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STARLIKE AND CONVEXITY PROPERTIES FOR p-VALENT HYPERGEOMETRIC FUNCTIONS

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Abstract. Given the hypergeometric function $F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n$, we place conditions on a, b and c to guarante that $z^p F(a, b; c; z)$ will be in various subclasses of p-valent starlike and p-valent convex functions. Operators related to the hypergeometric function are also examined.

1. Introduction

Let $S(p)$ be the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1, 2, \dots\}) \tag{1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in S(p)$ is called p-valent starlike of order α if $f(z)$ satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{2}$$

for $0 \le \alpha < p$, $p \in N$ and $z \in U$. By $S^*(p, \alpha)$ we denote the class of all p-valent starlike functions of order α. By $S_p^*(\alpha)$ denote the subclass of $S^*(p, \alpha)$ consisting of functions $f(z) \in S(p)$ for which

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha \tag{3}$$

for $0 \le \alpha < p$, $p \in N$ and $z \in U$. Also a function $f(z) \in S(p)$ is called p-valent convex of order α if $f(z)$ satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{4}$$

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for $0 \le \alpha < p$, $p \in N$ and $z \in U$. By $K(p,\alpha)$ we denote the class of all p-valent convex functions of order α. It follows from (2) and (4) that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S(p,\alpha). \tag{5}$$

Also by $K_p(\alpha)$ denote the subclass of $K(p,\alpha)$ consisting of functions $f(z) \in S(p)$ for which

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| < p - \alpha \tag{6}$$

for $0 \le \alpha < p$, $p \in N$ and $z \in U$.

By $T(p)$ we denote the subclass of $S(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \ge 0; \quad p \in N). \tag{7}$$

By $T^*(p,\alpha)$, $T_p^*(\alpha)$, $C(p,\alpha)$ and $C_p(\alpha)$ we denote the classes obtained by taking interesctions, respectively, of the classes $S^*(p,\alpha)$, $S_p^*(\alpha)$, $K(p,\alpha)$ and $K_p(\alpha)$ with the class $T(p)$

$$\begin{aligned} T^*(p,\alpha) &= S^*(p,\alpha) \cap T(p), \\ T_p^*(\alpha) &= S_p^*(\alpha) \cap T(p), \\ C(p,\alpha) &= K(p,\alpha) \cap T(p), \end{aligned}$$

and

$$C_p(\alpha) = K_p(\alpha) \cap T(p).$$

The class $S^*(p,\alpha)$ was studied by Patil and Thakare [**5**]. The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were studied by Owa [**4**].

For $a, b, c \in C$ and $c \ne 0, -1, -2, \dots$, the (Gaussian) hypergeometric function is defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \qquad (z \in U), \tag{8}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdot \ldots \cdot(\lambda+n-1) & (n \in N). \end{cases} \tag{9}$$

The series in (8) represents an analytic function in U and has an analytic continuation throughout the finite complex plane except at most for the cut $[1,\infty)$. We note that $F(a,b;c;1)$ converges for $\mathrm{Re}(a-b-c) > 0$ and is related to the Gamma function by

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{10}$$

Corresponding to the function $F(a,b;c;z)$ we define

$$h_p(a,b;c;z) = z^p F(a,b;c;z). \tag{11}$$

We observe that for a function $f(z)$ of the form (1), we have

$$h_p(a,b;c;z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n. \tag{12}$$

In [**7**] Silverman gave necessary and sufficient conditions for $zF(a,b;c;z)$ to be in $T^*(1,\alpha) = T^*(\alpha)$ and $C(1,\alpha) = C(\alpha)$ and has also examined a linear operator acting on hypergeometric functions. For the other interesting developments for $zF(a,b;c;z)$ in connection with various subclasses of univalent functions, the reader can refer to the works of Carlson and Shaffer [**1**], Merkes and Scott [**3**] and Ruscheweyh and Singh [**6**].

In the present paper, we determine necessary and sufficient conditions for $h_p(a,b;c;z)$ to be in $T^*(p,\alpha)$ and $C(p,\alpha)$. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Main Results

To establish our main results we shall need the following lemmas.

Lemma 1 ([**4**])**.** *Let the function $f(z)$ defined by* (1)*.*

(i) *A sufficient condition for $f(z) \in S(p)$ to be in the class $S_p^*(\alpha)$ is that*

$$\sum_{n=p+1}^{\infty} (n-\alpha)\,|a_n| \le (p-\alpha).$$

(ii) *A sufficient condition for $f(z) \in S(p)$ to be in the class $K_p(\alpha)$ is that*

$$\sum_{n=p+1}^{\infty} \frac{n}{p}(n-\alpha)\,|a_n| \le p-\alpha.$$

Lemma 2 ([**4**])**.** *Let the function $f(z)$ be defined by* (7)*. Then*

(i) *$f(z) \in T(p)$ is in the class $T^*(p,\alpha)$ if and only if*

$$\sum_{n=p+1}^{\infty} (n-\alpha)a_n \le p-\alpha.$$

(ii) *$f(z) \in T(p)$ is in the class $C(p,\alpha)$ if and only if*

$$\sum_{n=p+1}^{\infty} \frac{n}{p}(p-\alpha)a_n \le p-\alpha.$$

Lemma 3 ([**2**])**.** *Let $f(z) \in T(p)$ be defined by* (7)*. Then $f(z)$ is p-valent in U if*

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le p.$$

In addition, $f(z) \in T_p^*(\alpha) \Leftrightarrow f(z) \in T^*(p,\alpha)$, $f(z) \in K_p(\alpha) \Leftrightarrow f(z) \in K(p,\alpha)$ and $f(z) \in S_p^*(\alpha) \Leftrightarrow f(z) \in S^*(p,\alpha)$.

Theorem 1. *If $a,b > 0$ and $c > a+b+1$, then a sufficient condition for $h_p(a,b;c;z)$ to be in $S_p^*(\alpha)$, $0 \le \alpha < p$, is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab}{(p-\alpha)(c-a-p-1)}\right] \le 2. \tag{13}$$

Condition (13) is necessary and sufficient for F_p defined by $F_p(a,b;c;z) = z^p(2-F(a,b;c;z))$ to be in $T^*(p,\alpha)(T_p^*(\alpha))$.

Proof. Since $h_p(a,b;c;z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$, according to Lemma 1(i), we only need to show that

$$\sum_{n=p+1}^{\infty} (n-\alpha)\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \le p-\alpha.$$

Now

$$\sum_{n=p+1}^{\infty} (n-\alpha)\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + (p-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{14}$$

Noting that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ and then applying (10), we may express (14) as

$$\begin{aligned}
&\frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (p-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\
&\quad = \frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c+a)\Gamma(c-b)} + (p-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]\\
&\quad = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{c-a-b-1} + p - \alpha\right] - (p-\alpha).
\end{aligned}$$

But this last expression is bounded above by $p-\alpha$ if and only if (13) holds.

Since $F_p(a,b;c;z) = z^p - \sum_{n=p+1}^{\infty}\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$, the necessity of (13) for F_p to be in $T_p^*(\alpha)$ and $T^*(p,\alpha)$ follows from Lemma 2(i). □

Remark 1. Condition (13) with $\alpha = 0$ is both necessary and sufficient for F_p to be in the class T_p^*.

In the next theorem, we find constraints on a, b and c that lead to necessary and sufficient conditions for $h_p(a,b;c;z)$ to be in the class $T^*(p,\alpha)$.

Theorem 2. *If $a,b > -1$, $c > 0$ and $ab < 0$, then a necessary and sufficient condition for $h_p(a,b;c;z)$ to be in $T^*(p,\alpha)(T_p^*(\alpha))$ is that $c \ge a+b+1-\frac{ab}{p-\alpha}$. The condition $c \ge a+b+1-\frac{ab}{p}$ is necessary and sufficient for $h_p(a,b;c;z)$ to be in T_p^*.*

Proof. Since

$$
\begin{aligned}
h_p(a,b;c;z) &= z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n \\
&= z^p + \frac{ab}{c} \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} z^n \\
&= z^p - \left|\frac{ab}{c}\right| \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} z^n,
\end{aligned}
\tag{15}
$$

according to Lemma 2(i), we must show that

$$
\sum_{n=p+1}^{\infty} (n-\alpha) \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \le \left|\frac{c}{ab}\right| (p-\alpha). \tag{16}
$$

Note that the left side of (16) diverges if $c \le a+b+1$. Now

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (n+p+1-\alpha) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad = \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (p-\alpha)\frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&\quad = \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (p-\alpha)\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]
\end{aligned}
$$

Hence, (16) is equivalent to

$$
\begin{aligned}
&\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[1 + (p-\alpha)\frac{(c-a-b-1)}{ab}\right] \\
&\le (p-\alpha)\left[\frac{c}{|ab|} + \frac{c}{ab}\right] = 0.
\end{aligned}
\tag{17}
$$

Thus, (17) is valid if and only if

$$
1 + (p-\alpha)\frac{(c-a-b-1)}{ab} \le 0,
$$

or, equivalently,

$$
c \ge a+b+1 - \frac{ab}{p-\alpha}.
$$

Another application of Lemma 2(i) when $\alpha = 0$ completes the proof of Theorem 2. ∎

Our next theorems will parallel Theorems 1 and 2 for the p-valent convex case.

Theorem 3. *If $a, b > 0$ and $c > a + b + 2$, then a sufficient condition for $h_p(a, b; c; z)$ to be in $K_p(\alpha)$, $0 \leq \alpha < p$, is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{(2p+1-\alpha)}{p(p-\alpha)}\left(\frac{ab}{c-a-b-1}\right) + \frac{(a)_2(b)_2}{p(p-\alpha)(c-a-b-2)_2}\right] \leq 2. \tag{18}$$

Condition (18) *is necessary and sufficient for $F_p(a, b; c; z) = z^p(2 - F(a, b; c; z))$ to be in $C(p, \alpha)(C_p(\alpha))$.*

Proof. In view of Lemma 1(ii), we only need to show that

$$\sum_{n=p+1}^{\infty}(n-\alpha)\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \leq p(p-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+p+1)(n+p+1-\alpha)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&= \sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + (2p-\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&\qquad + p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&= \sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + (2p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\
&\qquad + p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&= \sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + (2p+1-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\
&\qquad + p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}} + (2p+1-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\
&\qquad + p(p-\alpha)\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.
\end{aligned} \tag{19}$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (19) as

$$\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c+a)\Gamma(c-b)} + (2p+1-\alpha)\frac{ab}{c} \cdot \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + p(p-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].$$

Upon simplification, we see that this last expression is bounded above by $p(p-\alpha)$ if and only if (18) holds. That (18) is also necessary for F_p to be in $C(p,\alpha)(C_p(\alpha))$ follows from Lemma 2(ii). □

Theorem 4. *If $a, b > -1$, $ab < 0$ and $c > a+b+2$, then a necessary and sufficient condition for $h_p(a, b; c; z)$ to be in $C(p,\alpha)(C_p(\alpha))$ is that*

$$(a)_2(b)_2 + (2p+1-\alpha)ab(c-a-b-2) + p(p-\alpha)(c-a-b-2)_2 \geq 0. \tag{20}$$

Proof. Since $h_p(a, b; c; z)$ has the form (15), we see from Lemma 2(ii) that our conclusion is equivalent to

$$\sum_{n=p+1}^{\infty} n(n-\alpha)\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \leq \left|\frac{c}{ab}\right| p(p-\alpha). \tag{21}$$

Note that $c > a+b+2$ if the left-hand side of (21) converges. Writing

$$(n+p+1)(n+p+1-\alpha) = (n+1)^2 + (2p-\alpha)(n+1) + p(p-\alpha),$$

we see that

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} n(n-\alpha)\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}\\
&\quad= \sum_{n=0}^{\infty}(n+p+1)(n+p+1-\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}\\
&\quad= \sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (2p-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\
&\qquad + p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}\\
&\quad= \frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + (2p+1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\
&\qquad + p(p-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\
&\quad= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big[(a+1)(b+1) + (2p+1-\alpha)(c-a-b-2)\\
&\qquad + \frac{p(p-\alpha)}{ab}(c-a-b-2)_2\Big] - \frac{p(p-\alpha)c}{ab}.
\end{aligned}$$

This last expression is bounded above by $\left|\frac{c}{ab}\right| p(p-\alpha)$ if and only if

$$(a+1)(b+1) + (2p+1-\alpha)(c-a-b-2) + \frac{p(p-\alpha)}{ab}(c-a-b-2)_2 \leq 0,$$

which is equivalent to (20). □

Putting $p = 1$ in Theorem 4, we obtain the following corollary.

Corollary 1. *If $a, b > -1$, $ab < 0$ and $c > a+b+2$, then a necessary and sufficient condition for $h_1(a, b; c; z)$ to be in $C(1, \alpha)(C(\alpha))$ is that*

$$(a)_2(b)_2 + (3-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geq 0.$$

Remark 2. We note that Corollary 1 corrects the result obtained by Silverman [**7**, Theorem 4].

3. Integral Operator

In this section, we obtain similar results in connection with a particular integral operator $G_p(a, b; c; z)$ acting on $F(a, b; c; z)$ as follows

$$\begin{aligned} G_p(a,b;c;z) &= p\int_0^z t^{p-1}F(a,b;c;z)\mathrm{d}t \\ &= z^p + \sum_{n=1}^{\infty}\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}z^{n+p}. \end{aligned} \tag{22}$$

We note that $\frac{zG'_p}{p} = h_p$.

Theorem 5.

(i) *If $a, b > 0$ and $c > a+b$, then a sufficient condition for $G_p(a, b; c; z)$ defined by (22) to be in $S^*(p)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c)\Gamma(c-b)} \leq 2. \tag{23}$$

(ii) *If $a, b > -1, c > 0$, and $ab < 0$, then $G_p(a, b; c; z)$ defined by (22) is in $T(p)$ or $S(p)$ if only if $c > \max\{a, b\}$.*

Proof. Since

$$G_p(a,b;c;z) = z^p + \sum_{n=1}^{\infty}\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}z^{n+p},$$

we note that

$$\begin{aligned} \sum_{n=1}^{\infty}(n+p)\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n} &= p\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= p\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right] \end{aligned}$$

is bounded above by p if and only if (23) holds.

To prove (ii), we apply Lemma 3 to

$$G_p(a,b;c;z) = z^p - \frac{|ab|}{c}\sum_{n=p+1}^{\infty}\left(\frac{p}{n}\right)\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}z^n.$$

It suffices to show that

$$\sum_{n=p+1}^{\infty}\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \le \frac{c}{|ab|}$$

or, equivalently,

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} = \frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \le \frac{c}{|ab|}.$$

But this is equivalent to

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \ge -1,$$

which is true if and only if $c > \max\{a,b\}$. This completes the proof of Theorem 5. □

Now $G_p(a,b;c;z) \in K_p(\alpha)(K(p,\alpha))$ if and only if

$$\frac{z}{p}G_p'(a,b;c;z) = h_p(a,b;c;z) \in S_p^*(\alpha)(S^*(p,\alpha)).$$

This follows upon observing that $\dfrac{zG_p'}{p} = h_p$, $\dfrac{z}{p}G_p'' = h_p' - \dfrac{1}{p}G_p'$, and so

$$1 + \frac{zG_p''}{G_p} = \frac{zh_p'}{h_p}.$$

Thus any p-valent starlike about h_p leads to a p-valent convex about G_p. Thus from Theorems 1 and 2, we have

Theorem 6.

(i) *If $a,b > 0$ and $c > a+b+1$, then a sufficient condition for $G_p(a,b;c;z)$ defined in Theorem 5 to be in $K_p(\alpha)(0 \le \alpha < p)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{ab}{(p-\alpha)(c-a-b-1)}\right] \le 2.$$

(ii) *If $a,b > -1, ab < 0$, and $c > a+b+2$, then a necessary and sufficient condition for $G_p(a,b;c;z)$ to be in $C(p,\alpha)(C_p(\alpha))$ is that*

$$c \ge a+b+1 - \frac{ab}{(p-\alpha)}.$$

Remark 3. Putting $p = 1$ in all the above results, we obtain the results obtained by Silverman [**7**].

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