Regan [**2**], Gatica and al [**4**], Huaxing and Tadeusz [**7**], Labovskii [**9**] and O'Regan [**12**].

Our purpose in this section is to apply the previous results to give some theorems of existence for singular functional differential equations.

**Theorem 5.1.** *Consider the initial value problem for singular functional differential equations*

$$(5.1) \qquad \begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in (\sigma, \sigma + b],\ (b > 0) \\ x_\sigma = \varphi \end{cases}$$

*where* $g : [\sigma, \sigma + b] \times \mathcal{C} \to E$ *is completely continuous,* $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ *is continuous,* $\psi > 0$ *on* $(\sigma, \sigma + b)$ *and* $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$. *Then,* (5.1) *has at least one noncontinuable solution.*

*Proof.* Let $\rho : [\sigma - r, \sigma + b] \to [\sigma - r, \sigma + a]$ defined by

$$\rho(\tau) = \begin{cases} \sigma + \displaystyle\int_\sigma^\tau \frac{\mathrm{d}s}{\psi(s)} & \text{if } \ \tau \in [\sigma, \sigma + b] \\ \tau & \text{if } \ \tau \in [\sigma - r, \sigma], \end{cases}$$

then $\rho$ is bijective and continuous.

For all $\tau \in [\sigma, \sigma + b]$ and $\theta \in [-r, 0]$, put

$$u(\rho(\tau + \theta)) = x(\tau + \theta).$$

Then, for all $\tau \in (\sigma, \sigma + b]$,

$$x'(\tau) = u'(\rho(\tau))\rho'(\tau) = u'(\rho(\tau))\frac{1}{\psi(\tau)}.$$

Hence $u'(\rho(\tau)) = \psi(\tau)x'(\tau) = g(\tau, x_\tau)$, and thus

$$u'(t) = g(\rho^{-1}(t), x_{\rho^{-1}(t)}), \qquad t \in (\sigma, \sigma + a], \qquad \text{and}$$

$$x_{\rho^{-1}(t)}(\theta) = x(\rho^{-1}(t) + \theta) = u(\rho(\rho^{-1}(t) + \theta)).$$

Consider the function $\gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a]$ defined by $\gamma(t, \theta) = \rho(\rho^{-1}(t) + \theta)$. It is clear that $\gamma \in \Gamma_{\sigma,a}$, $x_{\rho^{-1}(t)} = u \circ \gamma(t, \cdot)$ and

$$u_\sigma(\theta) = u(\sigma + \theta) = u(\gamma(\sigma, \theta)) = u(\rho(\rho^{-1}(\sigma) + \theta)) = x_\sigma(\theta) = \varphi(\theta).$$

□

Finally, the initial value problem (5.1) is equivalent to following problem

$$(5.2)\qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \varphi, \end{cases}$$

where $f(t,\phi) = g(\rho^{-1}(t), \phi)$, $f$ is also completely continuous and by Theorem 4.1, the problem (5.2) has at least one noncontinuable solution $u$ on $[\sigma - r, \sigma + \alpha)$ with $\alpha \leq a$. It easy to see that the problem (5.1) has also a noncontinuable solution $x$ on $[\sigma - r, \sigma + \beta)$ where $\alpha, \beta$ are fasten by the relation $\alpha = \int_\sigma^{\sigma+\beta} \frac{\mathrm{d}s}{\psi(s)}$.

An important criteria given by the following theorem assure the existence of global solutions of (5.1).

**Theorem 5.2.** *Assume the conditions of Theorem 5.1 are satisfied. Suppose further that*

(1) *for all* $(\tau, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$

$$|g(\tau, \phi)|_E \leq m(\tau) h(\|\phi\|),$$

*where* $m : [\sigma, \sigma + b] \to \mathbb{R}^+$ *and* $h : \mathbb{R}^+ \to \mathbb{R}^+_*$ *are continuous,* $h$ *nondecreasing on* $\mathbb{R}^+$ *and* $\int_\sigma^{\sigma+b} \frac{m(s)}{\psi(s)}\, \mathrm{d}s < \int_{\|\varphi\|}^{+\infty} \frac{\mathrm{d}s}{h(s)}$, *or*

(2) *for all* $(\tau, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$

$$|g(\tau, \phi)|_E \leq h(\tau, |\phi(0)|_E),$$

*where* $h : [\sigma, \sigma + b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and the set of solutions to the singular ordinary differential equation*

$$\begin{cases} \psi(\tau) y'(\tau) = h(\tau, y(\tau)) \\ y(\lambda) = \mu \end{cases}$$

*is bounded on* $C([\lambda, \sigma + b]; \mathbb{R})$, *or*

(3) $(\xi)$ *there are three functions* $V \in C([\sigma, \sigma + b] \times \mathcal{C}; \mathbb{R}^+)$, $a_1, a_2 \in C(\mathbb{R}^+; \mathbb{R}^+)$ *with* $\lim_{s \to +\infty} a_1(s) = +\infty$ *and* $a_1(\|\phi\|) \leq V(t, \phi) \leq a_2(\|\phi\|)$ *for all* $(t, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$

$(\xi')$ *for any* $0 < \beta \leq b$ *and for any solution* $x$ *of* (5.1) *on* $[\sigma - r, \sigma + \beta)$, *we have for all* $t \in (\sigma, \sigma + \beta)$

$$\begin{aligned} D^+ V(t, x_t(\sigma, \varphi)) &:= \limsup_{k \to 0^+} \frac{1}{k} [V(t + k, x_{t+k}(t, \varphi)) - V(t, x_t(\sigma, \varphi))] \\ &\leq [\psi(t)]^{-1} h(t, V(t, x_t(\sigma, \varphi)) \end{aligned}$$

*where* $h : [\sigma, \sigma + b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and the set of solutions of the singular ordinary differential equation*

$$\begin{cases} \psi(t) y'(t) = h(t, y(t)) \\ y(\lambda) = \mu \end{cases}$$

*is bounded on* $C([\lambda, \sigma + b]; \mathbb{R})$.

*Then* (5.1) *has at least one global solution on* $[\sigma - r, \sigma + b]$.

*Proof.*

– Suppose that the condition (1) is verified.

Let $x$ (resp. $u$) be a solution of (5.1) (resp. (5.2)) on $[\sigma - r, \sigma + \beta_1)$ (resp. on $[\sigma - r, \sigma + \alpha_1))$ with $\alpha_1 = \int_\sigma^{\sigma+\beta_1} \frac{\mathrm{d}s}{\psi(s)}$. By using the same argument seen in the Theorem 4.2, we obtain

$$\lim_{\tau\to\sigma+\beta_1} x(\tau) = \lim_{t\to\sigma+\alpha_1} u(t) \text{ exist.}$$

Take

$$x^1(\tau) = \begin{cases} x(\tau) & \text{if } \tau \in [\sigma - r, \sigma + \beta_1) \\ \lim\limits_{\tau'\to\sigma+\beta_1} x(\tau') & \text{if } \tau = \sigma + \beta_1. \end{cases}$$

Then $x^1$ is a solution to (5.1) on $[\sigma - r, \sigma + \beta_1]$. If $\beta_1 < b$, consider the problem

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in [\sigma + \beta_1, \sigma + b] \\ x_{\sigma+\beta_1} = x^1_{\sigma+\beta_1}, \end{cases}$$

then this problem has a solution $x^2$ on $[\sigma + \beta_1 - r, \sigma + \beta_1 + \beta_2]$. Define

$$z(t) = \begin{cases} x^1(t) & \text{if} \quad t \in [\sigma - r, \sigma + \beta_1] \\ x^2(t) & \text{if} \quad t \in [\sigma + \beta_1, \sigma + \beta_1 + \beta_2]. \end{cases}$$

Then $z$ is a solution of (5.1) on $[\sigma - r, \sigma + \beta_1 + \beta_2]$. Repeating this method, we can get a global solution of (5.1) on $[\sigma - r, \sigma + b]$.

– Suppose that the condition (2) is verified.

Let $x$ be a solution of (5.1) on $[\sigma-r, \sigma+\beta_1)$. Take $m(\tau) = |x(\tau)|_E$, $\tau \in [\sigma, \sigma+\beta)$, then $m(\sigma) \le \|\varphi\| = y(\sigma)$ (with $\mu = \|\varphi\|$) and for all $\tau \in (\sigma, \sigma + \beta)$

$$\psi(\tau)D^+m(\tau) := \psi(\tau)\limsup_{k\to 0^+}\frac{m(\tau + k) - m(\tau)}{k} \le \psi(\tau)\,|x'(\tau)|_E \le h(\tau, m(\tau)).$$

Consequently, by [**10**] we obtain $m(\tau) \le y_{\max}(\tau)$ (where $y_{\max}$ is a maximal solution of singular the ordinary differential equation). Let $M = \sup\{y_{\max}(\tau) : \tau \in [\sigma, \sigma + b]\}$. Note that $|x(\tau)|_E \le M,\ \tau \in [\sigma, \sigma + \beta_1)$. We shall prove that $\lim\limits_{\tau\to\sigma+\beta_1} x(\tau)$ exists. For $\sigma < \tau < \tau' < \sigma + \beta$ we have

$$\begin{aligned} |x(\tau) - x(\tau')|_E &\le \int_\tau^{\tau'} [\psi(s)]^{-1}\,|g(s, x_s)|_E\,\mathrm{d}s \\ &\le \int_\tau^{\tau'} [\psi(s)]^{-1} h(s, |x(s)|_E)\mathrm{d}s \le M' \int_\tau^{\tau'} [\psi(s)]^{-1}\mathrm{d}s, \end{aligned}$$

where $M' = \max\{h(s, t)\ : s \in [\sigma, \sigma + b],\ t \le M\}$.

For any $\varepsilon > 0$, we can find $\eta > 0$ such that

$$\left|\int_\tau^{\tau'} \frac{\mathrm{d}s}{\psi(s)}\right| < \frac{\varepsilon}{M'}$$

whenever $|\tau-\tau'|<\eta$, now for any $\tau<\tau'$ such that $|\tau-(\sigma+\beta)|<\frac{\eta}{2}$ and $|\tau'-(\sigma+\beta)|<\frac{\eta}{2}$, then

$$|x(\tau)-x(\tau')|_E \le M'\int_\tau^{\tau'}[\psi(s)]^{-1}\mathrm{d}s<\varepsilon.$$

The rest of the proof is identical to the condition (1).

– Suppose that the condition (3) is verified.
  Let $m(t)=V(t,x_t)$. By $(\xi)$ we obtain

$$m(\sigma)\le a_2(\|x_\sigma\|)=a_2(\|\varphi\|):=y(\sigma)$$

and by $(\xi')$ we have

$$\psi(t)D^+m(t)\le h(t,m(t)), \qquad t\in(\sigma,\sigma+\beta).$$

Then, $m(t)\le y_{\max}(t)$ where $y_{\max}$ is a maximal solution of the singular ordinary differential equation. Hence

$$a_1(\|x_t\|)\le V(t,\phi)=m(t)\le y_{\max}(t)\le M.$$

But $\lim\limits_{s\to+\infty}a_1(s)=+\infty$, there exists $M'>0$ such that $M<a_1(M')$, so $\|x_t\|\le M'$. Let $M_1=\sup\{|g(t,\varphi)|_E : t\in(\sigma,\sigma+\beta),\ \|\varphi\|<M'\}$, then the rest of the proof is similar to the condition (2). □

**Theorem 5.3.** *Consider the initial value problem for singular functional differential equations*

$$(5.3)\qquad \begin{cases} \psi(\tau)x^{(n)}(\tau)=g(\tau,x_\tau,x'_\tau,\dots,x_\tau^{(n-1)}), & \tau\in(\sigma,\sigma+b],\ (b>0)\\ x_\sigma=\varphi\in C^{(n-1)}([-r,0];E), \end{cases}$$

*where $g:[\sigma,\sigma+b]\times\mathcal{C}^n\to E$ is completely continuous, $\psi:[\sigma,\sigma+b]\to\mathbb{R}^+$ is continuous, $\psi>0$ on $(\sigma,\sigma+b]$ and $a:=\int_\sigma^{\sigma+b}\frac{\mathrm{d}s}{\psi(s)}<+\infty$. Then, (5.3) has at least one noncontinuable solution.*

*Proof.* Let $\rho$ and $\gamma$ be the functions defined in Theorem 5.1. For all $\tau\in[\sigma,\sigma+b]$ and $\theta\in[-r,0]$, put

$$u(\rho(\tau+\theta))=\begin{pmatrix} u^1(\rho(\tau+\theta))\\ u^2(\rho(\tau+\theta))\\ \vdots\\ u^n(\rho(\tau+\theta))\end{pmatrix}=\begin{pmatrix} x(\tau+\theta)\\ x'(\tau+\theta)\\ \vdots\\ x^{(n-1)}(\tau+\theta)\end{pmatrix}.$$

Using the same technique as in the proof of Theorem 5.1, the problem (5.3) becomes equivalent to the following problem

$$(5.4)\qquad \begin{cases} u'(t)=F_1(t,u\circ\gamma(t,\cdot))+F_2(t,u\circ\gamma(t,\cdot)), & t\in(\sigma,\sigma+a]\\ u_\sigma=(\varphi,\varphi',\dots,\varphi^{(n-1)}), \end{cases}$$

where $F_1, F_2 : [\sigma, \sigma + a] \times C([-r, 0]; E^n) \to E^n$ are defined by

$$F_1(t,\phi) = \begin{pmatrix} \psi(\rho^{-1}(t))\phi_2(0) \\ \vdots \\ \psi(\rho^{-1}(t))\phi_n(0) \\ 0 \end{pmatrix}, \qquad F_2(t,\phi) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(\rho^{-1}(t), \phi_1, \phi_2, \dots, \phi_n) \end{pmatrix}$$

with $\phi_1, \phi_2, \dots, \phi_n$ which are the components of $\phi \in C([-r, 0]; E^n)$. It is easy to see that $F_1(t, \cdot)$ is Lipschitz and $F_2$ is completely continuous, then $F = F_1 + F_2$ verifies the condition $(l)$ and by Theorem 4.1, the problem (5.4) has at least one noncontinuable solution and therefore, the problem (5.3) has also at least a noncontinuable solution. □

**Theorem 5.4.** *Consider the initial value problem for singular functional equations*

$$(5.5) \qquad \begin{cases} \psi(\tau)x'(\tau) = g(\tau, x(\tau - r_1), \dots, x(\tau - r_m)), & \tau \in ]\sigma, \sigma + b],\ (b > 0) \\ x_\sigma = \varphi & \text{on } [-r, 0] \text{ with} \\ & r = \max\limits_{1 \le i \le m}(r_i),\ r_i \ge 0 \end{cases}$$

*where $g : [\sigma, \sigma + b] \times E^2 \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$.*
*Then,* (5.5) *has at least one noncontinuable solution.*

*Proof.* Using the same argument as in the proof of Theorem 4.1, we can see that the problem (5.5) is equivalent to the following problem

$$(5.6) \qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \varphi, \end{cases}$$

where $f(t, \phi) = g(\rho^{-1}(t), \phi(-r_1), \dots, \phi(-r_m))$. □

**Theorem 5.5.** *Consider the initial value problem for singular functional equations*

$$(5.7) \qquad \begin{cases} \psi(\tau)x''(\tau) = g(\tau, x(\tau), x(\tau - r_1), x'(\tau), x'(\tau - r_2)), & \tau \in (\sigma, \sigma + b] \\ x_\sigma = \varphi,\ x'_\sigma = \varphi', & \text{on } [-r, 0] \text{ with} \\ & r = \max(r_1, r_2) \end{cases}$$

*where $g : [\sigma, \sigma + b] \times E^2 \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$.*
*Then,* (5.7) *has at least one noncontinuable solution.*

*Proof.* The problem (5.7) is equivalent to the following problem

$$(5.8) \qquad \begin{cases} u'(t) = F_1(t, u \circ \gamma(t, \cdot)) + F_2(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \Phi & \text{with } \Phi = (\varphi, \varphi'), \end{cases}$$

where $F_1, F_2 : [\sigma, \sigma + a] \times C([-r, 0]; E^2) \to E^2$ are defined by

$$F_1(t, \phi) = \begin{pmatrix} \psi(\rho^{-1}(t))\phi_2(0) \\ 0 \end{pmatrix}$$

and

$$F_2(t, \phi) = \begin{pmatrix} 0 \\ g(\rho^{-1}(t), \phi_1(0), \phi_1(-r_1)\phi_2(0), \phi_2(-r_2)) \end{pmatrix}$$

with $\phi : \theta \in [-r, 0] \to \phi(\theta) = (\phi_1(\theta), \phi_2(\theta)),\ \phi_1, \phi_2 \in \mathcal{C}$. □

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