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ON SMALL INJECTIVE, SIMPLE-INJECTIVE AND QUASI-FROBENIUS RINGS

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Abstract. Let R be a ring. A right ideal I of R is called small in R if $I + K \neq R$ for every proper right ideal K of R. A ring R is called *right small finitely injective* (briefly, *SF-injective*) (resp., *right small principally injective* (briefly, *SP-injective*) if every homomorphism from a small and finitely generated right ideal (resp., a small and principally right ideal) to R_R can be extended to an endomorphism of R_R. The class of right SF-injective and SP-injective rings are broader than that of right small injective rings (in [**15**]). Properties of right SF-injective rings and SP-injective rings are studied and we give some characterizations of a QF-ring via right SF-injectivity with ACC on right annihilators. Furthermore, we answer a question of Chen and Ding.

1. Introduction

Throughout the paper R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-module. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-module. We use J (resp. Z_r, S_r) for the Jacobson radical (resp. the right singular ideal, the right socle of R) and $E(M_R)$ for the injective hull of M_R. If X is a subset of R, the right (resp. left) annihilator of X in R is denoted by $r_R(X)$ (resp. $l_R(X)$) or simply $r(X)$ (resp. $l(X)$) if no confusion appears. If N is a submodule of M (resp. proper submodule) we denote by $N \leq M$ (resp. $N < M$). Moreover, we write $N \leq^{\mathrm{e}} M$, $N \ll M$, $N \leq^{\oplus} M$ and $N \leq^{\max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand and a maximal submodule of M, respectively. A module M is called *uniform* if $M \neq 0$ and every non-zero submodule of M is essential in M. M is *finite dimensional* (or has *finite rank*) if $E(M)$ is a finite direct sum of indecomposable submodules; or equivalently, if M has an essential submodule which is a finite direct sum of uniform submodules.

A module M_R is called *F-injective* (resp., *P-injective*) if every right homomorphism from a finitely generated (resp., principal) right ideal to M_R can be extended to an R-homomorphism from R_R to M_R. A ring R is called right F-injective (resp.,

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right P-injective) if R_R is F-injective (resp., P-injective). R is called *right min-injective* if every right R-homomorphism from a minimal right ideal to R can be extended to an endomorphism of R_R. A ring R is said to be a right PF-ring if the right R_R is an injective cogenerator in the category of right R-modules. A ring R is called QF-ring if it is right (or left) Artinian and right (or left) self-injective.

In [**15**], a module M_R is called *small injective* if every homomorphism from a small right ideal to M_R can be extended to an R-homomorphism from R_R to M_R. A ring R is called right small injective if R_R is small injective. Under small injective condition, Shen and Chen ([**15**]) gave some new characterizations of QF rings and right PF rings. In [**18**], authors showed some characterizations of Jacobson radical J via small injectivity. They proved that J is Noetherian as a right R-module if and only if every direct sum of small injective right R-modules is small injective if and only if $E^{(\mathbb{N})}$ is small injective for every small injective module E_R.

In 1966, Faith proved that R is QF if and only if R is right self-injective and satisfies ACC on right annihilators. Then around 1970, Björk proved that R is QF if and only if R is right F-injective and satisfies ACC on right annihilators. In this paper, we show that R is QF if and only if R is a semiregular and right SF-injective ring with ACC on right annihilators if and only if R is a semilocal and right SF-injective ring with ACC on right annihilators if and only if R is a right SF-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$. We also give some characterizations of rings whose R-homomorphism from a small, finitely generated right ideal to R with a simple image, can be extended to an endomorphism of R_R. Furthermore, we prove that if R is a right perfect, right simple-injective and left pseudo-coherent ring, then R is QF. Some known results are obtained as corollaries.

A general background material can be found in [**1**], [**7**], [**19**].

2. On SP(SF)-injective rings

Definition 2.1. A module M_R is called *small principally injective* (briefly, *SP-injective*) if every homomorphism from a small and principal right ideal to M_R can be extended to an R-homomorphism from R_R to M_R. A module M_R is called *small finitely injective* (briefly, *SF-injective*) if every homomorphism from a small and finitely generated right ideal to M_R can be extended to an R-homomorphism from R_R to M_R. A ring R is called right SP-injective (resp., right SF-injective) if R_R is SP-injective (resp., SF-injective).

The following implications are obvious:

$$\begin{array}{cccccc} & & small\ injective & & & \\ & \nearrow & & \searrow & & \\ injective & & & & SF-injective & \longrightarrow SP-injective \\ & \searrow & & \nearrow & & \\ & & F-injective & & & \end{array}$$

Lemma 2.2. *The following conditions are equivalent for a ring R:*

(1) *R is right SP-injective.*

(2) *$lr(a) = Ra$ for all $a \in J$.*

(3) *$r(a) \leq r(b)$, where $a \in J,\ b \in R$, implies $Rb \leq Ra$.*

(4) *$l(bR \cap r(a)) = l(b) + Ra$ for all $a \in J$ and $b \in R$.*

(5) *If $\gamma : aR \to R,\ a \in J$, is an R-homomorphism, then $\gamma(a) \in Ra$.*

Proof. A similar proving to [**10**, Lemma 5.1]. □

We also have:

Lemma 2.3. *A ring R is right SF-injective if and only if it satisfies the following two conditions:*

(1) *$l(T \cap T') = l(T) + l(T')$ for all small, finitely generated right ideals T and T'.*

(2) *R is right SP-injective.*

Proof. $(\Rightarrow)$: Assume that R is right SF-injective. If T and T' are small, finitely generated right ideals, then $T + T'$ is a small finitely generated right ideal. Let $b \in l(T \cap T')$ and then we define $\alpha : T + T' \to R$ via $\alpha(t + t') = bt$, for all $t \in T$ and $t' \in T'$, so $\alpha = a.$, for some $a \in R$ by hypothesis. Then $b - a \in l(T)$ and $a \in l(T')$. Hence $b \in l(T) + l(T')$. Thus (1) holds. (2) is clear.

$(\Leftarrow)$: We can prove it by induction on the number of generators of T and T'. □

Corollary 2.4. *Let R be a right SP-injective ring such that $l(T \cap T') = l(T) + l(T')$ for all right ideals T and T' of R where T is small, finitely generated. Then every R-homomorphism $\varphi : I \to R$ extends to $R \to R$ where I is a small right ideal and the image $\varphi(I)$ is finitely generated.*

Proposition 2.5. *A direct product $R = \prod_{i \in I} R_i$ of rings R_i is right SF-injective (resp., right SP-injective) if and only if R_i is right SF-injective (resp., right SP-injective) for each $i \in I$.*

Proof. Assume that $R = \prod_{i \in I} R_i$ is right SF-injective. For each $i \in I$, we take any $a_i \in J(R_i)$ and $b_i \in R_i$ such that $r_{R_i}(a_i) \leq r_{R_i}(b_i)$. Let $a = (a_j)_{j \in I}$, $b = (b_j)_{j \in I}$, where $a_j = 0, b_j = 0,\ \forall j \neq i$ and $a_j = a_i, b_j = b_i$ if $j = i$. Then $a \in J(R), b \in R$ and $r_R(a) \leq r_R(b)$. So $b \in Ra$ since R is right SP-injective. Therefore $b_i \in R_i a_i$. Thus R_i is right SP-injective. On the other hand, for all small, finitely generated right ideals T_i and T_i' of R_i, $\iota_i(T_i),\ \iota_i(T_i')$ are small, finitely generated right ideals of R, where $\iota_i : R_i \hookrightarrow R$ is the inclusion for each $i \in I$. By hypothesis, $l_R(\iota_i(T_i) \cap \iota_i(T_i')) = l_R(\iota_i(T_i)) + l_R(\iota_i(T_i'))$. This implies that $l_{R_i}(T_i \cap T_i') = l_{R_i}(T_i) + l_{R_i}(T_i')$. Thus R_i is right SF-injective by Lemma 2.3.

Conversely, $R = \prod_{i \in I} R_i$, where R_i is right SF-injective. For each $a = (a_i)_{i \in I} \in J(R)$ and $b = (b_i)_{i \in I} \in R$ such that $r_R(a) \leq r_R(b)$, then for each $i \in I$, $a_i \in J(R_i)$ and $r_{R_i}(a_i) \leq r_{R_i}(b_i)$. Since R_i is right SF-injective, $b_i \in R_i a_i$. Hence $b \in Ra$. If T and T' are small, finitely generated right ideals of R, then we can prove that $l_R(T \cap T') = l_R(T) + l_R(T')$. Thus R is right SF-injective. □

A ring R is called *left minannihilator* if $lr(K) = K$ for every minimal left ideal K of R.

Proposition 2.6. *Let R be a right SP-injective ring. Then:*

(1) *R is right mininjective and left minannihilator.*

(2) *$J \leq Z_r$.*

Proof. (1) Since every minimal one-sided ideal of R is either nilpotent or a one-sided direct summand of R, each right SP-injective ring is right mininjective and left minannihilator.

(2) If $a \in J$ we will show that $r(a) \leq^{e} R_R$. In fact, let $b \in R$ such that $bR \cap r(a) = 0$. By Lemma 2.2, $R = l(b) + Ra$, so $l(b) = R$ because $a \in J$. Hence $b = 0$. This proves that $a \in Z_r$. □

A ring R is called *right Kasch* if every simple right R-module embeds in R_R.

Proposition 2.7. *Let R be a right Kasch ring. Then:*

(1) *If R is right SP-injective, then:*
   a) *The map $\psi : T \mapsto l(T)$ from the set of maximal right ideals T of R to the set of minimal left ideals of R is a bijection. And the inverse map is given by $K \mapsto r(K)$, where K is a minimal left ideal of R.*
   b) *For $k \in R$, Rk is minimal iff kR is minimal, in particular $S_r = S_l$.*

(2) *If R is right SF-injective, then $\mathrm{rl}(I) = I$ for every small, finitely generated right ideal I of R. In particular, R is left SP-injective.*

Proof. (1) a): By Proposition 2.6 (1) and [**10**, Theorem 2.32]. For b), if Rk is minimal, then $r(k)$ is maximal by a). This means kR is minimal. Conversely, by [**10**, Theorem 2.21].

(2): Firstly, we have $J = \mathrm{rl}(J)$ because R is right Kasch. Let T be a right small, finitely generated ideal of R. Therefore, $T \leq \mathrm{rl}(T) \leq \mathrm{rl}(J) = J$. If $b \in \mathrm{rl}(T)\backslash T$, take I such that $T \leq I \leq^{\max} (bR + T)$. Since R is right Kasch, we can find a monomorphism $\sigma : (bR + T)/I \to R$, and then define $\gamma : bR + T \to R$ via $\gamma(x) = \sigma(x + I)$. Since $bR + I$ is a small, right finitely generated ideal of R and R is right SF-injective, it follows that $\gamma = c$, where $c \in R$. Hence $cb = \sigma(b + I) \neq 0$ because $b \notin I$. But if $t \in T$, then $ct = \sigma(t + I) = 0$ because $T \leq I$, so $c \in l(I)$. Since $b \in \mathrm{rl}(T)$ this gives $cb = 0$, a contradiction. Thus $T = \mathrm{rl}(T)$. It is clear that R is left SP-injective. □

Recall that a ring R is called semiregular if R/J is von Neumann regular and idempotents can be lifted modulo J. Note that if R is semiregular, then for every finitely generated right ideal I of R, $R = H \oplus K$, where $H \leq I$ and $I \cap K \ll R$.

Motivated by [**15**, Lemma 3.1] we have the following result.

Lemma 2.8. *If R is a semiregular ring and I is a right ideal of R, then the following conditions are equivalent:*

(1) *Every homomorphism from a finitely generated right ideal to I can be extended to an endomorphism of R_R.*

(2) *Every homomorphism from a small, finitely generated right ideal to I can be extended to an endomorphism of R_R.*

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1): Let $f : K \to I$ be an R-homomorphism, where K is a finitely generated right ideal. Since R is semiregular, then $R = H \oplus L$, where $H \leq K$ and $K \cap L \ll R$. Hence $R = K + L$ and $K = H \oplus (K \cap L)$, $K \cap L$ is a small, finitely generated right ideal of R. Thus there exists an endomorphism g of R_R such that $g(x) = f(x)$ for all $x \in K \cap L$. We construct a homomorphism $\varphi : R_R \to R_R$ defined by $\varphi(r) = f(k) + g(l)$ for any $r = k + l,\ k \in K,\ l \in L$. Now we show that φ is well defined. Indeed, if $k_1 + l_1 = k_2 + l_2$, where $k_i \in K,\ l_i \in L,\ i = 1, 2$, then $k_1 - k_2 = l_1 - l_2 \in K \cap L$. Hence $f(k_1 - k_2) = g(l_1 - l_2)$, which implies that $\varphi(k_1 + l_1) = \varphi(k_2 + l_2)$. Thus φ is an endomorphism of R_R such that $\varphi_{|K} = f$. $\square$

Let I be an ideal of R. A ring R is called right I-semiregular if for every $a \in I$, $aR = eR \oplus T$, where $e^2 = e$ and $T \leq I_R$.

Corollary 2.9. *Let R be a right Z_r-semiregular ring. Then R is right SF-injective if and only if R is right F-injective.*

It is well-known if R is semiperfect and right small injective with $S_r \leq^{\mathrm{e}} R_R$, then R is right self-injective. This result is proved by Yousif and Zhou (see [**20**, Theorem 2.11]). In [**15**, Theorem 3.4], they showed that a semilocal (or semiregular) ring R is right self-injective if and only if R is right small injective. From Lemma 2.8 we also have a similar result.

Theorem 2.10. *Let R be a semiregular ring. Then*

(1) *R is right P-injective if and only if R is right SP-injective.*

(2) *R is right F-injective if and only if R is right SF-injective.*

Because a semiperfect ring is semiregular, we have:

Corollary 2.11. *Let R be a semiperfect ring. Then*

(1) *R is right P-injective if and only if R is right SP-injective.*

(2) *R is right F-injective if and only if R is right SF-injective.*

Next we obtain some characterizations of QF-ring via right SF-injectivity with ACC on right annihilators. The following theorem extends [**15**, Theorem 3.8].

Theorem 2.12. *For a ring R, the following conditions are equivalent:*

(1) *R is QF.*

(2) *R is a semiregular and right SF-injective ring with ACC on right annihilators.*

(3) *R is a semilocal and right SF-injective ring with ACC on right annihilators.*

(4) *R is a right SF-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$.*

Proof. It is obvious that (1) $\Rightarrow$ (2), (3), (4).

(2) $\Rightarrow$ (1): By Theorem 2.10, R is right F-injective. Thus R is QF by [**3**, Theorem 4.1].

(3) $\Rightarrow$ (1): Since R satisfies ACC on right annihilators, Z_r is nilpotent and so $Z_r \leq J$. Therefore, $J = Z_r$ is nilpotent by Proposition 2.6. Hence R is semiprimary.

(4) $\Rightarrow$ (1): By [**13**, Theorem 2.1] or [**14**, Lemma 2.11], R is semiprimary. $\square$

Corollary 2.13. *Let R be a ring. Then R is QF if and only if R is a semilocal, left and right SP-injective ring with ACC on right annihilators.*

Remark. The condition "semilocal" in Theorem 2.12 can not be omitted, since the ring of integers $\mathbb{Z}$ is SP-injective, Noetherian, but $\mathbb{Z}$ is not QF.

The following result extends [**11**, Theorem 2.2].

Proposition 2.14. *If R is right SP-injective and $R/\operatorname{Soc}(R_R)$ has ACC on right annihilators, then J is nilpotent.*

Proof. Here we use a similar argument to that one in [**2**, Theorem 3]. Suppose that $R/\operatorname{Soc}(R_R)$ has ACC on right annihilators. Let $S = \operatorname{Soc}(R_R)$ and $\overline{R} = R/S$. For any $a_1, a_2, \ldots$ in J, since

$$r_{\bar{R}}(\bar{a}_1) \leq r_{\bar{R}}(\bar{a}_2\bar{a}_1) \leq \ldots,$$

by hypothesis there exists a positive integer m such that

$$r_{\bar{R}}(\bar{a}_m \ldots \bar{a}_2\bar{a}_1) = r_{\bar{R}}(\bar{a}_{m+k} \ldots \bar{a}_2\bar{a}_1)$$

for $k = 0, 1, 2, \ldots$. Now for any positive integer n, since $a_{n+1}a_n \ldots a_1 \in J \leq Z_r$, $r(a_{n+1}a_n \ldots a_1) \leq^{\mathrm{e}} R_R$. Hence $S \leq r(a_{n+1}a_n \ldots a_1)$. We claim that

$$r_{\bar{R}}(\bar{a}_n \ldots \bar{a}_2\bar{a}_1) \leq r(a_{n+1}a_n \ldots a_1)/S \leq r_{\bar{R}}(\bar{a}_{n+1} \ldots \bar{a}_2\bar{a}_1).$$

In fact, assume $b + S \in r_{\bar{R}}(\bar{a}_n \ldots \bar{a}_2\bar{a}_1)$. Then we have $a_n \ldots a_1 b \in S$. But since $S \leq r(a_{n+1})$, we get $a_{n+1}a_n \ldots a_1 b = 0$. Thus $b \in r(a_{n+1}a_n \ldots a_1)$, and so $b + S \in r(a_{n+1}a_n \ldots a_1)/S$. Now the other inclusion $r(a_{n+1}a_n \ldots a_1)/S \leq r_{\bar{R}}(\bar{a}_{n+1} \ldots \bar{a}_2\bar{a}_1)$ is obvious.

By this fact, it follows that

$$r(a_{m+1}a_m \ldots a_1)/S = r(a_{m+2}a_{m+1} \ldots a_1)/S$$

because $r_{\bar{R}}(\bar{a}_m \ldots \bar{a}_2\bar{a}_1) = r_{\bar{R}}(\bar{a}_{m+2} \ldots \bar{a}_2\bar{a}_1)$. Therefore

$$r(a_{m+1}a_m \ldots a_1) = r(a_{m+2}a_{m+1}a_m \ldots a_1),$$

and hence $(a_{m+1}a_m \ldots a_1)R \cap r(a_{m+2}) = 0$. But $r(a_{m+2})$ is an essential right ideal of R, and so $a_{m+1}a_m \ldots a_1 = 0$. Hence J is right T-nilpotent and the ideal $(J+S)/S$ of the ring $\bar{R} = R/S$ is also right T-nilpotent. By [**1**, Proposition 29.1], $(J+S)/S$ is nilpotent, and so there is a positive integer t such that $J^t \leq S$. Hence $J^{t+1} \leq SJ$. Thus J is nilpotent. $\square$

Theorem 2.15. *If R is a semilocal and right SF-injective ring such that R/S_r is right Goldie, then R is QF.*

Proof. By Proposition 2.14, J is nilpotent, and hence R is semiprimary. Hence R is right F-injective by Theorem 2.10. This implies that R is right GPF (i.e., R is semiperfect, right P-injective with $S_r \leq^{\mathrm{e}} R_R$) and so R is right Kasch by [**11**, Corollary 2.3]. Therefore R is left P-injective by [**3**, Proposition 4.1]. Thus R is QF by [**10**, Theorem 3.38]. □

Corollary 2.16. *If R is a semilocal and right SF-injective ring satisfying ACC on essential right ideals, then R is QF.*

Now we consider rings whose small and finitely generated right ideals are projective. We have the following result.

Theorem 2.17. *For a ring R the following conditions are equivalent:*

(1) *Every small and finitely generated right ideal of R is projective.*

(2) *Every quotient module of a SF-injective module is SF-injective.*

(3) *Every quotient module of a F-injective module is SF-injective.*

(4) *Every quotient module of a small injective module is SF-injective.*

(5) *Every quotient module of an injective module is SF-injective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(1) $\Rightarrow$ (2): Assume that E_R is SF-injective and $\pi : E \to B$ is an epimorphism. Let $f : I \to B$ be an R-homomorphism, where I is a small and finitely generated right ideal of R.

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \overset{\iota}{\hookrightarrow} & R \\ & & \xrightarrow{f}\downarrow & & \\ E & \xrightarrow{\pi} & B & \longrightarrow & 0 \end{array}$$

where ι is the inclusion.

By (1), I is projective. Therefore there exists an R-homomorphism $h : I \to E$ such that $\pi h = f$. Now since E is SF-injective, there is an R-homomorphism $h' : R \to E$ such that $h'\iota = h$. Let $h'' = \pi h' : R \to B$, then $h''\iota = f$. This means B_R is SF-injective.

(5) $\Rightarrow$ (1): For every small and finitely generated right ideal I of R, we consider the epimorphism $h : A \to B$ and R-homomorphism $\alpha : I \to B$.

Since $B = h(A) \overset{\psi}{\cong} A/\operatorname{Ker} h \overset{\iota_1}{\hookrightarrow} E(A)/\operatorname{Ker} h$, where ι_1 is the inclusion and $\psi(h(a)) = a + \operatorname{Ker} h$, for all $a \in A$. Then let $j = \iota_1\psi$. We consider the following diagram:

$$\begin{array}{ccccc} & & I & \overset{\iota}{\hookrightarrow} & R \\ & \overset{\varphi}{\swarrow} & \xrightarrow{\alpha}\downarrow & & \\ E \xrightarrow{h} & & B & \longrightarrow & 0 \\ & & \xrightarrow{j}\downarrow & & \\ E(A) \xrightarrow{p} & & E(A)/\operatorname{Ker} h & \longrightarrow & 0 \end{array}$$

where ι is the inclusion and p is the natural epimorphism.

By (5), $E(A)/\operatorname{Ker} h$ is SF-injective and then there exists an R-homomorphism $\alpha' : R \to E(A)/\operatorname{Ker} h$ such that $\alpha'\iota = j\alpha$. Since R_R is projective, there is an R-homomorphism $\alpha'' : R \to E(A)$ such that $p\alpha'' = \alpha'$. Let $h' = \alpha''\iota : I \to E(A)$. It is easy to see that $h'(I) \leq A$, so there exists an R-homomorphism $\varphi : I \to A$ such that $\varphi(x) = h'(x)$, for all $x \in I$.

Now we claim that $h\varphi = \alpha$. In fact, for each $x \in I$ we have

$$j(\alpha(x)) = \alpha'(\iota(x)) = \alpha'(x) = p(\alpha''(x)) = p(h'(x)) = p(\varphi(x)).$$

Since α is the epimorphism, $\alpha(x) = h(a)$ for some $a \in A$. Therefore $j(\alpha(x)) = j(h(a)) = a + \operatorname{Ker} h$, and so $a + \operatorname{Ker} h = \varphi(x) + \operatorname{Ker} h$, $h(a - \varphi(x)) = 0$. Hence $h\varphi(x) = h(a) = \alpha(x)$. Thus I is projective. $\square$

Example 2.18. i) Let $R = F[x_1, x_2, \ldots]$, where F is a field and x_i are commuting indeterminants satisfying the relations: $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all i and j. Then R is a commutative, semiprimary F-injective ring. But R is not a self-injective ring (see [**10**, Example 5.45]). Thus R is SF-injective, but R is not a small injective ring. Because if R is small injective, then R is self-injective by [**15**, Theorem 3.4], a contradiction.

ii) Let F be a field and assume that $a \mapsto \bar{a}$ is an isomorphism $F \to \overline{F} \subseteq F$, where the subfield $\overline{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$ (see [**10**, Example 2.5]). Then R is a right SP-injective (since R is right P-injective) and semiprimary ring but not a right SF-injective ring. If R is a right SF-injective ring, then R is right F-injective by Theorem 2.10. This is a contradiction by [**10**, Example 5.22]. Moreover, R is not left SP-injective since R is not left mininjective.

iii) The ring of integers $\mathbb{Z}$ is a commutative ring with $J = 0$. So R is small injective, but R is not P-injective.

3. On simple-FJ-injective rings

Definition 3.1. A ring R is called right simple-FJ-injective if every right R-homomorphism from a small, finitely generated right ideal to R with a simple image, can be extended to an endomorphism of R_R.

We have the implications *simple−injective* ⇒ *simple−J-injective* ⇒ *simple−FJ-injective*. But the converses in general are not true. By Example 2.18(i), R is commutative, semiprimary and simple-FJ-injective. But R is not simple-J--injective. In fact, if R is simple-J-injective then R is simple-injective by [**15**, Corollary 3.6]. Hence R is self-injective by [**10**, Theorem 6.47]. This is a contradiction.

Lemma 3.2. *If R is right simple-FJ-injective, then R is right mininjective and a left minannihilator.*

Proof. We can prove it as in Proposition 2.6. $\square$

Lemma 3.3. *A ring R is right simple-FJ-injective a ring if and only if every R-homomorphism $f : I \to R$ extends to $R_R \to R_R$, where I is a small, finitely generated right ideal and $f(I)$ is finitely generated, semisimple.*

Proof. Write $f(I) = \bigoplus_{i=1}^{n} S_i$ where S_i is a simple right ideal. Let $\pi_i : \bigoplus_{i=1}^{n} S_i \to S_i$ be the projection for each i. Since R is right simple-FJ-injective, $\pi_i f = c_i.$, for some $c_i \in R$ and for each i. Thus $f = c.$, with $c = c_1 + \ldots + c_n$. □

Proposition 3.4. *Let R be a right simple-FJ-injective and right Kasch ring. Then*

(1) $\mathrm{rl}(I) = I$ *for every small, finitely generated right ideal I of R.*

(2) $S_r = S_l$.

Proof. By Proposition 2.7. □

In [**20**], a ring R is called right $(I - K) - m$-injective if for any m-generated right ideal $U \leq I$ and any R-homomorphism $f : U_R \to K_R$, $f = c.$, for some $c \in R$, where I, K are two right ideals of R and $m \geq 1$.

Lemma 3.5 ([**20**], Lemma 2.5)**.** *If R is a right $(J, S_r) - 1$-injective, right Kasch and semiregular ring, then $l(J)$ is an essential left ideal of ${}_RR$.*

Lemma 3.6. *Let R be a right simple-FJ-injective and semiregular ring. Then every R-homomorphism $f : K \to R$ extends to $R_R \to R_R$ where K is a finitely generated right ideal and $f(K)$ is simple.*

Proof. Let $f : K \to I$ be an R-homomorphism, where K is a finitely generated right ideal and $f(K)$ is simple. Since R is semiregular, then $K = eR \oplus L$, where $e^2 = e \in R$ and $L \leq J$. So L is a small, finitely generated right ideal of R. It is easy to see that $K = eR \oplus (1-e)L$. Therefore $(1-e)L$ is a small, finitely generated right ideal of R. By hypothesis, there exists an endomorphism g of R_R such that $g(x) = f(x)$ for all $x \in (1 - e)L$. We construct a homomorphism $\varphi : R_R \to R_R$ defined by $\varphi(x) = f(ex) + g((1 - e)x)$ for any $x \in R$. Then $\varphi_{|K} = f$. □

Proposition 3.7. *Let R be a right simple-FJ-injective ring. Then*

(1) *If R is semiregular and e is a local idempotent of R, then* $\mathrm{Soc}(eR)$ *is either* 0 *or simple and essential in eR_R.*

(2) *If R is semiperfect, then the following conditions are equivalent*
   a) $\mathrm{Soc}(eR) \neq 0$ *for each local idempotent e.*
   b) S_r *is finitely generated and essential in R_R.*

Proof. (1) Suppose that $\mathrm{Soc}(eR) \neq 0$ and let aR be a simple right ideal of eR. If $0 \neq b \in eR$ such that $aR \cap bR = 0$, then we construct an R-homomorphism $\gamma : aR \oplus bR \to eR$ by $\gamma(ax + by) = ax$, for all $x, y \in R$. Therefore $\mathrm{Im}\,\gamma = aR$ is simple. By Lemma 3.6, $\gamma = c.$ for some $c \in R$. Let $c' = ece \in eRe$. So $(e - c')a = ea - eca = 0$. On the other hand, $\mathrm{End}(eR_R) \cong eRe$ is local. It implies that c' is invertible in eRe, but $c'b = eceb = ecb = 0$ and so $b = 0$, which is a contradiction. Hence $aR \cap bR \neq 0$, $aR \leq bR$ since aR is simple. Thus $\mathrm{Soc}(eR)$ is simple and essential in eR_R.

(2) If $1 = e_1 + \ldots + e_n$, where the e_i are orthogonal local idempotents, then $S_r = \bigoplus_{i=1}^n \mathrm{Soc}(e_iR)$ and $a) \Rightarrow b)$ follows from (1). The converse is clear. □

Proposition 3.8. *Let R be a semiperfect, right simple-FJ-injective ring with $\mathrm{Soc}(eR) \neq 0$ for each local idempotent $e \in R$. Then:*

(1) $\mathrm{rl}(I) = I$ *for every finitely generated right ideal I of R, so R is left P-injective.*

(2) *R is left and right Kasch.*

(3) $S_r = S_l = r(J) = l(J)$ *is essential in ${}_RR$ and in R_R.*

(4) $J = Z_r = Z_l = r(S) = l(S)$, *with $S_r = S_l = S$.*

(5) *R is left and right finitely cogenerated.*

Proof. (2): by [**12**, Theorem 3.7] and (1) by Proposition 3.4 and [**20**, Lemma 1.4].
(3): $S_r = S_l = S$ is essential in ${}_RR$ and in R_R by Proposition 3.4, Lemma 3.5 and Proposition 3.7. $S = r(J) = l(J)$ because R is left and right Kasch.
(4): follows from (2) and (3).
(5): follows from Proposition 3.7 and [**10**, Theorem 5.31]. □

Remark. There exists a semiprimary and right simple-FJ-injective ring, but it can not be right simple-injective. On the other hand, there is a ring R that is right simple-FJ-injective but not right SP-injective (see [**20**, Example 1.7]).

From the above proposition, we have the following result.

Proposition 3.9. *If R is a right simple-FJ-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$, then R is QF.*

Proof. By [**13**, Theorem 2.1] or [**14**, Lemma 2.11], R is semiprimary. Hence R is left and right mininjective by Proposition 3.8. Thus R is QF. □

Corollary 3.10 ([**14**], Theorem 2.15)**.** *If R is a right simple-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$ then R is QF.*

Recall that a ring R is called right pseudo-coherent if $r(S)$ is finitely generated for every finite subset S of R (see [**3**]). Chen and Ding [**5**] proved that if R is a left perfect, right simple-injective and right (or left) pseudo-coherent ring, then R is QF. They gave a question: If R is a right simple-injective ring which is also right perfect and right (or left) pseudo-coherent, is R a QF ring? The following results are motivated by this question.

Firstly, we have the following result

Lemma 3.11 (Osofsky's Lemma)**.** *If R is a left perfect ring in which J/J^2 is right finitely generated, then R is right Artinian.*

Theorem 3.12. *Assume that R is left perfect, right simple-FJ-injective. If R is right (or left) pseudo-coherent ring, then R is QF.*

Proof. Since R is left perfect, $\mathrm{Soc}(eR) \neq 0$ for each local idempotent $e \in R$. Thus by Proposition 3.8, $J = r(S) = l(S)$ with $S = S_r = S_l = r(J) = l(J)$ is a finitely generated left and right ideal. Hence by hypothesis, R is left (or right) pseudo-coherent, and so J is a finitely generated left (or right) ideal. If J is a finitely generated right R-module, then J/J^2 is too. Consequently, R is right Artinian by Lemma 3.11. If J is a finitely generated left R-module, then J is nilpotent by [**10**, Lemma 5.64], and so R is semiprimary. Hence R is left Artinian by Lemma 3.11. Thus R is QF. □

Corollary 3.13 ([**5**], Theorem 2.6)**.** *Assume that R is left perfect, right simple--injective. If R a is right (or left) pseudo-coherent ring, then R is QF.*

We consider a ring which is right simple-FJ-injective and left pseudo-coherent.

Theorem 3.14. *If R is a right perfect, right simple-FJ-injective and left pseudo--coherent ring then R is QF.*

Proof. Since R is right perfect and left pseudo-coherent, R satisfies DCC on finitely generated left ideals. Hence if $A \subseteq R$, $l(A) = l(A_0)$ for some finite subset A_0 of A. It follows that R satisfies DCC on left annihilators, and hence R has ACC on right annihilators. Therefore R is semiprimary by [**6**, Proposition 1]. Thus R is QF by Theorem 3.12. □

Corollary 3.15. *If R is a right perfect, right simple-injective and left pseudo--coherent ring, then R is QF.*

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