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DIFFERENTIAL SUBORDINATION FOR MEROMORPHIC MULTIVALENT QUASI-CONVEX FUNCTIONS

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Abstract. We introduce new classes of meromorphic multivalent quasi-convex functions and find some sufficient differential subordination theorems for such classes in punctured unit disk with applications in fractional calculus.

1. Introduction and preliminaries

Let $\Sigma^{+}_{p,\alpha}$ be the class of functions $F(z)$ of the form

$$F(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^{n+\alpha-1}, \qquad \alpha \geq 1, \quad p = 1, 2, \dots,$$

which are analytic in the punctured unit disk $U := \{z \in \mathbb{C},\ 0 < |z| < 1\}$. Let $\Sigma^{-}_{p,\alpha}$ be the class of functions of the form

$$F(z) = \frac{1}{z^p} - \sum_{n=0}^{\infty} a_n z^{n+\alpha-1}, \qquad \alpha \geq 1, \quad a_n \geq 0$$

which are analytic in the punctured unit disk U. Now let us recall the principle of subordination between two analytic functions: Let the functions f and g be analytic in $\triangle := \{z \in \mathbb{C},\ |z| < 1\}$. Then we say that the function f is *subordinate* to g if there exists a Schwarz function w, analytic in $\triangle$ such that

$$f(z) = g(w(z)), \qquad z \in \triangle.$$

We denote this subordination by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z).$$

If the function g is univalent in $\triangle$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\triangle) \subset g(\triangle).$$

Now, let $\phi : \mathbb{C}^3 \times \triangle \to \mathbb{C}$ and let h be univalent in $\triangle$. Assume that p, ϕ are analytic and univalent in $\triangle$. If p satisfies the differential superordination

$$h(z) \prec \phi(p(z)), zp'(z), z^2p''(z); z), \tag{1}$$

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then p is called a solution of the differential superordination. (If f is subordinate to g, then g is called superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1). A univalent function q such that $p \prec q$ for all subordinants p of (1) is said to be the best subordinant.

Let Σ_p^+ be the class of analytic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n, \qquad \text{in } U.$$

And let Σ_p^- be the class of analytic functions of the form

$$f(z) = \frac{1}{z^p} - \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \geq 0, \quad n = 0, 1, \dots \quad \text{in } U.$$

A function $f \in \Sigma_p^+(\Sigma_p^-)$ is meromorphic multivalent starlike if $f(z) \neq 0$ and

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in U.$$

Similarly, the function f is meromorphic multivalent convex if $f'(z) \neq 0$ and

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad z \in U.$$

Moreover, a function f is a called meromorphic multivalent quasi-convex function if there is a meromorphic multivalent convex function g such that

$$-\Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0.$$

A function $F \in \Sigma_{p,\alpha}^+(\Sigma_{p,\alpha}^-)$ such that $F(z) \neq 0$ is called meromorphic multivalent starlike if

$$-\Re\left\{\frac{zF'(z)}{F(z)}\right\} > 0, \qquad z \in U.$$

And the function F is meromorphic multivalent convex if $F'(z) \neq 0$ and

$$-\Re\left\{1 + \frac{zF''(z)}{F'(z)}\right\} > 0, \qquad z \in U.$$

A function $F \in \Sigma_{p,\alpha}^+(\Sigma_{p,\alpha}^-)$ is called a meromorphic multivalent quasi-convex function if there is a meromorphic multivalent convex function G such that $G'(z) \neq 0$ and

$$-\Re\left\{\frac{(zF'(z))'}{G'(z)}\right\} > 0.$$

In the present paper, we establish some sufficient conditions for functions $F \in \Sigma_{p,\alpha}^+$ and $F \in \Sigma_{p,\alpha}^-$ to satisfy

$$-\frac{(z^pF'(z))'}{G'(z)} \prec q(z), \tag{2}$$

where q is a given univalent function in U. Moreover, we give applications for these results in fractional calculus. We shall need the following known results.

Lemma 1.1 ([**1**]). *Let q be convex univalent in the unit disk $\triangle$. Let ψ be a function and number $\gamma \in \mathbb{C}$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right\} > 0.$$

If p is analytic in $\triangle$ and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2 ([**2**]). *Let q be univalent in the unit disk $\triangle$ and let θ be analytic in a domain D containing $q(\triangle)$. If $zq'(z)\theta(q)$ is starlike in $\triangle$ and*

$$z\psi'(z)\theta(\psi(z)) \prec zq'(z)\theta(q(z)),$$

then $\psi(z) \prec q(z)$ and q is the best dominant.

2. Subordination theorems

In this section, we establish some sufficient conditions for subordination of analytic functions in the classes $\Sigma^+_{p,\alpha}$ and $\Sigma^-_{p,\alpha}$.

Theorem 2.1. *Let the function q be convex univalent in U such that $q'(z) \neq 0$ and*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right\} > 0, \qquad \gamma \neq 0. \tag{3}$$

Suppose that $-\frac{(z^pF'(z))'}{G'(z)}$ is analytic in U. If $F \in \Sigma^+_{p,\alpha}$ satisfies the subordination

$$-\frac{(z^pF'(z))'}{G'(z)}\left\{\psi + \gamma\left[\frac{z(z^pF'(z))''}{(z^pF'(z))'} - \frac{zG''(z)}{G'(z)}\right]\right\} \prec \psi q(z) + \gamma z q'(z),$$

then

$$-\frac{(z^pF'(z))'}{G'(z)} \prec q(z),$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z) := -\frac{(z^pF'(z))'}{G'(z)}, \qquad z \in U.$$

It can easily observed that

$$\begin{aligned}\psi p(z) + \gamma z p'(z) &= -\frac{(z^pF'(z))'}{G'(z)}\left\{\psi + \gamma\left[\frac{z(z^pF'(z))''}{(z^pF'(z))'} - \frac{zG''(z)}{G'(z)}\right]\right\}\\ &\prec \psi q(z) + \gamma z q'(z).\end{aligned}$$

Then, using the assumption of the theorem the assertion of the theorem follows by an application of Lemma 1.1. $\square$

Corollary 2.1. *Assume that* (3) *holds. Let the function q be univalent in U. Let $n = 1$, if q satisfies*

$$-\frac{(zF'(z))'}{G'(z)}\left\{\psi+\gamma\left[\frac{z(zF'(z))''}{(zF'(z))'}-\frac{zG''(z)}{G'(z)}\right]\right\}\prec \psi q(z)+\gamma zq'(z),$$

then

$$-\frac{(zF'(z))'}{G'(z)}\prec q(z),$$

and q is the best dominant.

Theorem 2.2. *Let the function q be univalent in U such that $q(z)\neq 0$, $z\in U$, $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $F\in\Sigma^-_{p,\alpha}$ satisfies the subordination*

$$a\left[\frac{z(z^pF'(z))''}{(z^pF'(z))'}-\frac{zG''(z)}{G'(z)}\right]\prec a\frac{zq'(z)}{q(z)},$$

then

$$-\frac{(z^pF'(z))'}{G'(z)}\prec q(z)$$

and q is the best dominant.

Proof. Let the function ψ be defined by

$$\psi(z):=-\frac{(z^pF'(z))'}{G'(z)},\qquad z\in U.$$

By setting

$$\theta(\omega):=\frac{a}{\omega},\qquad a\neq 0,$$

it can be easily observed that θ is analytic in $\mathbb{C}-\{0\}$. By straightforward computation we have

$$\begin{aligned}a\frac{z\psi'(z)}{\psi(z)}&=a\left[\frac{z(z^pF'(z))''}{(z^pF'(z))'}-\frac{zG''(z)}{G'(z)}\right]\\&\prec a\frac{zq'(z)}{q(z)}.\end{aligned}$$

Then, by using the assumption of the theorem, the assertion of the theorem follows by an application of Lemma 1.2. □

Corollary 2.2. *Assume that q is convex univalent in U. Let $p = 1$, if $F\in\Sigma^-_{p,\alpha}$ and*

$$a\left[\frac{z(zF'(z))''}{(zF'(z))'}-\frac{zG''(z)}{G'(z)}\right]\}\prec a\frac{zq'(z)}{q(z)},$$

then

$$-\frac{(zF'(z))'}{G'(z)}\prec q(z)$$

and q is the best dominant.

3. Applications.

In this section, we introduce some applications of section (2) containing fractional integral operators. Assume that $f(z) = \sum_{n=0}^{\infty} \varphi_n z^n$ and let us begin with the following definition.

Definition 3.1 ([**3**])**.** For a function f, the fractional integral of order α is defined by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} \mathrm{d}\zeta; \qquad \alpha > 0,$$

where the function f is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when$(z-\zeta) > 0$. Note that $I_z^\alpha f(z) = f(z) \times \frac{z^{\alpha-1}}{\Gamma(\alpha)}$, for $z > 0$ and 0 for $z \leq 0$ (see [**4**]).

From Definition 3.1, we have

$$I_z^\alpha f(z) = f(z) \times \frac{z^{\alpha-1}}{\Gamma(\alpha)} = \frac{z^{\alpha-1}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \varphi_n z^n = \sum_{n=0}^{\infty} a_n z^{n+\alpha-1}$$

where $a_n := \frac{\varphi_n}{\Gamma(\alpha)}$, for all $n = 0, 1, 2, 3, \ldots$, thus

$$\frac{1}{z^p} + I_z^\alpha f(z) \in \Sigma_{p,\alpha}^+ \qquad \text{and} \qquad \frac{1}{z^p} - I_z^\alpha f(z) \in \Sigma_{p,\alpha}^-(\varphi_n \geq 0).$$

Then we have the following results:

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold, then*

$$-\frac{(z^p(\frac{1}{z^p} + I_z^\alpha f(z))')'}{(\frac{1}{z^p} + I_z^\alpha g(z))'} \prec q(z),$$

where $F(z) = \frac{1}{z^p} + I_z^\alpha f(z)$, $G(z) = \frac{1}{z^p} + I_z^\alpha g(z)$ *and q is the best dominant.*

Theorem 3.2. *Let the assumptions of Theorem 2.2 hold, then*

$$-\frac{(z^p(\frac{1}{z^p} - I_z^\alpha f(z))')'}{(\frac{1}{z^p} - I_z^\alpha g(z))'} \prec q(z),$$

where $F(z) = \frac{1}{z^p} - I_z^\alpha f(z)$, $G(z) = \frac{1}{z^p} - I_z^\alpha g(z)$ *and q is the best dominant.*

Let $F(a, b; c; z)$ be the Gauss hypergeometric function (see [**5**]) defined for $z \in U$ by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where the Pochhammer symbol is defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

We need the following definitions of fractional operators in the Saigo type of fractional calculus (see [**6**],[**7**]).

Definition 3.2. For $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta) \mathrm{d}\zeta,$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon})(z \to 0), \qquad \epsilon > \max\{0, \beta-\eta\} - 1$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

From Definition 3.2 with $\beta < 0$, we have

$$\begin{aligned}
I_{0,z}^{\alpha,\beta,\eta} f(z) &= \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta) \mathrm{d}\zeta \\
&= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n (-\eta)_n}{(\alpha)_n (1)_n} \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} \left(1-\frac{\zeta}{z}\right)^n f(\zeta) \mathrm{d}\zeta \\
&:= \sum_{n=0}^{\infty} B_n \frac{z^{-\alpha-\beta-n}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{n+\alpha-1} f(\zeta) \mathrm{d}\zeta \\
&= \sum_{n=0}^{\infty} B_n \frac{z^{-\beta-1}}{\Gamma(\alpha)} f(\zeta) \\
&:= \frac{\overline{B}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \varphi_n z^{n-\beta-1}
\end{aligned}$$

where $\overline{B} := \sum_{n=0}^{\infty} B_n$. Denote $a_n := \frac{\overline{B}\varphi_n}{\Gamma(\alpha)}$, for all $n = 2, 3, \dots$, and let $\alpha = -\beta$, thus

$$\frac{1}{z^p} + I_{0,z}^{\alpha,\beta,\eta} f(z) \in \Sigma_{p,\alpha}^{+} \qquad \text{and} \qquad \frac{1}{z^p} - I_{0,z}^{\alpha,\beta,\eta} f(z) \in \Sigma_{p,\alpha}^{-}, \qquad (\varphi_n \geq 0).$$

Then we have the following results:

Theorem 3.3. *Let the assumptions of Theorem 2.1 hold, then*

$$-\frac{(z^p(\frac{1}{z^p} + I_{0,z}^{\alpha,\beta,\eta} f(z))')'}{(\frac{1}{z^p} + I_{0,z}^{\alpha,\beta,\eta} g(z))'} \prec q(z), U$$

where $F(z) = \frac{1}{z^p} + I_{0,z}^{\alpha,\beta,\eta} f(z)$, $G(z) = \frac{1}{z^p} - I_{0,z}^{\alpha,\beta,\eta} g(z)$ *and* q *is the best dominant.*

Theorem 3.4. *Let the assumptions of Theorem 2.2 hold, then*

$$-\frac{(z^p(\frac{1}{z^p} - I_{0,z}^{\alpha,\beta,\eta} f(z))')'}{(\frac{1}{z^p} - I_{0,z}^{\alpha,\beta,\eta} g(z))'} \prec q(z),$$

where $F(z) = \frac{1}{z^p} - I_{0,z}^{\alpha,\beta,\eta} f(z)$, $G(z) = \frac{1}{z^p} - I_{0,z}^{\alpha,\beta,\eta} g(z)$ *and* q *is the best dominant.*

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