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THE DUAL SPACE OF THE SEQUENCE SPACE bv_p $(1 \leq p < \infty)$

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Abstract. The sequence space bv_p consists of all sequences (x_k) such that $(x_k - x_{k-1})$ belongs to the space l_p. The continuous dual of the sequence space bv_p has recently been introduced by Akhmedov and Basar [Acta Math. Sin. Eng. Ser., **23(10)**, 2007, 1757–1768]. In this paper, we show a counterexample for case $p = 1$ and introduce a new sequence space d_∞ instead of d_1 and show that $bv_1{}^* = d_\infty$. Also we have modified the proof for case $p > 1$. Our notations improve the presentation and are confirmed by last notations $l_1{}^* = l_\infty$ and $l_p{}^* = l_q$.

1. Priliminaries, background and notation

Let ω denote the space of all complex-valued sequences, i.e., $\omega = \mathbb{C}^{\mathbb{N}}$ where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Any vector subspace of ω which contains ϕ, the set of all finitely non-zero sequences, is called a sequence space. The continuous dual of a sequence space λ which is denoted by λ^* is the set of all bounded linear functionals on λ. The space bv_p is the set of all sequences of p-bounded variation and is defined by

$$bv_p = \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{\infty} |x_k - x_{k-1}|^p \right)^{\frac{1}{p}} < \infty \right\} \qquad (1 \leq p < \infty)$$

and

$$bv_\infty = \left\{ x = (x_k) \in \omega : \sup_{k \in n} |x_k - x_{k-1}| < \infty \right\}$$

where $x_{-1} = 0$.

Now, let

$$\|x\|_{bv_p} = \left(\sum_{k=0}^{\infty} |x_k - x_{k-1}|^p \right)^{\frac{1}{p}}$$

and

$$\|x\|_{bv_\infty} = \sup_{k \in \mathbb{N}} |x_k - x_{k-1}|.$$

Then bv_p and bv_∞ are Banach spaces with these norms and except the case $p = 2$, the space bv_p is not a Hilbert space for $1 \leq p \leq \infty$. If we define a sequence

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$b^{(k)} = (b_n^{(k)})_{n=0}^{\infty}$ of elements of the space bv_p for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)} = \begin{cases} 0, & \text{if } n < k \\ 1, & \text{if } n \geq k \end{cases}$$

then the sequence $(b^{(k)})_{k=0}^{\infty}$ is a Schauder basis for bv_p and any $x \in bv_p$ has a unique representation of the form

$$x = \sum_{k=0}^{\infty} \lambda_k b^{(k)}$$

where $\lambda_k = (x_k - x_{k-1})$ for all $k \in \mathbb{N}$.

2. A counterexample

In [**1**, Theorem 2.3] for case $p = 1$ suppose $f = (3, -1, 0, 0, 0, \ldots)$, i.e.,

$$f_0 = f(e^0) = 3, \quad f_1 = f(e^1) = -1, \quad f_k = f(e^k) = 0 \text{ for all } k \geq 2.$$

Trivially $f \in bv_1^*$ and

$$f(x) = f\left(\sum_{k=0}^{\infty} (\Delta x)_k b^{(k)}\right) = 2(\Delta x)_0 - (\Delta x)_1.$$

So

$$\|f\| = \sup_{\|x\|_{bv_1}=1} |f(x)| = \sup_{\sum_{i=0}^{\infty} |(\Delta x)_i|=1} |2(\Delta x)_0 - (\Delta x)_1| = 2. \tag{1}$$

Now inequality (2.5) in [**1**, Theorem 2.3] asserts that $\|f\| \geq \sup\limits_{k,n \in \mathbb{N}} |\sum_{j=k}^{n} f_j| = 3$ which is a contradiction.

3. The Spaces d_∞ and d_q $(1 < q < \infty)$

In this section, we introduce two sequence spaces and show that they are Banach spaces and then we give the main theorem of the paper. Let

$$d_\infty = \left\{a = (a_k)_{k=0}^{\infty} \in \omega : \|a\|_{d_\infty} = \sup_{k \in \mathbb{N}} \left|\sum_{j=k}^{\infty} a_j\right| < \infty\right\}$$

and

$$d_q = \left\{a = (a_k)_{k=0}^{\infty} \in \omega : \|a\|_{d_q} = \left(\sum_{k=0}^{\infty} |\sum_{j=k}^{\infty} a_j|^q\right)^{\frac{1}{q}} < \infty\right\}, \quad (1 < q < \infty).$$

Theorem 3.1. *d_∞ is a sequence space with usual coordinatewise addition and scalar multiplication and $\|\cdot\|_{d_\infty}$ is a norm on d_∞.*

Proof. We only show that $\|\cdot\|_{d_\infty}$ is a norm on d_∞. Let

$$D = \begin{bmatrix} 1\ 1\ 1\ 1\ 1 \cdots \\ 0\ 1\ 1\ 1\ 1 \cdots \\ 0\ 0\ 1\ 1\ 1 \cdots \\ 0\ 0\ 0\ 1\ 1 \cdots \\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots \end{bmatrix}.$$

Then

$$Da = \begin{bmatrix} 1\ 1\ 1\ 1\ 1 \cdots \\ 0\ 1\ 1\ 1\ 1 \cdots \\ 0\ 0\ 1\ 1\ 1 \cdots \\ 0\ 0\ 0\ 1\ 1 \cdots \\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\infty} a_j \\ \sum_{j=1}^{\infty} a_j \\ \sum_{j=2}^{\infty} a_j \\ \sum_{j=3}^{\infty} a_j \\ \vdots \end{bmatrix}.$$

So $\|a\|_{d_\infty} = \sup_{k\in\mathbb{N}} \left|\sum_{j=k}^{\infty} a_j\right| = \sup_{k\in\mathbb{N}} \left|(Da)_k\right| = \left\|Da\right\|_{l_\infty}$. Now, if $a \in d_\infty$ then $\|Da\|_{l_\infty} = \|a\|_{d_\infty} < \infty$ hence $Da \in l_\infty$. Also if $Da \in l_\infty$, then $\|a\|_{d_\infty} = \|Da\|_{l_\infty} < \infty$ hence $a \in d_\infty$. So $a \in d_\infty$ if and only if $Da \in l_\infty$. Now since

(I) $0 \le \|Da\|_{l_\infty} = \|a\|_{d_\infty} < \infty$
(II) $\|a+b\|_{d_\infty} = \|Da+Db\|_{l_\infty} \le \|Da\|_{l_\infty} + \|Db\|_{l_\infty} = \|a\|_{d_\infty} + \|b\|_{d_\infty}$
(III) $\|\alpha \cdot a\|_{d_\infty} = \|\alpha \cdot Da\|_{l_\infty} = |\alpha| \cdot \|Da\|_{l_\infty} = |\alpha| \cdot \|a\|_{d_\infty}$

$\|\cdot\|_{d_\infty}$ is a norm on d_∞. □

Theorem 3.2. *d_∞ is a Banach space.*

Proof. Let $(a^{(n)})_{n=0}^{\infty}$ is a Cauchy sequence in d_∞. So for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for all $n, m \ge N$

$$\|a^{(n)} - a^{(m)}\|_{d_\infty} < \varepsilon.$$

So

$$\|Da^{(n)} - Da^{(m)}\|_{l_\infty} = \|a^{(n)} - a^{(m)}\|_{d_\infty} < \varepsilon.$$

So the sequence $(Da^{(n)})_{n=0}^{\infty}$ is Cauchy in l_∞. So there exists $a \in l_\infty$ such that $Da^{(n)} \to a$ in l_∞. So $\|Da^{(n)} - DD^{-1}a\|_{l_\infty} \to 0$ and $\|a^{(n)} - D^{-1}a\|_{d_\infty} \to 0$

Furthermore, $D^{-1}a \in d_\infty$ since $DD^{-1}a = a \in l_\infty$. □

Theorem 3.3. *bv_1^* is isometrically isomorphic to d_∞.*

Proof. Define $T : bv_1^* \to d_\infty$ and $Tf = (f(e^{(0)}), f(e^{(1)}), f(e^{(2)}), \dots)$ where

$$e^{(k)} = (0, \dots, 0, \underbrace{1}_{k^{\text{th}} term}, 0, \dots).$$

Trivially, T is linear and injective since

$$Tf = 0 \Rightarrow f = 0.$$

T is surjective since if $\tilde{g} = (g_0, g_1, g_2, g_3, \ldots) \in d_\infty$ then if we define $f : bv_1 \to \mathbb{C}$ by

$$f(x) = \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} g_j.$$

Then $f \in bv_1^*$. Trivially, since f is linear and

$$\begin{aligned} |f(x)| &= \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} g_j \right| \le \sum_{k=0}^{\infty} |(\Delta x)_k| \cdot \left| \sum_{j=k}^{\infty} g_j \right| \\ &\le \sum_{k=0}^{\infty} |(\Delta x)_k| \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} g_j \right| = \sum_{k=0}^{\infty} |(\Delta x)_k| . \|\tilde{g}\|_{d_\infty} \\ &= \|\tilde{g}\|_{d_\infty} . \|x\|_{bv_1} \end{aligned}$$

and $Tf = \tilde{g}$, so T is surjective. Now we show that T is norm preserving, we have

$$\begin{aligned} |f(x)| &= \left| f\left(\sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} e^{(j)} \right) \right| = \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} f(e^{(j)}) \right| \\ &\le \sum_{k=0}^{\infty} |(\Delta x)_k| \left| \sum_{j=k}^{\infty} f(e^{(j)}) \right| \le \sum_{k=0}^{\infty} |(\Delta x)_k| \cdot \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} f(e^{(j)}) \right| \\ &\le \|x\|_{bv_1} \cdot \|Tf\|_{d_\infty}. \end{aligned}$$

So

$$(*) \qquad \|f\| \le \|Tf\|_{d_\infty}$$

On the other hand, $\left|\sum_{j=k}^{\infty} f(e^{(j)})\right| = \left|f(b^{(k)})\right| \le \|f\| \cdot \|b^{(k)}\|_{bv_1} = \|f\|$. So $\left|\sum_{j=k}^{\infty} f(e^{(j)})\right| \le \|f\|$ for all $k \in \mathbb{N}$.

So

$$\sup_{k\in\mathbb{N}} |\sum_{j=k}^{\infty} f(e^{(j)})| \le \|f\|,$$

i.e.,

$$(\dagger) \qquad \|Tf\|_{d_\infty} \le \|f\|$$

by $(*)$ and $(\dagger)$ we are done. □

Theorem 3.4. *d_q $(1 \le q < \infty)$ is a sequence space with usual coordinatewise addition and scalar multiplication and $\|.\|_{d_q}$ is a norm on d_q.*

Proof. With notations of Theorem 3.1 , $\|a\|_{d_q} = \|Da\|_{l_q}$ and $a \in d_q \Leftrightarrow Da \in l_q$. The continuation of the proof is similar to Theorem 3.1. □

Theorem 3.5. *d_q $(1 \le q < \infty)$ is a Banach space.*

Proof. The proof is similar to proof of Theorem 3.2 and we omit it. □

Theorem 3.6. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then bv_p^* is isometrically isomorphic to d_q.*

Proof. Define $A : bv_p^* \to d_q$ by $f \mapsto Af = (f(e^{(0)}), f(e^{(1)}), f(e^{(2)}), \ldots)$. Trivially A is linear. Additionally, since $Af = 0 = (0, 0, 0, \ldots)$ implies $f = 0$, A is injective. A is surjective since if $a = (a_k) \in d_q$ and define f on the space bv_p such that

$$f(x) = \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} a_j.$$

Then f is linear. Since

$$\begin{aligned} |f(x)| &\le \sum_{k=0}^{\infty} \left| (\Delta x)_k \sum_{j=k}^{\infty} a_j \right| \\ &\le \left[\sum_{k=0}^{\infty} \left| (\Delta x)_k \right|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_j \right|^q \right]^{\frac{1}{q}} = \|x\|_{bv_p} \cdot \|a\|_{d_q}, \end{aligned}$$

it yields to $\|f\| \le \|a\|_{d_q} < \infty$. So $f \in bv_p^*$ and $Af = a$.

Now, we show that A is norm preserving. Let $f \in bv_p^*$ and $x = (x_k)_{k=0}^{\infty} \in bv_p$, then

$$\begin{aligned} |f(x)| &= \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} f(e^{(j)}) \right| \le \sum_{k=0}^{\infty} \left| (\Delta x)_k \sum_{j=k}^{\infty} f(e^{(j)}) \right| \\ &\le \left[\sum_{k=0}^{\infty} \left| (\Delta x)_k \right|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} f(e^{(j)}) \right|^q \right]^{\frac{1}{q}} = \|x\|_{bv_p} \cdot \|Af\|_{d_q}. \end{aligned}$$

So

$$(*) \qquad \|f\| \le \|Af\|_{d_q}.$$

On the other hand, suppose $f \in bv_p^*$ and $x^{(n)} = \left(x_k^{(n)}\right)_{k=0}^{\infty}$ are such that

$$(\Delta x^{(n)})_k = \begin{cases} \left| \sum_{j=k}^{\infty} f(e^{(j)}) \right|^{q-1} \operatorname{sgn}\left(\sum_{j=k}^{\infty} f(e^{(j)}) \right), & \text{if } (0 \le k \le n) \\ 0, & \text{if } k > n. \end{cases}$$

We note that $\sum_{j=k}^{\infty} f(e^{(j)}) = f(b^{(k)})$. So $x^{(n)} \in bv_p$ since $\Delta x^{(n)} \in l_p$.

Then it is clear that

$$\Delta x^{(n)} =$$

$$\left(\left| \sum_{j=0}^{\infty} f(e^{(j)}) \right|^{q-1} \operatorname{sgn}\left(\sum_{j=0}^{\infty} f(e^{(j)}) \right), \ldots, \left| \sum_{j=n}^{\infty} f(e^{(j)}) \right|^{q-1} \operatorname{sgn}\left(\sum_{j=n}^{\infty} f(e^{(j)}) \right), 0, 0, \ldots \right).$$

So

$$x^{(n)} = \left(\underbrace{\left|\sum_{j=0}^{\infty} f(e^{(j)})\right|^{q-1} \operatorname{sgn}\Big(\sum_{j=0}^{\infty} f(e^{(j)})\Big)}_{b_0}, b_0 + \underbrace{\left|\sum_{j=1}^{\infty} f(e^{(j)})\right|^{q-1} \operatorname{sgn}\Big(\sum_{j=1}^{\infty} f(e^{(j)})\Big)}_{b_1}, \right.$$

$$\left. \ldots, \underbrace{\sum_{k=0}^{n} b_k}_{t=n+1^{th} term}, t, t, t, \ldots \right).$$

So if we let $f_k = f(e^{(k)})$, then

$$\begin{aligned} f(x^{(n)}) = b_0 f_0 +& \\ & b_0 f_1 + b_1 f_1 + \\ & b_0 f_2 + b_1 f_2 + b_2 f_2 + \\ & b_0 f_3 + b_1 f_3 + b_2 f_3 + b_3 f_3 + \\ & \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & b_0 f_n \ + \ b_1 f_n \ + b_2 f_n \ + b_3 f_n + \ldots + b_n f_n + \\ & b_0 f_{n+1} + b_1 f_{n+1} + b_2 f_{n+1} + b_3 f_{n+1} + \ldots + b_n f_{n+1} + \\ & b_0 f_{n+2} + b_1 f_{n+2} + b_2 f_{n+2} + b_3 f_{n+2} + \ldots + b_n f_{n+2} + \\ & b_0 f_{n+3} + b_1 f_{n+3} + b_2 f_{n+3} + b_3 f_{n+3} + \ldots + b_n f_{n+3} + \\ & \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & = \sum_{k=0}^{n} \left|\sum_{j=k}^{\infty} f_j\right|^q. \end{aligned}$$

So

$$\sum_{k=0}^{n} \left|\sum_{j=k}^{\infty} f_j\right|^q = f(x^{(n)}) = |f(x^{(n)})| \leq \|f\| \cdot \|x^{(n)}\|_{bv_p} = \|f\| \cdot \left[\sum_{k=0}^{n} \left|\sum_{j=k}^{\infty} f_j\right|^q\right]^{\frac{1}{p}}.$$

Since

$$\begin{aligned} \|x^{(n)}\|_{bv_p} = \|\Delta x^{(n)}\|_{l_p} = \left[\sum_{k=0}^{\infty} |\Delta x_k^{(n)}|^p\right]^{\frac{1}{p}} &= \left[\sum_{k=0}^{n} |\Delta x_k^{(n)}|^p\right]^{\frac{1}{p}} \\ &= \left[\sum_{k=0}^{n} \left|\left|\sum_{j=k}^{\infty} f_j\right|^{q-1} \operatorname{sgn}\left(\sum_{j=k}^{\infty} f_j\right)\right|^p\right]^{\frac{1}{p}} \\ &= \left[\sum_{k=0}^{n} \left|\sum_{j=k}^{\infty} f_j\right|^q\right]^{\frac{1}{p}}. \end{aligned}$$

So

$$\left[\sum_{k=0}^{n}\left|\sum_{j=k}^{\infty} f_j\right|^q\right]^1 \leq \|f\| \cdot \left[\sum_{k=0}^{n}|\sum_{j=k}^{\infty} f_j|^q\right]^{\frac{1}{p}}.$$

So

$$(\dagger) \qquad \|f\| \geq \left[\sum_{k=0}^{n}\left|\sum_{j=k}^{\infty} f_j\right|^q\right]^{\frac{1}{q}} = \|Af\|_{d_q}.$$

Therefore, by combinig the results $(*)$ and $(\dagger)$, A is norm preserving. Hence bv_p^* is isometrically isomorphic to d_q. □

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