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A TRANSFORMATION FORMULA FOR A SPECIAL BILATERAL BASIC HYPERGEOMETRIC ${}_{12}\psi_{12}$ SERIES

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Abstract. In this short note, we shall make use of decomposition of series to derive a transformation formula for a bilateral basic hypergeometric ${}_{12}\psi_{12}$ series.

1. Introduction

Throughout this note, we shall adopt some definitions and notations from [**1**]. The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n = 1, 2, \cdots ,$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

In this paper, during the process of the computations we shall also make use of the following notation:

$$(a;q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \qquad n = 1, 2, \ldots .$$

For products of q-shifted factorials, we use the short notation

$$(a_1,\ a_2,\ \ldots,\ q_r; q)_n = (a_1;q)_n\,(a_2;q)_n \ldots (a_r;q)_n$$

where n is an integer or infinity. Basic and bilateral basic hypergeometric series are defined by

$${}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

and

$${}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n,$$

respectively.

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In this short note, we make use of the idea decomposition of series to derive a formula for a bilateral basic hypergeometric ${}_{12}\psi_{12}$ series.

2. Main results

In the proof of Theorem 1, we use of the following very-well-poised ${}_8\phi_7$ transformation formula:

(1)
$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & y^{\frac12}, & -y^{\frac12}, & (yq)^{\frac12}, & -(yq)^{\frac12}, & x \\ & a^{\frac12}, & -a^{\frac12}, & aqy^{-\frac12}, & -aqy^{-\frac12}, & aq^{\frac12}y^{-\frac12}, & -aq^{\frac12}y^{-\frac12}, & aq/x \end{matrix}; q; \frac{a^2q}{y^2x}\right]
$$
$$
= \frac{(aq, a^2q/y^2; q)_\infty}{(aq/y, a^2q/y; q)_\infty}\ {}_2\phi_1\left[\begin{matrix} y, & xy/a \\ & aq/x \end{matrix}; q, \frac{a^2q}{y^2x}\right]
$$

provided $|\frac{a^2q}{y^2x}| < 1$, which is equivalent to [**1**, Equation (3.4.7)] by a substitution of variables.

Theorem 1. *For $|q| < 1$ and $|q^3/y^2x| < 1$, we have*

(2)
$$
{}_{12}\psi_{12}\left[\begin{matrix} q^{5/2}, & -q^{5/2}, & y^{1/2}, & qy^{1/2}, & -y^{1/2}, & -qy^{1/2}, & (yq)^{1/2}, \\ q^{1/2}, & -q^{1/2}, & q^2y^{-1/2}, & q^3y^{-1/2}, & -q^2y^{-1/2}, & -q^3y^{-1/2}, & q^{3/2}y^{-1/2}, \end{matrix}\right.
$$
$$
\left.\begin{matrix} q^{3/2}y^{1/2}, & -(yq)^{1/2}, & -q^{3/2}y^{1/2}, & x, & xq \\ q^{5/2}y^{-1/2}, & -q^{3/2}y^{-1/2}, & -q^{5/2}y^{-1/2}, & q^2/x, & q^3/x \end{matrix}; q^2, \frac{q^6}{y^4x^2}\right]
$$
$$
= \frac{(q^2, q^3/y^2; q)_\infty}{(q^2/y, q^3/y; q)_\infty}\ {}_2\phi_1\left[\begin{matrix} y, & xy/q \\ & q^2/x \end{matrix}; q, \frac{q^3}{y^2x}\right].
$$

Proof. We first write out the left-hand side of (1) explicitly:

(3)
$$
\sum_{n=0}^{\infty} \frac{(a, qa^{1/2}, -qa^{1/2}, y^{1/2}, -y^{1/2}, (yq)^{1/2}, -(yq)^{1/2}, x; q)_n}{(q, a^{1/2}, -a^{1/2}, aqy^{-1/2}, -aqy{-1/2}, aq^{1/2}y^{-1/2}, -aq^{1/2}y^{-1/2}, aq/x; q)_n}\left(\frac{a^2q}{y^2x}\right)^n.
$$

Letting $a = q$ in (3) and after some elementary manipulations, we get

$$
\sum_{n=0}^{\infty} \frac{(q^3;q^2)_n(y^{1/2}, -y^{1/2}, (yq)^{1/2}, -(yq)^{1/2}, x; q)_n}{(q;q^2)_n(q^2y^{-1/2}, -q^2y^{-1/2}, q^{3/2}y^{-1/2}, -q^{3/2}y^{-1/2}, q^2/x; q)_n}\left(\frac{q^3}{y^2x}\right)^n
$$
$$
= \sum_{n=0}^{\infty} \frac{(q^3;q^2)_{2n}(y^{1/2}, -y^{1/2}, (yq)^{1/2}, -(yq)^{1/2}, x; q)_{2n}}{(q;q^2)_{2n}(q^2y^{-1/2}, -q^2y^{-1/2}, q^{3/2}y^{-1/2}, -q^{3/2}y^{-1/2}, q^2/x; q)_{2n}}\left(\frac{q^3}{y^2x}\right)^{2n}
$$
$$
+ \sum_{n=0}^{\infty} \frac{(q^3;q^2)_{2n+1}(y^{1/2}, -y^{1/2}, (yq)^{1/2}, -(yq)^{1/2}, x; q)_{2n+1}}{(q;q^2)_{2n+1}(q^2y^{-1/2}, -q^2y^{-1/2}, q^{3/2}y^{-1/2}, -q^{3/2}y^{-1/2}, q^2/x; q)_{2n+1}}\left(\frac{q^3}{y^2x}\right)^{2n+1}
$$
$$
= \sum_{n=0}^{\infty} \frac{(q^3;q^2)_{2n}(y^{1/2}, -y^{1/2}, (yq)^{1/2}, -(yq)^{1/2}, x; q)_{2n}}{(q;q^2)_{2n}(q^2y^{-1/2}, -q^2y^{-1/2}, q^{3/2}y^{-1/2}, -q^{3/2}y^{-1/2}, q^2/x; q)_{2n}}\left(\frac{q^3}{y^2x}\right)^{2n}
$$
$$
+ \sum_{n=-\infty}^{-1} \frac{(q^3;q^2)_{2n}(y^{1/2}, -y^{1/2}, (yq)^{1/2}, -(yq)^{1/2}, x; q)_{2n}}{(q;q^2)_{2n}(q^2y^{-1/2}, -q^2y^{-1/2}, q^{3/2}y^{-1/2}, -q^{3/2}y^{-1/2}, q^2/x; q)_{2n}}\left(\frac{q^3}{y^2x}\right)^{2n}.
$$

According to the definition of bilateral basic hypergeometric series and combining the two sums of above, the consequence is just the left-hand side of (2). By (1) the desired result is immediate. □

Note that the left-hand side of (2) can be written in the following compact form:

$$\sum_{n=-\infty}^{\infty} \frac{(1-q^{1+4n})}{(1-q)} \frac{(y;q)_{4n}}{(y^3/y;q)_{4n}} \frac{(x;q)_{2n}}{(q^2/x;q)_{2n}} \left(\frac{q^3}{y^2x}\right)^{2n}.$$

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References

1. Gasper G. and Rahman M., *Basic Hypergeometric Series*, 2nd Edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.

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