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ON QUADRATIC INTEGRAL EQUATIONS OF URYSOHN TYPE IN FRÉCHET SPACES

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ABSTRACT. In this paper, we investigate the existence of a unique solution on a semiinfinite interval for a quadratic integral equation of Urysohn type in Fréchet spaces using a nonlinear alternative of Leray-Schauder type for contractive maps.

1. INTRODUCTION

In this paper, we establish the existence of the unique solution, defined on a semi-infinite interval $J = [0, +\infty)$ for a quadratic integral equation of Urysohn type, namely

$$(1) \qquad x(t) = f(t) + (Ax)(t) \int_0^T u(t, s, x(s))\,\mathrm{d}s, \qquad t \in J := [0, +\infty),$$

where $f : J \to \mathbb{R}$, $u : J \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are given functions and $A : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is an appropriate operator. Here $C(J, \mathbb{R})$ denotes the space of continuous functions $x : J \to \mathbb{R}$.

Integral equations arise naturally from many applications in describing numerous real world problems, see, for instance, books by Agarwal et al. [**1**], Agarwal and O'Regan [**2**], Corduneanu [**8**], Deimling [**13**], O'Regan and Meehan [**18**] and the references therein. On the other hand, also quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can been countered very often in many applications; see for instance the book by Chandrasekher [**7**] and the research papers by Banas et al. [**3, 4**], Benchohra and Darwish [**6**], Darwish [**9, 10, 11, 12**], Hu et al. [**15**], Kelly [**16**], Leggett [**17**], Stuart [**19**] and the references therein. In [**3**] Banas et al. established the existence of monotonic solutions of a Volterra counter part of equation (1) by means of a technique associated with measure of noncompactness.

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The same technique have been applied to a class of quadratic Urysohn integral equation over an unbounded interval by Banas and Olszowy [**5**].

In this paper, we investigate the question of unique solvability of equation (1). Motivated by the previous papers considered for integral equations on a bounded interval, here we extend these results to semi-infinite intervals for a class of quadratic integral equations. The method we are going to use is to reduce the existence of the unique solution for the quadratic integral equation (1) to the search for the existence of the unique fixed-point of an appropriate operator on the Fréchet space $C(J,\mathbb{R})$ by applying a nonlinear alternative of Leray-Schauder type for contraction maps due to Frigon and Granas [**14**].

2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$, there exists $M_n>0$ such that

$$\|y\|_n\leq M_n \qquad \text{for all } y\in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows : For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$defined by : $x\sim_n y$ if and only if $\|x-y\|_n=0$ for $x,y\in X$. We denote $X^n=(X|_{\sim_n},\|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows: For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and define $Y^n=\{[x]_n : x\in Y\}$. We denote $\overline{Y^n}$, $\mathrm{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1\leq\|x\|_2\leq\ \|x\|_3\leq\dots \qquad \text{for every } x\in X.$$

Definition 2.1 ([**14**])**.** A function $f:X\to X$ is said to be a contraction if for each $n\in\mathbb{N}$ there exists $k_n\in(0,1)$ such that:

$$\|f(x)-f(y)\|_n\leq k_n\ \|x-y\|_n \qquad \text{for all } x,y\in X.$$

Theorem 2.2 ([**14**])**.** *Let Ω be a closed subset of a Fréchet space X such that $0\in\Omega$ and $F:\Omega\to X$ a contraction such that $F(\Omega)$ is bounded. Then either*

(C1) *F has a unique fixed point or*
(C2) *there exist $\lambda\in(0,1)$, $n\in\mathbb{N}$ and $u\in\partial\Omega^n$ such that $\|u-\lambda F(u)\|_n=0$.*

3. Main Theorem

In this section, we will study equation (1) assuming that the following assumptions are satisfied:

(a_1) $f:J\to\mathbb{R}$ is a continuous function.

(a_2) For each $n \in \mathbb{N}$ there exists $L_n > 0$ such that

$$|(Ax)(t) - (A\bar{x})(t)| \leq L_n \, |x(t) - \bar{x}(t)|$$

for each $x,\ \bar{x} \in C(J,\mathbb{R})$ and $t \in [0,n]$.

(a_3) There exist nonnegative constants a and b such that

$$|(Ax)(t)| \leq a + b\,|x(t)|$$

for each $x \in C(J,\mathbb{R})$ and $t \in J$.

(a_4) $u : J \times J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and for each $n \in \mathbb{N}$ there exists a constant $L_n^* > 0$ such that

$$|u(t,s,x) - u(t,s,\bar{x})| \leq L_n^* \, |x - \bar{x}|$$

for all $(t,s) \in [0,n] \times [0,T]$ and $x,\ \bar{x} \in \mathbb{R}$.

(a_5) There exists a continuous nondecreasing function $\psi : J \to (0,\infty)$ and $p \in C(J,\mathbb{R}_+)$ such that

$$|u(t,s,x)| \leq p(s)\ \psi(|x|)$$

for each $(t,s) \in J \times [0,T]$ and $x \in \mathbb{R}$ and moreover there exists a constant $M_n,\ n \in \mathbb{N}$, such that

$$\frac{M_n}{\|f\|_n + T(a + b\,M_n)\ \psi(M_n)\ p^*} > 1, \tag{2}$$

where $p^* = \sup\{p(s) : s \in [0,T]\}$.

Theorem 3.1. *Suppose that hypotheses* $(a_1) - (a_5)$ *are satisfied. If*

$$(a + b\,M_n)\ L_n^*\,T + T L_n\ \psi(M_n) p^* < 1, \tag{3}$$

then the equation (1) *has a unique solution.*

Proof. For every $n \in \mathbb{N}$, we define in $C(J,\mathbb{R})$ the semi-norms by

$$\|y\|_n := \sup\ \{|y(t)| : t \in [0,n]\ \}.$$

Then $C(J,\mathbb{R})$ is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

Transform the problem (1) into a fixed-point problem. Consider the operator $\mathcal{F} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$(\mathcal{F}y)(t) = f(t) + (Ay)(t) \int_0^T u(t,s,y(s))\ \mathrm{d}s, \qquad t \in J.$$

Let y be a possible solution of the problem (1). Given $n \in \mathbb{N}$ and $t \leq n$, then with the view of $(a_1),\ (a_3),\ (a_5)$ we have

$$\begin{aligned} |y(t)| &\leq |f(t)| + |(Ay)(t)| \int_0^T |u(t,s,y(s))|\ \mathrm{d}s \\ &\leq |f(t)| + (a + b|y(t)|) \int_0^T p(s)\ \psi(|y(s)|)\ \mathrm{d}s \\ &\leq \|f\|_n + T(a + b\|y\|_n)\ \psi(\|y\|_n) p^*. \end{aligned}$$

Then

$$\frac{\|y\|_n}{\|f\|_n + T(a + b\|y\|_n)\ \psi(\|y\|_n)p^*} \le 1.$$

From (2) it follows that for each $n \in \mathbb{N}$

$$\|y\|_n \ne M_n.$$

Now, set

$$\Omega = \{y \in C(J, \mathbb{R}) : \|y\|_n \le M_n \text{ for all } n \in \mathbb{N}\}.$$

Clearly, Ω is a closed subset of $C(J, \mathbb{R})$. We shall show that $\mathcal{F} : \Omega \to C(J, \mathbb{R})$ is a contraction operator. Indeed, consider $y, \overline{y} \in \Omega$, for each $t \in [0, n]$ and $n \in \mathbb{N}$, from $(a_2) - (a_4)$ we have

$$\begin{aligned}
|(\mathcal{F}y)(t) - (\mathcal{F}\overline{y})(t)| &\le \left|(Ay)(t) \int_0^T u(t,s,y(s))\,\mathrm{d}s - (A\overline{y})(t) \int_0^T u(t,s,\overline{y}(s))\,\mathrm{d}s\right| \\
&\le \left|(Ay)(t) \int_0^T u(t,s,y(s))\,\mathrm{d}s - (Ay)(t) \int_0^T u(t,s,\overline{y}(s))\,\mathrm{d}s\right| \\
&\quad + \left|(Ay)(t) \int_0^T u(t,s,\overline{y}(s))\,\mathrm{d}s - (A\overline{y})(t) \int_0^T u(t,s,\overline{y}(s))\,\mathrm{d}s\right| \\
&\le |(Ay)(t)| \int_0^T |u(t,s,y(s)) - u(t,s,\overline{y}(s))|\,\mathrm{d}s \\
&\quad + |(Ay)(t) - (A\overline{y})(t)| \int_0^T |u(t,s,\overline{y}(s))|\,\mathrm{d}s \\
&\le (a + b\,|y(t)|)\ L_n^* \int_0^T |y(s) - \overline{y}(s))|\,\mathrm{d}s \\
&\quad + L_n\,|y(t) - \overline{y}(t)| \int_0^T p(s)\ \psi(|\overline{y}(s)|)\,\mathrm{d}s \\
&\le\ [(a + b\,M_n)\ L_n^*\ T + TL_n\ \psi(M_n)p^*]\ \|y - \overline{y}\|_n.
\end{aligned}$$

Therefore,

$$\|\mathcal{F}y - \mathcal{F}\overline{y}\|_n \le [(a + b\,M_n)\ L_n^*\ T + TL_n\ \psi(M_n)p^*]\ \ \|y - \overline{y}\|_n.$$

$\mathcal{F}$ is a contraction for all $n \in \mathbb{N}$. From the choice of Ω there is no $y \in \partial\Omega$ such that $y = \lambda\ \mathcal{F}(y)$ for some $\lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 2.2 does not hold. The nonlinear alternative of Leray-Schauder type [**14**] shows that $(C1)$ holds, and hence we deduce that the operator $\mathcal{F}$ has a unique fixed-point y in Ω which is a solution of Equation (1). This completes the proof. □

Example. Consider the quadratic integral equation of Urysohn type, namely

$$x(t) = \frac{1}{t+2} + \frac{|x(t)|}{1 + |x(t)|} \int_0^1 \frac{1}{t+2} \frac{1}{s+4} x(s)\,\mathrm{d}s, \qquad t \in J := [0, +\infty). \tag{4}$$

Set

$$
\begin{aligned}
f(t) &= \frac{1}{t+2}, && t \in J,\\
q(t) &= \frac{1}{t+2}, && t \in J,\\
p(s) &= \frac{1}{s+4}, && s \in [0,1],\\
(Ax)(t) &= \frac{x(t)}{1+x(t)}, && t \in J \text{ and } x \in C(J,\mathbb{R}_+),\\
\psi(x) &= x, && \text{for each } x \geq 0,\\
u(t,s,x) &= \frac{1}{t+2}\frac{1}{s+4}x, && \text{for each } (t,s) \in J \times [0,1], \text{ and } x \in \mathbb{R}.
\end{aligned}
$$

It is clear that equation (4) can be written as equation (1). Let us show that conditions $(a_1)-(a_5)$ hold. For each $n \in \mathbb{N},\ (t,s) \in [0,n]\times[0,1]$ and $x,\ \bar{x} \in \mathbb{R}$ we have

$$
\begin{aligned}
|u(t,s,x) - u(t,s,\bar{x})| &= |q(t)p(s)x - q(t)p(s)\bar{x}|\\
&\leq \frac{1}{t+2}\frac{1}{s+4}|x-\bar{x}| \leq \frac{1}{8}|x-\bar{x}|.
\end{aligned}
$$

Hence (a_4) is satisfied with $L_n^* = \frac{1}{8}$.

For each $n \in \mathbb{N},\ t \in [0,n],$ and $x,\ \bar{x} \in C([0,n],\mathbb{R}_+)$ we have

$$
|(Ax)(t) - (A\bar{x})(t)| = \left|\frac{x(t)}{1+x(t)} - \frac{\bar{x}(t)}{1+\bar{x}(t)}\right| = \frac{|x(t)-\bar{x}(t)|}{(1+x(t))(1+\bar{x}(t))} \leq |x(t)-\bar{x}(t)|.
$$

Hence (a_2) is satisfied with $L_n = 1$.

For each $n \in \mathbb{N},\ t \in [0,n],$ and $x \in C([0,n],\mathbb{R})$ we have

$$
|(Ax)(t)| = \frac{|x(t)|}{1+|x(t)|} \leq |x(t)|.
$$

Hence (a_3) holds with $a = 0$ and $b = 1$.

A simple calculation shows that conditions (2) and (3) hold for $M_n \in \left(\frac{4-\sqrt{8}}{2}, \frac{4+\sqrt{8}}{2}\right)$ and $M_n \in (0, \frac{8}{3})$, respectively.

Consequently from Theorem 3.1 Equation (4) has a unique solution.

Remark 3.2. Let us mention that our analysis is still applied to the following quadratic integral equations which were widely considered in the literature on bounded intervals and with the measure of noncompactness and appropriate fixed point theorems

$$
x(t) = f(t) + x(t)\int_0^T u(t,s,x(s))\,\mathrm{d}s, \qquad t \in J := [0,+\infty), \tag{5}
$$

and

$$
x(t) = f(t) + g(t,x(t))\int_0^T u(t,s,x(s))\,\mathrm{d}s, \qquad t \in J := [0,+\infty). \tag{6}
$$

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