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FOURIER COEFFICIENTS OF HILBERT CUSP FORMS ASSOCIATED WITH MIXED HILBERT CUSP FORMS

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Abstract. We express the Fourier coefficients of the Hilbert cusp form $\mathfrak{L}_h^* f$ associated with mixed Hilbert cusp forms f and h in terms of the Fourier coefficients of a certain periodic function determined by f and h. We also obtain an expression of each Fourier coefficient of $\mathfrak{L}_h^* f$ as an infinite series involving the Fourier coefficients of f and h.

1. Introduction

Mixed automorphic forms are automorphic forms defined by using an automorphy factor associated with an equivariant holomorphic map of Hermitian symmetric domains, and certain types of mixed automorphic forms occur as holomorphic forms of the highest degree on a family of Abelian varieties parametrized by a locally symmetric space (cf. [**7**]). When the Hermitian symmetric domains are Cartesian products of the Poincaré upper half plane $\mathcal{H}$, we obtain mixed Hilbert modular forms which generalize the usual Hilbert modular forms (see [**5**]).

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})^n$. Assume that there are a holomorphic map $\omega : \mathcal{H}^n \to \mathcal{H}^n$ and a homomorphism $\chi : \Gamma \to SL(2,\mathbb{R})^n$ such that ω is equivariant with respect to χ. If $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$ with $r_i \geq 0$ for each i, we denote by $J^{\mathbf{r}} : \Gamma \times \mathcal{H}^n \to \mathbb{C}$ the automorphy factor defining Hilbert modular forms for Γ of weight $\mathbf{r}$. Then mixed Hilbert modular forms for Γ of type $(\mathbf{r}, \mathbf{r}')$ are defined by using the automorphy factor $J^{\mathbf{r}} \cdot (J^{\mathbf{r}'} \circ (\chi, \omega))$. Hilbert cusp forms and mixed Hilbert cusp forms are defined with an additional condition on the cusps.

Let $\mathcal{S}_{\mathbf{k}}(\Gamma)$ and $\mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ be the spaces of Hilbert cusp forms of weight $\mathbf{k}$ and mixed Hilbert cusp forms of type $(\mathbf{m}, \mathbf{r})$, respectively, for Γ. Given an element $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$, we consider the associated linear map

$$\mathfrak{L}_h : \mathcal{S}_{\mathbf{k}}(\Gamma) \to \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$$

defined by $\mathfrak{L}_h(g) = gh$ for all $g \in \mathcal{S}_{\mathbf{k}}(\Gamma)$, and denoted by

$$\mathfrak{L}_h^* : \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi) \to \mathcal{S}_{\mathbf{k}}(\Gamma)$$

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corresponding adjoint linear map with respect to the Petersson inner products. This map determines a Hilbert cusp form $\mathfrak{L}_h^* f \in \mathcal{S}_{\mathbf{k}}(\Gamma)$ associated with a mixed Hilbert cusp form $f \in \mathcal{S}_{\mathbf{k+m},\mathbf{r}}(\Gamma, \omega, \chi)$.

In this paper we express the Fourier coefficients of the Hilbert cusp form $\mathfrak{L}_h^* f$ associated with mixed Hilbert cusp forms $f \in \mathcal{S}_{\mathbf{k+m},\mathbf{r}}(\Gamma, \omega, \chi)$ and $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ in terms of the Fourier coefficients of some periodic function determined by f and h. We also obtain an expression of each Fourier coefficient of $\mathfrak{L}_h^* f$ as an infinite series involving the Fourier coefficients of f and h.

2. Mixed Hilbert modular forms

We fix a positive integer n and let $\mathcal{H}^n$ be the Cartesian product of n copies of the Poincaré upper half plane $\mathcal{H}$. Then the usual operation of $SL(2, \mathbb{R})$ on $\mathcal{H}$ by linear fractional transformations induces an action of the n-fold product $SL(2, \mathbb{R})^n$ of $SL(2, \mathbb{R})$ on $\mathcal{H}^n$. Let F be a totally real number field with $[F : \mathbb{Q}] = n$, so that there are n embeddings

$$a \mapsto a^{(j)}, \quad F \hookrightarrow \mathbb{R} \tag{2.1}$$

for $1 \le j \le n$. These embeddings induce the injective homomorphism

$$\iota : SL(2, F) \to SL(2, \mathbb{R})^n \tag{2.2}$$

defined by

$$\iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right) \tag{2.3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, F)$. Throughout this paper we shall often identify an element γ of $SL(2, F)$ with its image $\iota(\gamma)$ in $SL(2, \mathbb{R})^n$ under the injection ι in (2.2). Given $z = (z_1, \dots, z_n) \in \mathcal{H}^n$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, F)$ with $\iota(\gamma)$ as in (2.3), we set

$$\gamma z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \dots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right).$$

Then the map $(\gamma, z) \mapsto \gamma z$ determines an action of $SL(2, F)$ on $\mathcal{H}^n$. For the same $z \in \mathcal{H}^n$, $\gamma \in SL(2, F)$, we set

$$J(\gamma, z) = J(\gamma, z)^{\mathbf{1}} = \prod_{j=1}^n \bigl(c^{(j)} z_j + d^{(j)}\bigr), \quad J(\gamma, z)^{\mathbf{r}} = \prod_{j=1}^n \bigl(c^{(j)} z_j + d^{(j)}\bigr)^{r_j}, \tag{2.4}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^n$ and $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$. Then for each $\mathbf{r} \in \mathbb{Z}^n$ we see easily that the map

$$(\gamma, z) \mapsto J(\gamma, z)^{\mathbf{r}} : SL(2, F) \times \mathcal{H}^n \to \mathbb{C}$$

is an automorphy factor, meaning that it satisfies the cocycle condition

$$J(\gamma\gamma', z)^{\mathbf{r}} = J(\gamma, \gamma' z)^{\mathbf{r}} J(\gamma', z)^{\mathbf{r}} \tag{2.5}$$

for all $z \in \mathcal{H}^n$ and $\gamma, \gamma' \in SL(2, F)$.

We now consider a discrete subgroup $\Gamma \subset SL(2,F)$ of $SL(2,\mathbb{R})^n$. Let $\chi : \Gamma \to SL(2,F)$ be a homomorphism and let $\omega : \mathcal{H}^n \to \mathcal{H}^n$ be a holomorphic map that is equivariant with respect to χ, such that,

$$\omega(\gamma z) = \chi(\gamma)\omega(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}^n$. We assume that the image $\chi(\Gamma)$ of Γ under χ is also a discrete subgroup of $SL(2,\mathbb{R})^n$ and the inverse image of the set of parabolic elements of $\chi(\Gamma)$ coincides with the set of parabolic elements of Γ, so there is a correspondence between parabolic elements of Γ and those of $\chi(\Gamma)$. Let $\mathbf{k} = (k_1, \ldots, k_n)$ and $\mathbf{m} = (m_1, \ldots, m_n)$ be elements of $\mathbb{Z}^n$ with $k_i, m_i \geq 0$ for each $i \in \{1, \ldots, n\}$. If $\gamma \in \Gamma \subset SL(2,F)$ and $z \in \mathcal{H}^n$, we set

$$J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma, z) = J(\gamma, z)^{\mathbf{k}} J(\chi(\gamma), \omega(z))^{\mathbf{m}},$$

where $J(\cdot,\cdot)$ is as in (2.4). Using the cocycle condition in (2.5), we see that the resulting map $J^{\mathbf{k},\mathbf{m}}_{\omega,\chi} : \Gamma \times \mathcal{H}^n \to \mathbb{C}$ is an automorphy factor satisfying the relation

$$J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma\gamma', z) = J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma, \gamma' z) \cdot J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma', z)$$

for all $\gamma, \gamma' \in \Gamma$ and $z \in \mathcal{H}^n$.

Let s be a cusp of Γ and σ an element of $SL(2,F) \subset SL(2,\mathbb{R})^n$ such that $\sigma(\infty) = s$. If we set

$$\Gamma^\sigma = \sigma^{-1}\Gamma\sigma \subset SL(2,\mathbb{R}),$$

then ∞ is a cusp of Γ^σ. We extend the homomorphism $\chi : \Gamma \to SL(2,F)$ to a map $\chi : \Gamma' \to SL(2,F)$ where

$$\Gamma' = \Gamma \cup \{\alpha \in SL(2,F) \mid \alpha(\infty) = s,\ s \text{ a cusp of } \Gamma\}.$$

We consider a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ satisfying

$$f(\gamma z) = J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma, z) f(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}^n$ and define the function $f \mid \sigma : \mathcal{H}^n \to \mathbb{C}$ by

$$(f \mid \sigma)(z) = J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\sigma, z)^{-1} f(\sigma z)$$

for all $z \in \mathcal{H}^n$. Then, we have

$$(f \mid \sigma)(\gamma z) = J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma, z)(f \mid \sigma)(z)$$

for all $\gamma \in \Gamma^\sigma$ and $z \in \mathcal{H}^n$. We set

$$\Lambda = \Lambda(\Gamma^\sigma) = \{\lambda \in F \mid \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma\} \tag{2.6}$$

which we identify with a subgroup of $\mathbb{R}^n$ via the natural embedding $F \hookrightarrow \mathbb{R}^n$ in (2.1) so that Λ is a lattice in $\mathbb{R}^n$. Let Λ^* be the corresponding dual lattice given by

$$\Lambda^* = \{\xi \in F \mid \mathrm{Tr}(\xi\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\},$$

where $\mathrm{Tr}(\xi\lambda) = \sum_{j=1}^n \xi_j\lambda_j$. Using the fact that ∞ is a cusp of Γ^σ and noting that

χ carries parabolic elements to parabolic elements, we see that the function $f \mid \sigma$ has a Fourier expansion at ∞ of the form

$$(f \mid \sigma)(z) = \sum_{\xi \in \Lambda^*} A_\xi \exp(2\pi \mathrm{i} \operatorname{Tr}(\xi z)). \tag{2.7}$$

This series is the Fourier expansion of f at the cusp s and the coefficients A_ξ are the Fourier coefficients of f at s.

Definition 2.1. Let $\Gamma \subset SL(2, F)$ be a discrete subgroup $SL(2, \mathbb{R})^n$ with cusp s and let $f : \mathcal{H}^n \to \mathbb{C}$ be a holomorphic function satisfying

$$f(\gamma z) = J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma, z) f(z)$$

for all $z \in \mathcal{H}^n$ and $\gamma \in \Gamma$.

(i) The function f is *regular at s* if the Fourier coefficients of f at s satisfy the condition that $\xi \geq 0$ whenever $A_\xi \neq 0$.

(ii) The function f *vanishes at s* if the Fourier coefficients of f at s satisfy the condition that $\xi > 0$ whenever $A_\xi \neq 0$.

Definition 2.2. Let Γ, χ and ω be as above and assume that the quotient space $\Gamma \backslash \mathcal{H}^n \cup \{\text{cusps}\}$ is compact. A *mixed Hilbert modular form of type* $(\mathbf{k}, \mathbf{m})$ associated with Γ, χ and ω is a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ satisfying the following conditions

(i) $f(\gamma z) = J^{\mathbf{k},\mathbf{m}}_{\omega,\chi}(\gamma, z) f(z)$ for all $\gamma \in \Gamma$.

(ii) f is regular at the cusps of Γ.

The holomorphic function f is a *mixed Hilbert cusp form* of the same type if (ii) is replaced with the following condition:

(ii)$'$ f vanishes at the cusps of Γ.

Remark 2.3. Mixed Hilbert modular forms of certain types occur naturally as holomorphic forms on a family of Abelian varieties parametrized by a Hilbert modular variety (see [**5**]). A special case of such a family and their connections with Hilbert modular forms were also investigated by Kifer and Skornyakov in [**3**].

3. Hilbert cusp forms associated with mixed Hilbert cusp forms

Let $\Gamma \subset SL(2, \mathbb{R})^n$, $\chi : \Gamma \to SL(2, \mathbb{R})^n$ and $\omega : \mathcal{H}^n \to \mathcal{H}^n$ be as in Section 2. Let $\mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ for $\mathbf{m}, \mathbf{r} \in \mathbb{Z}^n$ with nonnegative components be the space mixed of Hilbert cusp forms of type $(\mathbf{m}, \mathbf{r})$ for Γ associated with ω and χ in the sense of Definition 2.2. Note that a mixed Hilbert cusp form of type $(\mathbf{m}, \mathbf{0})$ with $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^n$ is simply a usual Hilbert cusp form of weight $\mathbf{m}$. We denote by $\mathcal{S}_{\mathbf{k}}(\Gamma)$ the space of Hilbert cusp forms of weight $\mathbf{k}$ for Γ.

We fix an element $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$. Then for each $g \in \mathcal{S}_{\mathbf{k}}(\Gamma)$, we see that the product gh is an element of $\mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$. Thus we obtain the linear map

$$\mathfrak{L}_h : \mathcal{S}_{\mathbf{k}}(\Gamma) \to \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$$

defined by

$$\mathfrak{L}_h(g) = gh$$

for all $g \in \mathcal{S}_{\mathbf{k}}(\Gamma)$. As it is well-known, the complex vector space $\mathcal{S}_{\mathbf{k}}(\Gamma)$ is equipped with the Petersson inner product given by

$$\langle g_1, g_2 \rangle = \int_{\Gamma\backslash\mathcal{H}^n} g_1(z)\overline{g_2(z)}(\operatorname{Im} z)^{\mathbf{k}} \mathrm{d}\mu(z) \tag{3.1}$$

for all $g_1, g_2 \in \mathcal{S}_{\mathbf{k}}(\Gamma)$, where

$$(\operatorname{Im} z)^{\mathbf{k}} = (y_1, \dots, y_n)^{\mathbf{k}} = \prod_{j=1}^n y_j^{k_j}, \qquad \mathrm{d}\mu(z) = \prod_{j=1}^n y_j^{-2} \mathrm{d}x_j \mathrm{d}y_j$$

for $\mathbf{k} = (k_1, \dots, k_n)$ and $z = x + \mathrm{i}\, y \in \mathcal{H}^n$ with $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We can also define the Petersson inner product on $\mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ by

$$\langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\Gamma\backslash\mathcal{H}^n} f_1(z)\overline{f_2(z)}(\operatorname{Im} z)^{\mathbf{k}+\mathbf{m}}(\operatorname{Im}\omega(z))^{\mathbf{r}} \mathrm{d}\mu(z) \tag{3.2}$$

for all $f_1, f_2 \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$. We denote by

$$\mathfrak{L}_h^* : \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi) \to \mathcal{S}_{\mathbf{k}}(\Gamma) \tag{3.3}$$

the adjoint linear map of $\mathfrak{L}_h$ with respect to the Petersson inner products in (3.1) and (3.2), so that

$$\langle \mathfrak{L}_h^* f, g \rangle = \langle\!\langle f, \mathfrak{L}_h g \rangle\!\rangle \tag{3.4}$$

for all $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ and $g \in \mathcal{S}_{\mathbf{k}}(\Gamma)$.

Let $\mathfrak{o}$ be the ring of integers in the totally real number field F with $[F : \mathbb{Q}] = n$ considered in Section 2 and let $\mathfrak{n}$ be a nonzero ideal of $\mathfrak{o}$. Then the *principal congruence subgroup* of level $\mathfrak{n}$ is the subgroup of $SL(2, \mathfrak{o})$ given by

$$\Gamma(\mathfrak{n}) = \{\gamma \in SL(2, \mathfrak{o}) \mid \gamma \equiv 1 \pmod{\mathfrak{n}}\},$$

which is regarded as a discrete subgroup of $SL(2, \mathbb{R})^n$ as usual. We set

$$\mathfrak{n}^* = \{r \in F \mid \operatorname{Tr}(r\mathfrak{n}) \subset \mathfrak{o}\},$$

and consider a totally positive element ν of $\mathfrak{n}^*$. Then the ν-th Poincaré series of weight $\mathbf{k}$ with respect to Γ is given by

$$P_{\mathbf{k},\nu}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} J(\gamma, z)^{-\mathbf{k}} \exp(2\pi\,\mathrm{i}\operatorname{Tr}(\nu(\gamma z))) \tag{3.5}$$

where $J(\gamma, z)$ is as in (2.4) and Γ_∞ is the subgroup of Γ consisting of the elements of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ (see [**2**, Section 1.13]).

We consider an element $\phi \in \mathcal{S}_{\mathbf{k}}(\Gamma)$ and write its Fourier expansion at ∞ considered in (2.7) in the form

$$\phi(z) = \sum_{\xi\in\Lambda^*} A_\xi(\phi) \exp(2\pi\,\mathrm{i}\operatorname{Tr}(\xi z))$$

for all $z \in \mathcal{H}^n$. Then we have

$$\langle \phi, P_{\mathbf{k},\nu} \rangle = A_\nu(\phi) \cdot \operatorname{vol}(\mathbb{R}^n/\Lambda) \cdot \prod_{j=1}^n \frac{\boldsymbol{\Gamma}(k_j - 1)}{(4\pi\nu_j)^{k_j-1}}, \tag{3.6}$$

where $\mathbf{\Gamma}$ is the Gamma-function and Λ is as in (2.6) (cf. [**2**]). In particular, the Fourier expansion of the image $\mathfrak{L}_h^* f$ of an element $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ under the map $\mathfrak{L}_h^*$ in (3.3) associated with $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ can be written in the form

$$\mathfrak{L}_h^* f(z) = \sum_{\xi \in \Lambda^*} A_\xi(\mathfrak{L}_h^* f) \exp(2\pi \,\mathrm{i}\, \mathrm{Tr}(\xi z)). \tag{3.7}$$

For the same f and h we also set

$$\Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z) = f(z)\overline{h(z)}(\operatorname{Im} z)^{\mathbf{m}}(\operatorname{Im} \omega(z))^{\mathbf{r}} \tag{3.8}$$

for all $z \in \mathcal{H}^n$.

Lemma 3.1. *If $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ and $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$, the Fourier coefficient $A_\xi(\mathfrak{L}_h^* f)$ of $\mathfrak{L}_h^* f \in \mathcal{S}_{\mathbf{k}}(\Gamma)$ in (3.7) for $\xi \in \Lambda^*$ is given by*

$$A_\xi(\mathfrak{L}_h^* f) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \prod_{j=1}^n \frac{(4\pi\xi_j)^{k_j-1}}{\mathbf{\Gamma}(k_j-1)} \int_{\Gamma\backslash\mathcal{H}^n} \Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z)\overline{P_{\mathbf{k},\xi}(z)}(\operatorname{Im} z)^{\mathbf{k}} d\mu(z), \tag{3.9}$$

where $\Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z)$ is as in (3.8) and $P_{\mathbf{k},\xi}(z)$ is the Poincaré series in (3.5).

Proof. Using (3.2), (3.4), (3.6) and (3.8), we see that

$$\begin{aligned} A_\xi(\mathfrak{L}_h^* f) \cdot \mathrm{vol}(\mathbb{R}^n/\Lambda) \cdot \prod_{j=1}^n \frac{\mathbf{\Gamma}(k_j-1)}{(4\pi\xi_j)^{k_j-1}} & \\ &= \langle \mathfrak{L}_h^* f, P_{\mathbf{k},\xi} \rangle = \langle\!\langle f, \mathfrak{L}_h P_{\mathbf{k},\xi} \rangle\!\rangle = \langle\!\langle f, h P_{\mathbf{k},\xi} \rangle\!\rangle \\ &= \int_{\Gamma\backslash\mathcal{H}^n} f(z)\overline{h(z)P_{\mathbf{k},\xi}(z)}(\operatorname{Im} z)^{\mathbf{k}+\mathbf{m}}(\operatorname{Im} \omega(z))^{\mathbf{r}} d\mu(z) \\ &= \int_{\Gamma\backslash\mathcal{H}^n} \Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z)\overline{P_{\mathbf{k},\xi}(z)}(\operatorname{Im} z)^{\mathbf{k}} d\mu(z); \end{aligned}$$

hence the lemma follows. □

4. Fourier coefficients

Let the mixed Hilbert cusp forms $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$, $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ and the associated functions $\mathfrak{L}_h^* f$, $\Phi_{f,h}^{\mathbf{m},\mathbf{r}}$ be as in Section 3. In this section we express the Fourier coefficients of $\mathfrak{L}_h^* f$ in terms of those of $\Phi_{f,h}^{\mathbf{m},\mathbf{r}}$. We also obtain an expression of each Fourier coefficient of $\mathfrak{L}_h^* f$ as an infinite series involving the Fourier coefficient of f and h.

Lemma 4.1. *The function $\Phi_{f,h}^{\mathbf{m},\mathbf{r}}$ in (3.8) satisfies the relation*

$$\Phi_{f,h}^{\mathbf{m},\mathbf{r}}(\gamma z) = J(\gamma, z)^{\mathbf{k}} \Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z) \tag{4.1}$$

for all $z \in \mathcal{H}^n$ and $\gamma \in \Gamma$.

Proof. Since $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$, $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$, given $z \in \mathcal{H}^n$ and $\gamma \in \Gamma$, we see that

$$\begin{aligned}
\Phi^{\mathbf{m},\mathbf{r}}_{f,h}(\gamma z) &= f(\gamma z)\overline{h(\gamma z)}(\operatorname{Im}\gamma z)^{\mathbf{m}}(\operatorname{Im}\chi(\gamma)\omega(z))^{\mathbf{r}} \\
&= J(\gamma, z)^{\mathbf{k}+\mathbf{m}} J(\chi(\gamma), \omega(z))^{\mathbf{r}} f(z)\overline{J(\gamma, z)}^{\mathbf{m}}\overline{J(\chi(\gamma), \omega(z))}^{\mathbf{r}}\overline{h(z)} \\
&\qquad \times |J(\gamma, z)|^{-2\mathbf{m}}(\operatorname{Im} z)^{\mathbf{m}}|J(\chi(\gamma), \omega(z))|^{-2\mathbf{r}}(\operatorname{Im}\omega(z))^{\mathbf{r}} \\
&= J(\gamma, z)^{\mathbf{k}} f(z)\overline{h(z)}(\operatorname{Im} z)^{\mathbf{m}}(\operatorname{Im}\omega(z))^{\mathbf{r}} \\
&= J(\gamma, z)^{\mathbf{k}}\Phi^{\mathbf{m},\mathbf{r}}_{f,h}(z),
\end{aligned}$$

which proves the lemma. □

By Lemma 4.1 the function $\Phi^{\mathbf{m},\mathbf{r}}_{f,h}$ satisfies

$$\Phi^{\mathbf{m},\mathbf{r}}_{f,h}(z + \lambda) = J\left(\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right), z\right)^{\mathbf{k}}\Phi_{f,h}(z) = \Phi_{f,h}(z)$$

for all $z \in \mathcal{H}^n$ and $\lambda \in \Lambda$, where Λ is as in (2.6). Thus $\Phi^{\mathbf{m},\mathbf{r}}_{f,h}$ has a Fourier expansion of the form

$$\Phi^{\mathbf{m},\mathbf{r}}_{f,h}(z) = \sum_{\xi \in \Lambda^*} A^{\mathbf{m},\mathbf{r}}_{f,h,\xi}(y) \exp(2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi x)). \tag{4.2}$$

Theorem 4.2. *Setting $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$, the ξ-th Fourier coefficient of the Hilbert cusp form $\mathfrak{L}^*_h f$ in the expansion* (3.7) *is given by*

$$A_\xi(\mathfrak{L}^*_h f) = \prod_{j=1}^{n} \frac{(4\pi\xi_j)^{k_j-1}}{\boldsymbol{\Gamma}(k_j - 1)} \int_{\mathbb{R}^n_+} A^{\mathbf{m},\mathbf{r}}_{f,h,\xi}(y) \exp(-2\pi\,\mathrm{Tr}(\xi y)) y^{\mathbf{k}-2}\mathrm{d}y, \tag{4.3}$$

where $A^{\mathbf{m},\mathbf{r}}_{f,h,\xi}(y)$ is as in (4.2).

Proof. Using (3.5) and the relation

$$\mathrm{d}\mu(z) = (\operatorname{Im} z)^{-\mathbf{2}}(\mathrm{i}/2)^n \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

with $\mathbf{2} = (2, \dots, 2) \in \mathbb{Z}^n$, the integral on the right hand side of (3.9) can be written as

$$\begin{aligned}
&\int_{\mathcal{F}} \Phi^{\mathbf{m},\mathbf{r}}_{f,h}(z)\overline{P_{\mathbf{k},\xi}(z)}(\operatorname{Im} z)^{\mathbf{k}}\mathrm{d}\mu(z) \\
&\qquad = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}} \Phi^{\mathbf{m},\mathbf{r}}_{f,h}(z) \exp(-2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi(\overline{\gamma z}))) \\
&\qquad\qquad \times \overline{J(\gamma, z)}^{-\mathbf{k}}(\operatorname{Im} z)^{\mathbf{k}-\mathbf{2}}(\mathrm{i}/2)^n \mathrm{d}z \wedge \mathrm{d}\overline{z},
\end{aligned} \tag{4.4}$$

where $\mathcal{F}$ is a fundamental domain of Γ. Given $\gamma \in \Gamma$, if we use the new variable $w = u + \mathrm{i}\,v = \gamma z$, the integral on the right-hand side of (4.4) is equal to

$$\begin{aligned}
&\int_{\gamma\mathcal{F}} \Phi^{\mathbf{m},\mathbf{r}}_{f,h}(\gamma^{-1}w) \exp(-2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi(\overline{w})))\overline{J(\gamma, \gamma^{-1}w)}^{-\mathbf{k}} \\
&\qquad \times (\operatorname{Im}\gamma^{-1}w)^{\mathbf{k}-\mathbf{2}}(\mathrm{i}/2)^n \mathrm{d}(\gamma^{-1}w) \wedge \mathrm{d}(\overline{\gamma^{-1}w}).
\end{aligned} \tag{4.5}$$

However, by using the cocycle condition (2.5), we have

$$\overline{J(\gamma, \gamma^{-1}w)}^{-\mathbf{k}} = \overline{J(\gamma^{-1}, w)}^{\,\mathbf{k}},$$

$$\begin{aligned}
(\operatorname{Im}\gamma^{-1}w)^{\mathbf{k-2}} &= |J(\gamma^{-1}, w)|^{-2(\mathbf{k-2})}(\operatorname{Im} w)^{\mathbf{k-2}} \\
&= J(\gamma^{-1}, w)^{-\mathbf{k+2}}\overline{J(\gamma^{-1}, w)}^{-\mathbf{k+2}}(\operatorname{Im} w)^{\mathbf{k-2}}, \\
(\mathrm{i}/2)^n d(\gamma^{-1}w) \wedge d(\overline{\gamma^{-1}w}) &= (\mathrm{i}/2)^n J(\gamma^{-1}, w)^{-\mathbf{2}}\mathrm{d}w \wedge \overline{J(\gamma^{-1}, w)}^{-\mathbf{2}}\mathrm{d}\overline{w} \\
&= J(\gamma^{-1}, w)^{-\mathbf{2}}\overline{J(\gamma^{-1}, w)}^{-\mathbf{2}}\mathrm{d}u \wedge \mathrm{d}v;
\end{aligned}$$

Hence (4.5) can be written in the form

$$\begin{aligned}
&\int_{\gamma\mathcal{F}} J(\gamma^{-1}, w)^{-\mathbf{k}}\Phi^{\mathbf{m,r}}_{f,h}(\gamma^{-1}w)\exp(-2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi(\overline{w})))(\operatorname{Im} w)^{\mathbf{k}}\mathrm{d}\mu(w) \\
&\qquad = \int_{\gamma\mathcal{F}} \Phi^{\mathbf{m,r}}_{f,h}(w)\exp(-2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi(\overline{w})))(\operatorname{Im} w)^{\mathbf{k}}\mathrm{d}\mu(w),
\end{aligned}$$

where we used (4.1). By taking the summation of this over $\gamma \in \Gamma_\infty\backslash\Gamma$, we see that the integral on the left-hand side of (4.4) is equal to

$$\int_{\widetilde{\mathcal{F}}} \Phi^{\mathbf{m,r}}_{f,h}(z)\exp(-2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi(\overline{z})))(\operatorname{Im} z)^{\mathbf{k}}\mathrm{d}\mu(z), \tag{4.6}$$

where $\widetilde{\mathcal{F}}$ is the subset of $\mathcal{H}^n$ given by

$$\widetilde{\mathcal{F}} = \bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma} \gamma\mathcal{F}. \tag{4.7}$$

From (4.7) we see that $\widetilde{\mathcal{F}}$ is a fundamental domain of Γ_∞ and therefore we have

$$\widetilde{\mathcal{F}} = \mathbb{R}^n_+ \times [0, \lambda_0] = \mathbb{R}^n_+ \times \prod_{j=1}^n [0, \lambda_{0,j}],$$

where $\lambda_{0,j} \in \mathbb{R}_+$ is the generator of the j-th component of the lattice Λ in $\mathbb{R}^n$ for each $j \in \{1, \dots, n\}$. Thus, using this and (4.2), the integral (4.6) can now be written as

$$\begin{aligned}
&\sum_{\nu\in\Lambda^*}\int_{\mathbb{R}^n_+}\int_{[0,\lambda_0]} A^{\mathbf{m,r}}_{f,h,\nu}(y)\exp(2\pi\,\mathrm{i}\,\mathrm{Tr}(\nu x)) \\
&\qquad\qquad \times \Phi^{\mathbf{m,r}}_{f,h}(z)\exp(-2\pi\,\mathrm{i}\,\mathrm{Tr}(\xi(x-iy))y^{\mathbf{k-2}}\mathrm{d}x\mathrm{d}y \\
&\qquad = \sum_{\nu\in\Lambda^*}\int_{[0,\lambda_0]}\exp(2\pi\,\mathrm{i}\,\mathrm{Tr}((\nu-\xi)x))\mathrm{d}x \\
&\qquad\qquad \times \int_{\mathbb{R}^n_+} A^{\mathbf{m,r}}_{f,h,\nu}(y)\exp(-2\pi\,\mathrm{Tr}(\xi y))y^{\mathbf{k-2}}\mathrm{d}y \\
&\qquad = \mathrm{vol}(\mathbb{R}^n/\Lambda)\int_{\mathbb{R}^n_+} A^{\mathbf{m,r}}_{f,h,\xi}(y)\exp(-2\pi\,\mathrm{Tr}(\xi y))y^{\mathbf{k-2}}\mathrm{d}y.
\end{aligned} \tag{4.8}$$

Thus we obtain (4.3) by replacing the integral on the right-hand side of (3.9) with (4.8); hence the proof of the theorem is complete. □

We now assume that the Fourier expansions of the mixed automorphic forms $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ and $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ can be written in the forms

$$f(z) = \sum_{\xi \in \Lambda^*} B_\xi \exp(2\pi \mathrm{i} \operatorname{Tr}(\xi z)), \tag{4.9}$$

$$h(z) = \sum_{\eta \in \Lambda^*} C_\eta \exp(2\pi \mathrm{i} \operatorname{Tr}(\eta z)). \tag{4.10}$$

Since χ carries parabolic elements to parabolic elements, we have

$$\operatorname{Im} \omega(z + \lambda) = \operatorname{Im} \omega(z)$$

for all $\lambda \in \Lambda$. Thus $(\operatorname{Im} \omega(z))^{\mathbf{r}}$ has a Fourier expansion as a function of $x \in \operatorname{Re} z$ of the form

$$(\operatorname{Im} \omega(z))^{\mathbf{r}} = \sum_{\nu \in \Lambda^*} W_\nu(y) \exp(2\pi \mathrm{i} \operatorname{Tr}(\nu x)) \tag{4.11}$$

for some functions $W_\nu(y)$ of $y = \operatorname{Im} z$.

Theorem 4.3. *Assume that the Fourier expansions of $f \in \mathcal{S}_{\mathbf{k}+\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ and $h \in \mathcal{S}_{\mathbf{m},\mathbf{r}}(\Gamma, \omega, \chi)$ are as in (4.9) and (4.10), respectively. Then the α-th Fourier coefficient of the Hilbert cusp form $\mathfrak{L}_h^* f$ in (3.7) is given by*

(4.12)

$$A_\alpha(\mathfrak{L}_h^* f) = \prod_{j=1}^{n} \frac{(\alpha_j)^{k_j - 1}}{\mathbf{\Gamma}(k_j - 1)(4\pi)^{m_j}} \\ \times \sum_{\eta,\nu} \frac{B_{\alpha+\eta-\nu} \overline{C}_\eta}{(\alpha + \eta - \nu/2)^{\mathbf{m}+\mathbf{k}-\mathbf{1}}} \int_{\mathbb{R}_+^n} W_\nu(\widetilde{t}) t^{\mathbf{m}+\mathbf{k}-\mathbf{2}} \exp(-|t|) \mathrm{d}t,$$

where $\widetilde{t} = (\widetilde{t}_1, \dots, \widetilde{t}_n)$ with $\widetilde{t}_j = (4\pi(\alpha_j + \eta_j - \nu_j/2))^{-1} t_j$ for $1 \le j \le n$ and $|t|$ denotes the sum of the components of $t \in \mathbb{R}_+^n$.

Proof. Using (4.9), (4.10) and (4.11), the function in (3.8) $\Phi_{f,h}^{\mathbf{m},\mathbf{r}}$ can be written in the form

$$\Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z) = y^{\mathbf{m}} \sum_{\xi,\eta,\nu} B_\xi \overline{C}_\eta W_\nu(y) \exp(2\pi \mathrm{i} \operatorname{Tr}((\xi + \nu - \eta)x)) \\ \times \exp(-2\pi \operatorname{Tr}((\xi + \eta)y)) \\ = y^{\mathbf{m}} \sum_{\alpha,\eta,\nu} B_{\alpha+\eta-\nu} \overline{C}_\eta W_\nu(y) \exp(-2\pi \operatorname{Tr}((\alpha + 2\eta - \nu)y)) \\ \times \exp(2\pi \mathrm{i} \operatorname{Tr}(\alpha x)),$$

where we have introduced a new index $\alpha = \xi + \nu - \eta$ so that $\xi = \alpha + \eta - \nu$. By comparing this with the Fourier expansion of $\Phi_{f,h}^{\mathbf{m},\mathbf{r}}(z)$ in (4.2), we obtain

$$A_{f,h,\alpha}^{\mathbf{m},\mathbf{r}}(y) = y^{\mathbf{m}} \sum_{\eta,\nu} B_{\alpha+\eta-\nu} \overline{C}_\eta W_\nu(y) \exp(-2\pi \operatorname{Tr}((\alpha + 2\eta - \nu)y))$$

for each $\alpha \in \Lambda^*$. Substituting this into (4.3), we have

$$\begin{aligned}
A_\alpha(\mathfrak{L}_h^* f) &= \prod_{j=1}^n \frac{(4\pi\alpha_j)^{k_j-1}}{\mathbf{\Gamma}(k_j-1)} \sum_{\eta,\nu} B_{\alpha+\eta-\nu}\overline{C}_\eta \\
&\quad\times \int_{\mathbb{R}_+^n} W_\nu(y) y^{\mathbf{m}+\mathbf{k}-\mathbf{2}} \exp(-2\pi \operatorname{Tr}((2\alpha+2\eta-\nu)y))\mathrm{d}y, \\
&= \prod_{j=1}^n \frac{(4\pi\alpha_j)^{k_j-1}}{\mathbf{\Gamma}(k_j-1)} \sum_{\eta,\nu} B_{\alpha+\eta-\nu}\overline{C}_\eta \int_{y_1,\ldots,y_n\geq 0} W_\nu(y) \prod_{j=1}^n y_j^{m_j+k_j-2} \\
&\quad\times \exp(-2\pi(2\alpha_j+2\eta_j-\nu_j)y_j)\mathrm{d}y_j.
\end{aligned} \tag{4.13}$$

For $1 \leq j \leq n$, using the new variable $t_j = (4\pi(\alpha_j+\eta_j-\nu_j/2))y_j$, we see that

$$\begin{aligned}
&\int_{y_j\geq 0} W_\nu(y) y_j^{m_j+k_j-2} \exp(-2\pi(2\alpha_j+2\eta_j-\nu_j)y_j)\mathrm{d}y_j \\
&\qquad = \frac{1}{(4\pi(\alpha_j+\eta_j-\nu_j/2))^{m_j+k_j-1}} \int_{t_j\geq 0} W_\nu(y^*) t_j^{m_j+k_j-2} \exp(-t_j)\mathrm{d}y_j,
\end{aligned}$$

where y^* is $y = (y_1,\ldots,y_n)$ with y_j replaced with $(4\pi(\alpha_j+\eta_j-\nu_j/2))^{-1}t_j$. Substituting this into (4.13), we obtain (4.12); hence the proof of the theorem is complete. □

Example 4.4. We consider the result of the previous theorem in the special case for $\mathbf{r} = \mathbf{0}$. Then the functions f, h, and $\mathfrak{L}_h^* f$ in Theorem 4.3 are simply usual Hilbert modular forms of weights $\mathbf{k}+\mathbf{m}$, $\mathbf{m}$, and $\mathbf{k}$, respectively. In this case we can consider an analog of the Dirichlet series of Rankin type $L_{f,h}^\xi(\mathbf{s})$ defined by

$$L_{f,h}^\xi(\mathbf{s}) = \sum_{\eta\in\Lambda^*} \frac{A_{\xi+\eta}\overline{B}_\eta}{(\xi+\eta)^{\mathbf{s}}}$$

for $\xi \in \Lambda^*$ and $\mathbf{s} \in \mathbb{C}^n$. When $\mathbf{r} = \mathbf{0}$, we may set $W_{\mathbf{0}}(y) = 1$ and $W_\nu(y) = 0$ for $\nu \neq \mathbf{0}$ in the series on the right hand side of (4.11); hence (4.12) can be written as

$$\begin{aligned}
A_\alpha(\mathfrak{L}_h^* f) &= \prod_{j=1}^n \frac{(\alpha_j)^{k_j-1}}{\mathbf{\Gamma}(k_j-1)(4\pi)^{m_j}} \\
&\quad\times \sum_{\eta\in\Lambda^*} \frac{B_{\alpha+\eta}\overline{C}_\eta}{(\alpha+\eta)^{\mathbf{m}+\mathbf{k}-\mathbf{1}}} \int_{\mathbb{R}_+^n} t^{\mathbf{m}+\mathbf{k}-\mathbf{2}} \exp(-|t|)\mathrm{d}t.
\end{aligned}$$

However, we see that

$$\int_{\mathbb{R}_+^n} t^{\mathbf{m}+\mathbf{k}-\mathbf{2}} \exp(-|t|)dt = \prod_{j=1}^n \mathbf{\Gamma}(m_j+k_j-1).$$

Thus the Fourier coefficient of $\mathfrak{L}_h^* f$ in (4.12) can be written in the form

$$A_\alpha(\mathfrak{L}_h^* f) = \left(\prod_{j=1}^n \frac{\mathbf{\Gamma}(m_j+k_j-1)(\alpha_j)^{k_j-1}}{\mathbf{\Gamma}(k_j-1)(4\pi)^{m_j}}\right) L_{f,h}^\xi(\mathbf{m}+\mathbf{k}-\mathbf{1})$$

for each $\alpha \in \Lambda^*$.

Remark 4.5. The method used in the proof of Theorem 4.3 was developed by Kohnen [**4**]. Results similar to those described in this section were obtained in [**8**] for modular forms of one variable and in [**6**] for Hilbert modular forms. The case of Siegel modular forms was considered in [**1**].

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