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HOMOROOT INTEGER NUMBERS

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Abstract. In this paper we first define homorooty between two integer numbers and study some of their properties. Thereafter we state some applications of the homorooty in studying and solving some Diophantine equations and systems, as an interesting and useful elementary method. Also by the homorooty, we state and prove the necessary and sufficient conditions for existence of finite solutions in a special case of the quartic equation and evaluate the bounds of its solutions.

1. Preliminaries

We first introduce a new notation and definition.

Definition 1.1. We say that two integer numbers a, b are homoroot if there exist integer numbers r_1, r_2 (the root of a, b) such that $a = r_1 + r_2$ and $b = r_1 r_2$. Two homoroot integer numbers a, b will be denoted by $\langle a, b\rangle \to \mathbb{Z}\langle r_1, r_2\rangle$ or simply by $\langle a, b\rangle \to \mathbb{Z}$.

By $\langle a, b\rangle \to \mathbb{N}$ we mean $\langle a, b\rangle \to \mathbb{Z}\langle r_1, r_2\rangle$ and $\{a, b, r_1, r_2\} \subseteq \mathbb{N}$. Thus if $a, b \in \mathbb{N}$ and $\langle a, b\rangle \to \mathbb{Z}$, then $\langle a, b\rangle \to \mathbb{N}$. The following properties for homoroot integers hold. We need these basic properties for studying the homorooty and solving some indeterminate equations.

(I)

$$\langle a, a+b\rangle \to \mathbb{Z} \Longleftrightarrow \langle a-2, b+1\rangle \to \mathbb{Z},$$
$$\langle a, -a+b\rangle \to \mathbb{Z} \Longleftrightarrow \langle a+2, b+1\rangle \to \mathbb{Z}.$$

(II) (The homorooty inequalities) Let b be a non-zero integer. Then

(a) $\langle a, b\rangle \to \mathbb{Z} \Longrightarrow |a| \leq |b+1|$.

(b) If $\langle a, b\rangle \to \mathbb{Z}$ and $|a| \neq |\frac{b}{i} + i|$, for $i = 1, \cdots, n \leq \sqrt{|b|}$, then $|a| < |\frac{b}{n} + n|$.

(c) moreover if $a, b \in \mathbb{N}$, then

$$\langle a, b\rangle \to \mathbb{N} \Longrightarrow 2\sqrt{b} \leq a \leq b+1.$$
$$\langle a, a+b\rangle \to \mathbb{N} \Longrightarrow a \leq b+4.$$
$$\langle a, -a+b\rangle \to \mathbb{Z} \Longrightarrow a \leq b.$$

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(III) (The homorooty lemma) For all integers a, b with $b \neq 0$, the following statements are equivalent:

(a) $\langle a, b\rangle \to \mathbb{Z}$,

(b) The equation $x^2 - ax + b$ has an integer root,

(c) $\langle \lambda a, \lambda^2 b\rangle \to \mathbb{Z}$ for every integer $\lambda \neq 0$,

(d) $a = r + \frac{b}{r}$ for an integer r such that $r|b$ and $1 \leq |r| \leq \sqrt{|b|}$,

(e) $\langle \lambda_0 a, \lambda_0^2 b\rangle \to \mathbb{Z}$ for an integer $\lambda_0 \neq 0$,

(f) $a^2 - 4b$ is a square integer,

(g) $\langle -a, b\rangle \to \mathbb{Z}$.

(IV) We have $\langle a, a-1\rangle \to \mathbb{Z}$ and $\langle a, 0\rangle \to \mathbb{Z}$, $\langle 0, a^2\rangle \to \mathbb{Z}$ for every a.

To explicate some of the above properties, note that if $a = r_1 + r_2$ and $b = r_1 r_2 \neq 0$, then $a = r_2 + \frac{b}{r_2}$ and $r_2|b$. Putting $r = r_2$ and considering $a = r + \frac{b}{r} = b/r + \frac{b}{b/r}$, there exists an integer $r|b$ such that $1 \leq |r| \leq \sqrt{|b|}$ and $a = \frac{b}{r} + r$. Therefore if $\langle a, b\rangle \to \mathbb{Z}$ and $b \neq 0$, then $|a| = \left|\frac{b}{i} + i\right|$ for an integer i such that $1 \leq i \leq \sqrt{|b|}$ (because $|a| = \left|\frac{b}{r} + r\right| = \left|\frac{b}{|r|} + |r|\right|$). On the other hand for all integers $b \neq 0$ and $1 < n \leq \sqrt{|b|}$ we have

$$|b+1| > \left|\frac{b}{2} + 2\right| > \cdots > \left|\frac{b}{n} + n\right|.$$

Therefore (II)-(a) and (II)-(b) are proved.

If $\langle a, -a+b\rangle \to \mathbb{Z}$, then $a = t_1 + t_2$, and $-a + b = t_1 t_2$ for some integers t_1, t_2. So $a+2 = (t_1+1)+(t_2+1)$ and $b+1 = (t_1+1)(t_2+1)$ thus $\langle a+2, b+1\rangle \to \mathbb{Z}$. Therefore if $a, b \in \mathbb{N}$, then $\langle a, -a+b\rangle \to \mathbb{Z}$ implies $\langle a+2, b+1\rangle \to \mathbb{N}$ so $a+2 \leq (b+1)+1$ (by (II)-(a)) and so $a \leq b$. Also if $a^2 - 4b = c^2$ for an integer c, then $4b = (a+c)(a-c)$ so $\langle 2a, 4b\rangle \to \mathbb{Z}$ and so $\langle a, b\rangle \to \mathbb{Z}$.

Now we determine all homoroot symmetric integer pairs.

Lemma 1.2 (The homoroot symmetric numbers)**.** *All integer pairs that agree in relations $\langle a, b\rangle \to \mathbb{Z}$ and $\langle b, a\rangle \to \mathbb{Z}$ are in the following forms*

$$(a, b) \in \{(4,4), (6,5), (5,6), (r, -r-1), (-r-1, r), (0, -r^2), (-r^2, 0)\},$$

where $r \in \mathbb{Z}$.

Proof. Let $\langle a, b\rangle \to \mathbb{Z}$ and $\langle b, a\rangle \to \mathbb{Z}$. If $ab \neq 0$, then $|a| \leq |b+1|$ and $|b| \leq |a+1|$ (by the homorooty inequality). So $|a| - 1 \leq |b| \leq |a| + 1$ and therefore $b = \pm a \pm 1$ or $b = \pm a$. Now by the elementary properties of the homorooty, it can be shown that (a, b) has one of the above forms (if $ab = 0$, then $a = -r^2$ or $b = -r^2$). □

2. Application of the homorooty in solving some Diophantine equations

In this section we introduce some applications of the homorooty for studying some quartic equations and systems as a useful elementary method.

When we say (x_0, y_0) is a solution of the indeterminate equation $f(x, y, z) = 0$, it means that there exists z_0 such that (x_0, y_0, z_0) is a solution of the equation. Sometimes by finding the values x, y of this equation, the value z is easily gained by simple algebraic operations. Hence, in these cases we refrain from finding the value z and write the solution by the form (x_0, y_0). We say that the equation $f(x, y, z) = 0$ is symmetric relative to x, y if $f(x_0, y_0, z_0) = 0$ implies $f(y_0, x_0, z_0) = 0$, in the symmetric equations, (x_0, y_0) is a solution if and only if (y_0, x_0) is so, thus we consider only one of the cases. These notes are discussed in equations and systems with more variables similarly.

Lemma 2.1. (i) *Let d be a positive integer constant. The general solution of the equation $x^2 - dy = z^2$ (The d-homorooty equation) is $(x = \frac{d_1t_1+d_2t_2}{2}, y = t_1t_2)$, where d_1, d_2 are all positive integers such that $d_1d_2 = d$ and t_1, t_2 are all integers for which $d_1t_1 + d_2t_2$ is even. Specially if $d = p$ is an odd prime number, then the above form can be written as:*

$$(r_1 + pr_2, 4r_1r_2)\ , \quad \left(r_1 + pr_2 + \frac{p+1}{2}, 4r_1r_2 + 2r_1 + 2r_2 + 1\right),$$

where r_1, r_2 run over all integers. Also if $d = 4$ (the homorooty equation), then $(x = r_1 + r_2, y = r_1r_2)$, where $r_1,\ r_2 \in \mathbb{Z}$.

(ii) *The general solution of the system*

$$\begin{cases} x^2 - 4y = z^2 \\ y^2 - 4x = w^2 \end{cases},$$

(The homorooty system) is

$$(x, y) = (4, 4), (6, 5), (r, -r - 1), (0, -r^2),$$

where $r \in \mathbb{Z}$ (up to symmetry).

(iii) *The only nonzero solution ($xy \neq 0$) of the system*

$$\begin{cases} x^2 - dy = z^2 \\ y^2 - dx = w^2 \end{cases},$$

(The d-homorooty system) for $d = \pm1, \pm2$ is $(x, y) = (d, d)$ and the general solution of the system for $d = -4$ is

$$(x, y) = (-4, -4), (-6, -5), (-r, r + 1), (0, r^2),$$

where $r \in \mathbb{Z}$.

(iv) *The general solution of the equation $x^2y^2 - 4x - 4y - z^2 = 0$ is*

$$(x, y) = (2, 2), (2, 3), (1, 5), (r, -r), (-1, -r), (0, -r^2),$$

where $r \in \mathbb{Z}$.

Proof. (i) It is clear, by (III)-(f) and this fact that $x_0^2 - dy_0 = z_0^2$ implies $\langle 2x_0, dy_0\rangle \to \mathbb{Z}$.

(ii) Apply (III)-(f) and Lemma 1.2.

(iii) Multiplying the system by $(4/d)^2$ and putting $X = 4x/d$, $Y = 4y/d$ the claim (by part (ii)) can be proved.

(iv) Put $X = xy$ and $Y = x + y$, since always $\langle Y, X\rangle \to \mathbb{Z}$ the equation is equivalent to the system

$$\begin{cases} X^2 - 4Y = z^2 \\ Y^2 - 4X = w^2 \end{cases},$$

then get the result, from part (ii).

□

The Diophantine equation in the following theorem is a special case of the quartic equation

$$\sum_{r,s=0}^{2} a_{r,s} x^r y^s = dz^2.$$

There is no general solution for the quartic equation as you can see in [**1**]. Now as an application of the homorooty, we state and prove the necessary and sufficient conditions for existence of finitely many solutions in the indeterminate equation and evaluate the bounds of its solutions.

Theorem 2.2. *The equation $x^2y^2 - \alpha x - \beta y = z^2$ where α, β are integer constants with $\alpha\beta \neq 0$ is given. If there is no integer γ satisfying the conditions, $\alpha\beta = 2\gamma^3$ and $2\gamma|\beta$ or $\alpha\beta = 2\gamma^3$ and $2\gamma|\alpha$, then the equation has finitely many nontrivial solutions $(\alpha x + \beta y \neq 0, xy \neq 0)$. Moreover, if (x_0, y_0) is a nontrivial solution of the equation (in this case), then*

$$|x_0| \leq 1/4(|\alpha\beta|\alpha^2 + 2\alpha^2 + 2|\alpha|), \quad \textit{if } |\alpha| \neq 1,$$
$$|y_0| \leq 1/4(|\alpha\beta|\beta^2 + 2\beta^2 + 2|\beta|), \quad \textit{if } |\beta| \neq 1.$$

In case $|\alpha| = 1$ we have $|x_0| \leq |\beta| + 1$, and if $|\beta| = 1$, we have $|y_0| \leq |\alpha| + 1$. The converse of the theorem is also valid.

Proof. Suppose (x_0, y_0) to be a nontrivial solution of the equation. So $\langle 2x_0y_0, \alpha x_0 + \beta y_0\rangle \to \mathbb{Z}$. Hence applying the homorooty inequality we have

$$|2x_0y_0| \leq |\alpha x_0 + \beta y_0 + 1| \leq |\alpha||x_0| + |\beta||y_0| + 1.$$

Therefore

$$(2|x_0| - |\beta|)(2|y_0| - |\alpha|) \leq |\alpha\beta| + 2.$$

Consider the following cases:

i) $|x_0| > |\beta|/2, |y_0| > |\alpha|/2$. Considering the above inequality we have

$$2|x_0| - |\beta| \leq |\alpha\beta| + 2\ ,\ 2|y_0| - |\alpha| \leq |\alpha\beta| + 2.$$

Therefore

$$|x_0| \leq 1/2(|\alpha\beta| + |\beta|) + 1\ ,\ |y_0| \leq 1/2(|\alpha\beta| + |\alpha|) + 1.$$

ii) $|x_0| \leq |\beta|/2$. We know that there exists z_0 such that

$$x_0^2y_0^2 - \alpha x_0 - \beta y_0 = z_0^2,$$

so there is $w_0 \geq 0$ with

$$\Delta = \beta^2 + 4\alpha x_0^3 + 4x_0^2z_0^2 = w_0^2.$$

Therefore

$$(w_0 - 2x_0z_0)(w_0 + 2x_0z_0) = \beta^2 + 4\alpha x_0^3.$$

But $\beta^2 + 4\alpha x_0^3 \neq 0$ (if it is not so, we have $(\alpha\beta/2)^2 = (-\alpha x_0)^3$, thus there exists an integer γ such that $\alpha\beta/2 = \gamma^3$, i.e, $\alpha\beta = 2\gamma^3$ so $\beta = -2\gamma x_0$, i.e, $2\gamma|\beta$, but it is a contradiction) so we have

$$w_0 \leq \max\{w_0 - 2x_0z_0, w_0 + 2x_0z_0\} \leq |\beta^2 + 4\alpha x_0^3|,$$

on the other hand from $y_0 = \dfrac{\beta \pm w_0}{2x_0^2}$ we get

$$2|y_0|x_0^2 = |\beta \pm w_0| \leq |\beta| + |w_0| \leq |\beta| + \beta^2 + 4|\alpha|(|\beta|/2)^3.$$

Therefore

$$|y_0| \leq 1/4(|\alpha\beta|\beta^2 + 2\beta^2 + 2|\beta|).$$

iii) $|y_0| \leq |\alpha/2|$. This case is similar to the case (ii)

$$\alpha^2 + 4\beta y_0^3 \neq 0, \qquad |x_0| \leq 1/4(|\alpha\beta|\alpha^2 + 2\alpha^2 + 2|\alpha|).$$

But for $|\alpha| \neq 1$ we have

$$|\beta/2| \leq 1/2(|\alpha\beta| + |\beta|) + 1 \leq 1/4(|\alpha\beta|\alpha^2 + 2\alpha^2 + 2|\alpha|),$$

and for $|\beta| \neq 1$ we have

$$|\alpha/2| \leq 1/2(|\alpha\beta| + |\alpha|) + 1 \leq 1/4(|\alpha\beta|\beta^2 + 2\beta^2 + 2|\beta|).$$

Therefore the theorem is proved.

Now suppose that there exists an integer γ with $\alpha\beta = 2\gamma^3$ and that $2\gamma|\alpha$ or $2\gamma|\beta$. For $r \in \mathbb{Z}$ at least one of the couples $(x_* = \frac{-\beta}{2\gamma} = \frac{-\gamma^2}{\alpha}, y_* = r)$ or $(x_* = r, y_* = \frac{-\alpha}{2\gamma} = \frac{-\gamma^2}{\beta})$, gives us infinitely many nontrivial integer solutions, and hence the converse of the theorem is also proved. □

Note. From the proof of the above theorem it is concluded that the number of solutions of the indeterminate equation $x^2y^2 - \alpha x - \beta y = z^2$, such that

$$\beta^2 + 4\alpha x^3 \neq 0, \qquad \alpha^2 + 4\beta y^3 \neq 0$$

is finite.

The following results one obtained from Theorem 2.2 and the above note.

Corollary 2.3. If α and β are positive integer numbers (resp. negative integer numbers), then the equation has finitely many positive integer solutions (resp. negative integer solutions).

Corollary 2.4. If $\alpha\beta$ is odd, then the equation has finitely many nontrivial integer solutions.

Now we want to study a more general system of equations than the homorooty system (which we call (α, β)-homorooty system).

Lemma 2.5. *The pair* (x_0, y_0) *is a solution of the system*

$$\begin{cases} x^2 - \alpha y = z^2 \\ y^2 - \beta x = w^2 \end{cases},$$

(α, β are integer constants with $\alpha\beta \neq 0$) if and only if there exists a solution of the form $(X = 2X_0, Y = 2Y_0)$ for the equation

$$X^2Y^2 - 4\beta\alpha^2(X+Y) = Z^2 \qquad \textit{with } \ 2x_0 = X_0 + Y_0, \ \alpha y_0 = X_0Y_0.$$

Proof. Assume (x_0, y_0) to be a solution of the system. Then $\langle 2x_0, \alpha y_0\rangle \to \mathbb{Z}$, i.e, there exist X_0, Y_0 such that $2x_0 = X_0 + Y_0$ and $\alpha y_0 = X_0Y_0$. Now replacing x_0, y_0 in the system, it gives

$$\left(\frac{X_0Y_0}{\alpha}\right)^2 - \beta\left(\frac{X_0+Y_0}{2}\right) = w_0^2 \Longleftrightarrow (2X_0)^2(2Y_0)^2 - 4\beta\alpha^2(2X_0+2Y_0) = (4\alpha w_0)^2.$$

Now, suppose $(2X_0, 2Y_0)$ to be a solution of the equation with $2|X_0+Y_0$, $\alpha|X_0Y_0$ $(X_0 + Y_0 = 2x_0, X_0Y_0 = \alpha y_0)$. So there exists z_0 such that

$$(2X_0)^2(2Y_0)^2 - 4\beta\alpha^2(2X_0 + 2Y_0) = z_0^2.$$

Therefore $(4\alpha)^2|z_0^2$ thus $z_0 = 4\alpha w_0$ (for some w_0). So we have $x_0^2 - \alpha y_0 = (\frac{X_0-Y_0}{2})^2$ and $y_0^2 - \beta x_0 = w_0^2$. □

Theorem 2.6. *The system*

$$\begin{cases} x^2 - \alpha y = z^2 \\ y^2 - \beta x = w^2 \end{cases},$$

where $\alpha\beta \neq 0$, has finitely many nonzero solutions $(x_0y_0 \neq 0)$ if and only if there is no integer γ with

$$\beta\alpha^2 = (2\gamma)^3, \ 4\gamma|\alpha\beta, \ \gcd(2\gamma, \alpha)|\gamma^2.$$

Proof. Assume that the system has infinitely many nonzero solutions. Considering the Lemma 2.5, the equation $X^2Y^2 - 4\beta\alpha^2(X+Y) = Z^2$ has infinitely many nonzero solutions of the form $(2X_*, 2Y_*)$, where $2|X_* + Y_*$ and $\alpha|X_*Y_*$. Therefore there exists at least one integer couple $(2X_*, 2Y_*) = (2X_0, 2Y_0)$ such that

$$(4\beta\alpha^2)^2 + 4(4\beta\alpha^2)(2X_0)^3 = 0 \quad \text{or} \quad (4\beta\alpha^2)^2 + 4(4\beta\alpha^2)(2Y_0)^3 = 0,$$

(by considering the previous note). So we have

i) $(2X_0)^2(2Y_0)^2 - 4\beta\alpha^2(2X_0 + 2Y_0) = (4\alpha r_0)^2$,
ii) $2|X_0 + Y_0$ and $\alpha|X_0Y_0$,
iii) $\beta\alpha^2 = (-2X_0)^3$ or $\beta\alpha^2 = (-2Y_0)^3$.

Now considering (ii) we infer that

$$X_0 + Y_0 = 2v_0;, \ X_0Y_0 = \alpha u_0 \Longrightarrow \alpha u_0 - 2X_0v_0 = -X_0^2,$$

so

$$\gcd(\alpha, -2X_0)|X_0^2.$$

If $\beta\alpha^2 = (-2X_0)^3$, then according to (i), we get $(4X_0Y_0 + 8X_0^2)^2 = (4\alpha r_0)^2$, hence $4\alpha | 4X_0Y_0 + 8X_0^2$ and so (ii) guarantees that $\alpha | 2X_0^2$ and so $-4X_0 | \alpha\beta$ (because, $\frac{\alpha\beta}{-4X_0} = \frac{2X_0^2}{\alpha}$). Therefore putting $\gamma = -X_0$, we get

$$\beta\alpha^2 = (2\gamma)^3, \quad 4\gamma | \alpha\beta, \quad \gcd(2\gamma, \alpha) | \gamma^2.$$

In case $\beta\alpha^2 = (-2Y_0)^3$, we can reach the above result likewise.

Conversely, suppose that there exists an integer γ which satisfies the above conditions, therefore we can see that the pairs $(x_* = \frac{\gamma^2 - \alpha r_*}{2\gamma},\ y_* = r_* - \frac{\alpha\beta}{4\gamma} = r_* - \frac{2\gamma^2}{\alpha})$, where r_* runs over the set of all solutions of the linear indeterminate equation $2\gamma t + \alpha r = \gamma^2$ $(r = r_*)$, give us infinitely many nonzero solutions for the system. $\square$

Note. According to the above theorem we have $\beta\alpha^2 = (2\gamma)^3$, $4\gamma | \alpha\beta$ and $\gcd(2\gamma, \alpha) | \gamma^2$ if and only if $\beta^2\alpha = (2\lambda)^3$, $4\lambda | \alpha\beta$ and $\gcd(2\lambda, \beta) | \lambda^2$ (for some γ, λ).

Corollary 2.7. If α and β are positive (negative) integer numbers, then the $\alpha\beta$-homorooty system has finitely many integer solutions on $\mathbb{N}$ (negative integers).

Corollary 2.8. If $4 \nmid \alpha\beta$, then the $\alpha\beta$-homorooty system has finitely many nonzero integer solutions.

Here we study another generalization of the homorooty system.

Theorem 2.9. *The following system (d-homorooty n-cyclic system)*

i) for $d = 1, 2, 4$ has only finitely many positive integer solutions and does not have any negative integer solution and (in this case) $d \leq x_k \leq \frac{n+1}{2} d$ for $1 \leq k \leq n$.

ii) for $d = -1, -2, -4$ has only finitely many negative integer solutions and does not have any positive integer solution and (in this case) $\frac{n+1}{2} d \leq x_k \leq d$ for $1 \leq k \leq n$.

$$\begin{cases} x_1^2 - dx_2 = y_1^2 \\ x_2^2 - dx_3 = y_2^2 \\ \cdots \\ x_{n-1}^2 - dx_n = y_{n-1}^2 \\ x_n^2 - dx_1 = y_n^2 \end{cases}$$

Proof. Suppose $d = 4$ (for $d = 4$ the system is called "homorooty n-cyclic system"), if $n = 2$, then it is clear, by Lemma . Let $n > 2$ and $(a_1, \cdots, a_n)$ be a positive integer solution of the homorooty n-cyclic system, therefore $\langle a_{k-1}, a_k \rangle \to \mathbb{N}$, $\langle a_n, a_1 \rangle \to \mathbb{N}$, for all $2 \leq k \leq n$, thus $a_{k-1} \leq a_k + 1$, $a_n \leq a_1 + 1$ and so

$$a_1 - k + 1 \leq a_k \leq a_1 + n - k + 1,$$

for all $2 \leq k \leq n$, i.e.,

$$a_1 - 1 \leq a_2 \leq a_1 + n - 1, \qquad a_1 - 2 \leq a_3 \leq a_1 + n - 2, \cdots,$$

$$(*) \qquad a_1 - n + 1 \leq a_n \leq a_1 + 1.$$

By $(*)$ we consider two cases:

Case 1. $a_2 = a_1 - 1$. If $a_k = a_1 - k + 1$, for all $2 \leq k \leq n$, then

$$\langle a_n = a_1 - n + 1, (a_1 - n + 1) + n - 1 \rangle \to \mathbb{N},$$

thus $a_1 - n + 1 \leq n + 3$ (by (II)-(c) of Section 1), i.e., $a_1 \leq 2n + 4$, and if there exist a (least) natural k_0 such that $a_{k_0} \neq a_1 - k_0 + 1$ (clearly $k_0 \geq 2$), then (by $(*)$) there exist an i such that $2 \leq i \leq n + 1$ and $a_{k_0} = a_1 - k_0 + i$. Since $a_{k_0 - 1} = a_1 - k_0 + 2$ we have $\langle a_1 - k_0 + 2, a_1 - k_0 + i \rangle \to \mathbb{N}$, i.e., $\langle a_1 - k_0 + 2, (a_1 - k_0 + 2) + i - 2 \rangle \to \mathbb{N}$, so $a_1 - k_0 + 2 \leq i + 2 \leq n + 3$ thus $a_1 \leq n + k_0 + 1 \leq 2n + 1$.

Case 2. $a_2 \neq a_1 - 1$. By $(*)$ we have $a_2 = a_1 + i$ where $0 \leq i \leq n - 1$ so $< a_1, a_1 + i > \to \mathbb{N}$ thus $a_1 \leq i + 4 \leq n + 3$.

On the other hand since $a_1^2 - 4a_2 \geq 0$, $a_2^2 - 4a_3 \geq 0, \cdots, a_{n-1}^2 - 4a_n \geq 0$ and $a_n^2 - 4a_1 \geq 0$, then combining them we get

$$2\sqrt{a_1} < a_n < \frac{1}{2^{2^n - 2}} a_1^{2^{n-1}}.$$

So $a_1 \geq 4$. Therefore we have proved that $4 \leq a_1 \leq 2n + 2$ but since this system is symmetric (circle symmetric with respect to $x_1, \cdots, x_n$), we have $4 \leq a_k \leq 2n + 2$, for all $1 \leq k \leq n$.

Now let $(a_1, \cdots, a_n)$ be a negative solution of the system so we have $-a_{k-1} \leq -a_k - 1$ and $-a_n \leq -a_1 - 1$, for all $2 \leq k \leq n$, thus $-a_1 \leq -a_n - n + 1 \leq -a_1 - n$ so $n \leq 0$, that is a contradiction.

For $d = \pm 1, \pm 2, -4$, multiplying the system by $(4/d)^2$ and putting $X_k = 4x_k/d$, $Y_k = 4y_k/d$, prove the claims (by the previous part). $\square$

Note. Since $(2n+2, 2n+1, \cdots, n+3)$ and $(4, \cdots, 4)$ are positive solutions of the homorooty n-cyclic system, then the bounds of the solutions in the above theorem are their best bounds.

Remark 2.1. We discussed a lot of equations. In case thet constant $(d, n, \alpha, \beta, \cdots)$ are determined, equations can be solvable very well by the procedure of the proofs of their related theorems. For example, all solutions of the homorooty 3-cyclic system are

$$(4, 4, 4), (8, 7, 6), (4, -5, 4), (3, -4, 4), (0, -r_1^2, -r_2 r_1^2 - r_2^2),$$

and the general solution of the equation $x^2 y^2 - x - y - z^2 = 0$ is

$$(1, 2), (-r, r), (0, -r^2).$$

It is worth noting that the Homorooty is conducive to study some indeterminate equations reformable into $f^2 - 4g = p^2$, where f, g, p are polynomials.

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