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A MULTI-STEP ITERATIVE METHOD FOR APPROXIMATING FIXED POINTS OF PRESIĆ-KANNAN OPERATORS

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Abstract. The convergence of a Presić type k-step iterative method for a new class of operators $f : X^k \to X$ satisfying a general Presić type contraction condition is proved. Our result is completing an existing list of Presić type iteration methods, see [Rus I. A., *An iterative method for the solution of the equation $x = f(x, \dots, x)$*, Rev. Anal. Numer. Theor. Approx., **10(1)** (1981), 95–100] and the recent [Ćirić L. B., Presić S. B., *On Presić type generalization of the Banach contraction mapping principle*, Acta Math. Univ. Comenianae, **76(2)** (2007), 143–147], having significant potential applications in the study of nonlinear difference equations.

1. Introduction

A dynamic field of research is today devoted to the study of nonlinear difference equations, as proved by a great number of very recent papers on related topics, with applications in economics, biology, ecology, genetics, psychology, sociology, probability theory and others (see for example [**4**], [**5**], [**7**], [**8**], [**10**], [**11**], [**12**], [**13**], [**18**], [**21**], [**22**] and the references therein). Beside some equations present in the titles of the cited papers, we could also mention some known difference equations, to be found for example in [**18**], [**21**] and the papers referred there:

- the generalized Beddington-Holt stock recruitment model:
$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}, \qquad x_0, x_1 > 0, \qquad n \in \mathbb{N},$$
where $a \in (0, 1)$, $b \in \mathbb{R}^*_+$ and $c, d \in \mathbb{R}_+$ with $c + d > 0$;
- the delay model of a perennial grass:
$$x_{n+1} = ax_n + (b + cx_{n-1})e^{x_n}, \qquad n \in \mathbb{N},$$
where $a, c \in (0, 1)$ and $b \in \mathbb{R}_+$;
- the flour beetle population model:
$$x_{n+3} = ax_{n+2} + bx_n e^{-(cx_{n+2}+dx_n)}, \qquad n \in \mathbb{N},$$
where $a, b, c, d \geq 0$ and $c + d > 0$.

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These suggest considering the k-th order nonlinear difference equation

$$x_{n+k} = f(x_n, \dots, x_{n+k-1}), \qquad n \in \mathbb{N}, \tag{1.1}$$

with the initial values $x_0, \dots, x_k \in X$, where (X, d) is a metric space, $k \in \mathbb{N}$, $k \geq 1$ and $f : X^k \to X$.

Equation (1.1) can be studied by means of a fixed point theory in view of the fact that $x^* \in X$ is a solution of (1.1) if and only if x^* is a fixed point of f, that is

$$x^* = f(x^*, \dots, x^*).$$

One of the most important results on this direction has been obtained by S. Presić in [**14**]:

Theorem 1 (S. Presić [**14**], 1965)**.** *Let (X, d) be a complete metric space, k a positive integer, $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^k \alpha_i = \alpha < 1$ and $f : X^k \to X$ a mapping satisfying*

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \alpha_1 d(x_0, x_1) + \cdots + \alpha_k d(x_{k-1}, x_k), \tag{P}$$

for all $x_0, \dots, x_k \in X$.

Then:

1) *f has a unique fixed point $x^* \in X$;*
2) *the sequence $\{x_n\}_{n \geq 0}$ defined by*

$$x_{n+1} = f(x_{n-k+1}, \dots, x_n), \quad n = k-1, k, k+1, \dots \tag{1.2}$$

converges to x^* for any $x_0, \dots, x_{k-1} \in X$.

Notice that Theorem 1 is an inspired generalization of the Contraction Mapping Principle of Banach, which can be derived for $k = 1$.

An important generalization of Theorem 1, probably not yet sufficiently exploited in applications, was proved in I. A. Rus [**17**], see also [**18**], for operators f fulfilling

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \varphi(d(x_0, x_1), \dots, d(x_{k-1}, x_k)), \tag{PR}$$

for any $x_0, \dots, x_k \in X$, where $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfies:

a) if $r, s \in \mathbb{R}_+^k$, $r \leq s$, then $\varphi(r) \leq \varphi(s)$;
b) if $t \in \mathbb{R}_+$, $t > 0$, then $\varphi(t, \dots, t) < t$;
c) φ is continuous;
d) $\sum\limits_{i=0}^{\infty} \varphi^i(r) < \infty$ for any $r \in \mathbb{R}_+^k$;
e) $\varphi(t, 0, \dots, 0) + \varphi(0, t, 0, \dots, 0) + \cdots + \varphi(0, \dots, 0, t) \leq \varphi(t, \dots, t)$ for any $t \in \mathbb{R}_+$.

Significant related results can be found in [**20**].

Another important generalization of Presić' result was recently obtained by L. Ćirić and S. Presić in [**6**], where the following contraction condition is considered:

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \lambda \max\{d(x_0, x_1), \dots, d(x_{k-1}, x_k)\} \tag{PC}$$

for any $x_0, \ldots, x_k \in X$, where $\lambda \in (0,1)$. It is not difficult to notice that $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$, $\varphi(t_1, \ldots, t_k) = \lambda \max\{t_1, \ldots, t_k\}$, corresponding to condition (PC), does not satisfy condition e) in the theorem of I. A. Rus.

The applicability of the result due to L. Ćirić and S. Presić to the study of global asymptotic stability of the equilibrium for the nonlinear difference equation (1.1) is revealed, for example, in the very recent paper [**5**].

Motivated by this background and also by the importance of the convergence of k-step iteration methods in the study of nonlinear equations (see, for example, the famous monograph [**13**] of J. M. Ortega and W. C. Rheinboldt), in this paper we prove the convergence of the k-step iteration method defined by (1.1) for a new class of Presić type operators, also providing an estimate of its rate of convergence.

Unlike the theorems mentioned above, our result does not generalize the Contraction Principle of Banach, but the independent (see [**15**]) one due to R. Kannan [**9**] who considers the condition:

$$d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))]$$

for any $x, y \in X$, where $f : X \to X$ and $k \in [0, \frac{1}{2})$.

In order to certify the validity of the main result, we shall also include a very simple example of operator $f : [0,1] \times [0,1] \to [0,1]$ which satisfies the new Presić-Kannan condition, but does not satisfy any of the previously mentioned Presić type conditions (P), (PR) or (PC).

In other words, a new class of Presić type operators, which cannot be approached by means of other Presić type theorems, is outlined. Therefore the convergence result proved in this paper has a significant potential applicability in the study of nonlinear difference equations.

2. The main result

In order to prove our main result, we need the following lemma given by S. Presić [**14**].

Lemma 1 (Presić, [**14**]). *Let $k \in \mathbb{N}$, $k \neq 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$ such that $\sum\limits_{i=1}^{k} \alpha_i = \alpha < 1$. If $\{\Delta_n\}_{n \geq 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \leq \alpha_1 \Delta_n + \alpha_2 \Delta_{n+1} + \ldots + \alpha_k \Delta_{n+k-1}, \qquad n \geq 1. \tag{2.1}$$

Then there exist $L > 0$ and $\theta \in (0,1)$ such that

$$\Delta_n \leq L \cdot \theta^n, \qquad \text{for all } n \geq 1. \tag{2.2}$$

The main result of this paper is the following theorem.

Theorem 2. *Let (X,d) be a complete metric space, k a positive integer, $a \in \mathbb{R}$ a constant such that $0 < ak(k+1) < 1$ and $f : X^k \to X$ a mapping satisfying the following contractive type condition:*

$$d(f(x_0,\dots,x_{k-1}), f(x_1,\dots,x_k)) \le a \sum_{i=0}^{k} d(x_i, f(x_i,\dots,x_i)) \tag{PK}$$

for any $x_0, x_1, \dots, x_k \in X$.
Then:

1) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*,\dots,x^*) = x^*$;*
2) *the sequence $\{y_n\}_{n\ge 0}$,*

$$y_{n+1} = f(y_n, y_n, \dots, y_n), \qquad n \ge 0, \tag{2.3}$$

converges to x^*;

3) *the sequence $\{x_n\}_{n\ge 0}$ with $x_0,\dots,x_{k-1} \in X$ and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), \qquad n \ge k, \tag{2.4}$$

also converges to x^* with a rate estimated by:

$$d(x_{n+1}, x^*) \le \frac{aL}{1-A} M\theta^n, \qquad n \ge 0, \tag{2.5}$$

where $M = \theta^{1-k} + 2\theta^{2-k} + \dots + k$, $A = \dfrac{ak(k+1)}{2}$, $L > 0$ and $\theta \in (0,1)$.

Proof. Let $F : X \to X$, $F(x) = f(x, x, \dots, x)$, $x \in X$. For any $x, y \in X$, one has:

$$\begin{aligned} d(F(x), F(y)) &= d(f(x,x,\dots,x), f(y,y,\dots,y)) \\ &\le d(f(x,\dots,x), f(x,\dots,x,y)) \\ &\quad + d(f(x,\dots,x,y), f(x,\dots,x,y,y)) + \dots \\ &\quad + d(f(x,y,\dots,y), f(y,\dots,y)). \end{aligned}$$

By (PK) it follows that

$$\begin{aligned} d(F(x),F(y)) \le a &\Big[\underbrace{d(x,f(x,\dots,x)) + \dots + d(x,f(x,\dots,x))}_{k \text{ times}} + d(y,f(y,\dots,y))\Big] \\ &+ a\Big[\underbrace{d(x,f(x,\dots,x)) + \dots + d(x,f(x,\dots,x))}_{k-1 \text{ times}} + \\ &\qquad + \underbrace{d(y,f(y,\dots,y)) + d(y,f(y,\dots,y))}_{2 \text{ times}}\Big] + \dots \\ &+ a\Big[d(x,f(x,\dots,x)) + \underbrace{d(y,f(y,\dots,y)) + \dots + d(y,f(y,\dots,y))}_{k \text{ times}}\Big], \end{aligned}$$

so

$$d(F(x), F(y)) \leq ad(x, f(x, \ldots, x))[k + (k-1) + \ldots + 1] \\ + ad(y, f(y, \ldots, y))[1 + 2 + \ldots + k]$$

and finally

$$d(F(x), F(y)) \leq a\frac{k(k+1)}{2}\left[d(x, f(x, \ldots, x)) + d(y, f(y, \ldots, y))\right].$$

Thus

$$d(F(x), F(y)) \leq a\frac{k(k+1)}{2}\left[d(x, F(x)) + d(y, F(y))\right] \tag{2.6}$$

for any $x, y \in X$.

As a was assumed to satisfy $0 < ak(k+1) < 1$, it follows that $0 < a\frac{k(k+1)}{2} < \frac{1}{2}$, so F is a Kannan operator. According to the fixed point theorem due to Kannan [**9**], there exists a unique $x^* \in X$ such that $F(x^*) = x^*$, namely

$$x^* = f(x^*, \ldots, x^*),$$

and this can be obtained as a limit of the sequence of successive approximations of F. We mean exactly the sequence $\{y_n\}_{n\geq 0}$ defined by (2.3).

Now we shall prove the convergence of the k-step method given by the above sequence $\{x_n\}_{n\geq 0}$ defined by relation (2.4). As we already know that f has a unique fixed point $x^* \in X$, we may write:

$$\begin{aligned} d(x_{n+1}, x^*) &= d(f(x_{n-k+1}, x_{n-k+2}, \ldots, x_n), f(x^*, x^*, \ldots, x^*)) \\ &\leq d(f(x_{n-k+1}, \ldots, x_n), f(x_{n-k+2}, \ldots, x_n, x^*)) \\ &\quad + d(f(x_{n-k+2}, \ldots, x_n, x^*), f(x_{n-k+3}, \ldots, x_n, x^*, x^*)) + \ldots \\ &\quad + d(f(x_n, x^*, \ldots, x^*), f(x^*, x^*, \ldots, x^*)), \end{aligned} \tag{2.7}$$

which yields

$$\begin{aligned} d(x_{n+1}, x^*) \leq{}& a\left[d(x_{n-k+1}, F(x_{n-k+1})) + \ldots + d(x_n, F(x_n)) + d(x^*, F(x^*))\right] \\ &+ a\left[d(x_{n-k+2}, F(x_{n-k+2})) + \ldots + d(x_n, F(x_n))\right. \\ &+ \left. d(x^*, F(x^*)) + d(x^*, F(x^*))\right] + \ldots \\ &+ a\left[d(x_n, F(x_n)) + d(x^*, F(x^*)) + \ldots + d(x^*, F(x^*))\right]. \end{aligned}$$

Since $d(x^*, F(x^*)) = 0$, this implies

$$\begin{aligned} d(x_{n+1}, x^*) \leq{}& a\left[1 \cdot d(x_{n-k+1}, F(x_{n-k+1}) + 2 \cdot d(x_{n-k+2}, F(x_{n-k+2}))\right. \\ &+ \left. \ldots + k \cdot d(x_n, F(x_n))\right]. \end{aligned} \tag{2.8}$$

For each $j \in \mathbb{N}$, the following holds

$$d(x_j, F(x_j)) \leq d(x_j, x^*) + d(x^*, F(x_j)). \tag{2.9}$$

Also, by (2.6), one has

$$\begin{aligned} d(x^*, F(x_j)) &= d(F(x^*), F(x_j)) \\ &\leq a\frac{k(k+1)}{2}\left[d(x^*, F(x^*)) + d(x_j, F(x_j))\right] \\ &= a\frac{k(k+1)}{2} d(x_j, F(x_j)). \end{aligned} \tag{2.10}$$

Thus (2.9) becomes

$$d(x_j, F(x_j)) \leq d(x_j, x^*) + a\frac{k(k+1)}{2} d(x_j, F(x_j)).$$

By denoting $A = \frac{ak(k+1)}{2}$, now we get

$$d(x_j, F(x_j)) \leq \frac{1}{1-A} d(x_j, x^*), \tag{2.11}$$

for each $j \in \mathbb{N}$.

Using (2.11) in inequality (2.8), we obtain

$$\begin{aligned} d(x_{n+1}, x^*) \leq \frac{a}{1-A} d(x_{n-k+1}, x^*) &+ \frac{2a}{1-A} d(x_{n-k+2}, x^*) + \ldots \\ &+ \frac{ka}{1-A} d(x_n, x^*). \end{aligned} \tag{2.12}$$

Now, by denoting

$$\Delta_n = d(x_n, x^*), \qquad n \geq 0,$$
$$\alpha_i = \frac{i \cdot a}{1-A}, \qquad i = \overline{1, k},$$

the above inequality (2.12) becomes

$$\Delta_{n+1} \leq \alpha_1 \Delta_{n-k+1} + \alpha_2 \Delta_{n-k+2} + \ldots + \alpha_k \Delta_n, \qquad n \geq k. \tag{2.13}$$

The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are all positive, as $0 < a < \frac{1}{k(k+1)}$. Besides,

$$\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \frac{ia}{1-A} = \frac{a}{1-A} \sum_{i=1}^{k} i = \frac{a}{1-A} \cdot \frac{k(k+1)}{2} = \frac{A}{1-A},$$

so, considering the conditions on a and implicitely on A, it is easy to prove that $\sum_{i=1}^{k} \alpha_i < 1$.

Now the conditions required in Lemma 1 are fulfilled. Consequently, there exist $L > 0$ and $\theta \in (0, 1)$ such that $\Delta_n \leq L\theta^n, n \geq 1$, namely such that

$$d(x_n, x^*) \leq L\theta^n, n \geq 1. \tag{2.14}$$

It follows immediately that $d(x_n, x^*) \to 0$ as $n \to \infty$, so the sequence $\{x_n\}_{n \geq 0}$ converges to x^*, the unique fixed point of the operator f.

The estimation (2.5) is easily obtained from (2.12), by repeatedly using inequality (2.14).

Now the proof is complete. □

Remark 1. In the particular case $k = 1$, from Theorem 2 we obtain Kannan's fixed point theorem for discontinuous mappings in [**9**].

A corresponding *data dependence* result can also be proved:

Theorem 3. *Let (X, d) be a complete metric space, k a positive integer, $f : X^k \to X$ as in Theorem 2 and $g : X^k \to X$ satisfying:*

ι) *g has at least one fixed point $x_g^* \in X$;*

$\iota\iota$) *there exists $\eta > 0$ such that for any $x \in X$*

$$d(f(x, \dots, x), g(x, \dots, x)) \le \eta.$$

Then

$$d(x_f^*, x_g^*) \le \left[1 + a \cdot \frac{k(k+1)}{2}\right] \eta, \tag{2.15}$$

where $F_f = \{x_f^*\}$.

Proof. By Theorem 2, condition ι) above guarantees the existence and uniqueness of the fixed point x_f^* for f. Thus we may write

$$\begin{aligned} d(x_f^*, x_g^*) &= d(f(x_f^*, \dots, x_f^*), g(x_g^*, \dots, x_g^*)) \\ &\le d(f(x_f^*, \dots, x_f^*), f(x_g^*, \dots, x_g^*)) + d(f(x_g^*, \dots, x_g^*), g(x_g^*, \dots, x_g^*)). \end{aligned}$$

By $\iota\iota$) we can write

$$\begin{aligned} d(x_f^*, x_g^*) \le \eta &+ d(f(x_f^*, \dots, x_f^*), f(x_f^*, \dots, x_f^*, x_g^*)) \\ &+ \dots + d(f(x_f^*, x_g^*, \dots, x_g^*), f(x_g^*, \dots, x_g^*)) \end{aligned}$$

and further on

$$\begin{aligned} d(x_f^*, x_g^*) \le \eta &+ a \left[d(x_f^*, f(x_f^*, \dots, x_f^*)) + \dots \right. \\ &\left. + d(x_f^*, f(x_f^*, \dots, x_f^*)) + d(x_g^*, f(x_g^*, \dots, x_g^*)) \right] \dots \\ &+ a \left[d(x_f^*, f(x_f^*, \dots, x_f^*)) + d(x_g^*, f(x_g^*, \dots, x_g^*)) + \dots \right. \\ &\left. + d(x_g^*, f(x_g^*, \dots, x_g^*)) \right]. \end{aligned}$$

After some elementary calculations

$$\begin{aligned} d(x_f^*, x_g^*) &\le \eta + a d(x_g^*, f(x_g^*, \dots, x_g^*))[1 + 2 + \dots + k] \\ &= \eta + a \frac{k(k+1)}{2} d(x_g^*, f(x_g^*, \dots, x_g^*)) \\ &= \eta + a \frac{k(k+1)}{2} d(g(x_g^*, \dots, x_g^*), f(x_g^*, \dots, x_g^*)), \end{aligned}$$

we finally get to

$$d(x_f^*, x_g^*) \le \left[1 + a \frac{k(k+1)}{2}\right] \eta.$$

□

Remark 2. Note that if we consider the conditions on a in Theorem 3, estimation (2.15) actually implies that

$$d(x_f^*, x_g^*) \le \frac{3}{2}\eta.$$

Obviously, $\left[1 + a\dfrac{k(k+1)}{2}\right]\eta$ as well as $\dfrac{3}{2}\eta$ tends to zero as $\eta \to 0$.

3. Conclusions

Before formulating a conclusion, let us present an elementary example of operator that can be approached by means of Theorem 2, whereas other known Presić type theorems cannot be applied.

Example 1. Let $f : [0,1] \times [0,1] \to [0,1]$ be defined by

$$f(x,y) = \begin{cases} \dfrac{1}{6}, & x < \dfrac{3}{4}, y \in [0,1] \\[2ex] \dfrac{1}{15}, & x \ge \dfrac{3}{4}, y \in [0,1]. \end{cases} \tag{3.1}$$

Then:

1) f is a Presić-Kannan operator, i.e., it satisfies condition (PK);
2) f is not a Presić operator, i.e., it does not satisfy condition (P);
3) f is not a Ćirić-Presić operator, i.e., it does not satisfy condition (PC);
4) f is not a Presić-Rus operator, i.e., it does not satisfy condition (PR).

Proof. 1) In the first part of the proof we will show that f is a Presić-Kannan operator. In this particular case condition (PK) becomes

$$\begin{aligned} |f(x_0,x_1) - f(x_1,x_2)| \le a[&|x_0 - f(x_0,x_0)| + |x_1 - f(x_1,x_1)| \\ &+ |x_2 - f(x_2,x_2)|], \end{aligned} \tag{Ex.P-K}$$

for any $x_0, x_1, x_2 \in [0,1]$, where $a \in [0, \frac{1}{6})$ is constant.

Considering the way of defining f, we may divide the domain $[0,1]\times[0,1]$ in four regions:

$$D_1 = \left\{(x,y)\,|\,0 \le x, y < \frac{3}{4}\right\} \quad D_2 = \left\{(x,y)\,|\,\frac{3}{4} \le x \le 1; 0 \le y < \tfrac{3}{4}\right\}$$
$$D_3 = \left\{(x,y)\,|\,\frac{3}{4} \le x, y \le 1\right\} \quad D_4 = \left\{(x,y)\,|\,0 \le x < \frac{3}{4}; \frac{3}{4} \le y \le 1\right\}.$$

Indeed, $[0,1]\times[0,1] = D_1 \cup D_2 \cup D_3 \cup D_4$.

With these notations, due to the way of defining f, we have to discuss 5 cases:

I. $(x_0, x_1) \in D_1$ or $(x_0, x_1) \in D_3$, while $x_2 \in [0,1]$.
Then $f(x_0,x_1) = f(x_1,x_2)$ and the left-hand side of (Ex.P-K) is equal to 0. Consequently, (Ex.P-K) holds for any x_0, x_1, x_2 in the specified domains and any $a \in \left[0, \frac{1}{6}\right)$.

II. $(x_0, x_1) \in D_2,\ x_2 < \dfrac{3}{4}$.

Then $f(x_0, x_1) = \dfrac{1}{15},\ f(x_1, x_2) = \dfrac{1}{6}$,

and $f(x_0, x_0) = \dfrac{1}{15},\ f(x_1, x_1) = \dfrac{1}{6},\ f(x_2, x_2) = \dfrac{1}{6}$.

Thus condition (Ex.P-K) becomes

$$\frac{1}{10} \leq a \left[\left| x_0 - \frac{1}{15} \right| + \left| x_1 - \frac{1}{6} \right| + \left| x_2 - \frac{1}{6} \right| \right], \tag{3.2}$$

but:

$$\frac{3}{4} \leq x_0 \leq 1 \Rightarrow \frac{41}{60} \leq x_0 - \frac{1}{15} \leq \frac{14}{15} \Rightarrow \left| x_0 - \frac{1}{15} \right| \geq \frac{41}{60};$$

$$0 \leq x_1 < \frac{3}{4} \Rightarrow -\frac{1}{6} \leq x_1 - \frac{1}{6} < \frac{7}{12} \Rightarrow \left| x_1 - \frac{1}{6} \right| \geq 0;$$

$$0 \leq x_2 < \frac{3}{4} \Rightarrow -\frac{1}{6} \leq x_2 - \frac{1}{6} < \frac{7}{12} \Rightarrow \left| x_2 - \frac{1}{6} \right| \geq 0,$$

so $\left| x_0 - \dfrac{1}{15} \right| + \left| x_1 - \dfrac{1}{6} \right| + \left| x_2 - \dfrac{1}{6} \right| \geq \dfrac{41}{60}$

and by (3.2) it follows that

$$\frac{1}{10} \leq a \frac{41}{60}.$$

Consequently, for (3.2) to hold, it is necessary that $a \geq \dfrac{6}{41}$.

III. $(x_0, x_1) \in D_2,\ x_2 \geq \dfrac{3}{4}$.

Then $f(x_0, x_1) = \dfrac{1}{15},\ \ f(x_1, x_2) = \dfrac{1}{6}$,

and $f(x_0, x_0) = \dfrac{1}{15},\ \ f(x_1, x_1) = \dfrac{1}{6},\ \ f(x_2, x_2) = \dfrac{1}{15}$.

Thus condition (Ex.P-K) becomes

$$\frac{1}{10} \leq a \left[\left| x_0 - \frac{1}{15} \right| + \left| x_1 - \frac{1}{6} \right| + \left| x_2 - \frac{1}{15} \right| \right]. \tag{3.3}$$

But:

$$\left| x_0 - \frac{1}{15} \right| \geq \frac{41}{60}, \quad \left| x_1 - \frac{1}{6} \right| \geq 0, \quad \left| x_2 - \frac{1}{15} \right| \geq \frac{41}{60},$$

so $\left| x_0 - \dfrac{1}{15} \right| + \left| x_1 - \dfrac{1}{6} \right| + \left| x_2 - \dfrac{1}{6} \right| \geq \dfrac{41}{30}$,

and by (3.3) it follows that

$$\frac{1}{10} \leq a \frac{41}{30}.$$

Consequently, for (3.3) to hold, it is necessary that $a \geq \dfrac{3}{41}$.

IV. $(x_0, x_1) \in D_4,\;\; x_2 < \dfrac{3}{4}$.

Similarly to case II, it follows that $a \geq \dfrac{6}{41}$.

V. $(x_0, x_1) \in D_4,\;\; x_2 \geq \dfrac{3}{4}$.

Similarly to case III, it follows that $a \geq \dfrac{3}{41}$.

The conclusion after analyzing these 5 cases is that f given by (3.1) is a Presić-Kannan operator, that is, it satisfies (PK) for any $x_0, x_1, x_2 \in [0, 1]$, with constant $a \in \left[\dfrac{6}{41}, \dfrac{1}{6}\right)$.

2) Now, we shall prove that f is not a Presić operator. In our particular case inequality (P) becomes

$$\text{(Ex.P)} \qquad |f(x_0, x_1) - f(x_1, x_2)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2|,$$

where $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_1 + \alpha_2 < 1$.

It suffices to take, for example, $x_0 = \dfrac{3}{4}$ and $x_1 = x_2 = \dfrac{7}{10}$. Then $f(x_0, x_1) = \dfrac{1}{15}$, while $f(x_1, x_2) = \dfrac{1}{6}$ and inequality (Ex.P) becomes

$$\left|\frac{1}{15} - \frac{1}{6}\right| \leq \alpha_1 \left|\frac{3}{4} - \frac{7}{10}\right| + \alpha_2 \left|\frac{7}{10} - \frac{7}{10}\right|,$$

which is equivalent to

$$\frac{1}{10} \leq \alpha_1 \frac{1}{20}. \tag{3.4}$$

Since $\alpha_1 < 1$, it is obvious that (3.4) will never hold. Thus f is not a Presić operator.

3) We shall prove that f is neither a Ćirić-Presić operator. In our particular case inequality (PC) becomes:

$$\text{(Ex.PC)} \qquad |f(x_0, x_1) - f(x_1, x_2)| \leq \lambda \max\{|x_0 - x_1|, |x_1 - x_2|\},$$

where $\lambda \in (0, 1)$.

For the same values as above, namely $x_0 = \dfrac{3}{4}$ and $x_1 = x_2 = \dfrac{7}{10}$, (Ex.PC) is:

$$\frac{1}{10} \leq \lambda \max\left\{\frac{1}{20}, 0\right\},$$

which again never holds since $\lambda \in (0, 1)$.

4) At last we shall prove that f does not satisfy the condition (PR) mentioned above. In our particular case this would imply the existence of a function $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ with the following properties:

a) $r = (r_1, r_2)$, $s = (s_1, s_2) \in \mathbb{R}_+^2$, $r \leq s$, $\;\varphi(r_1, r_2) \leq \varphi(s_1, s_2)$;

b) $\varphi(t,t) < t$, for any $t \in \mathbb{R}_+$, $t > 0$;
c) φ is continuous;
d) $\sum_{i=0}^{\infty} \varphi(r) < \infty$, for any $r \in \mathbb{R}_+^2$;
e) $\varphi(t,0) + \varphi(0,t) \leq \varphi(t,t)$, for any $t \in \mathbb{R}_+$,

such that the following also holds

$$|f(x_0,x_1) - f(x_1,x_2)| \leq \varphi(|x_0 - x_1|, |x_1 - x_2|) \tag{3.5}$$

for any $x_0, x_1, x_2 \in [0,1]$.

Letting $\varepsilon > 0$, $x_0 = \frac{3}{4} - \varepsilon < \frac{3}{4}$, $x_1 = \frac{3}{4} \geq \frac{3}{4}$ and $x_2 = \frac{3}{4} \in [0,1]$, we have $f(x_0,x_1) = \frac{1}{6}$, $f(x_1,x_2) = \frac{1}{15}$. Then (3.5) becomes

$$\frac{1}{10} \leq \varphi(|x_0 - x_1|, |x_1 - x_2|). \tag{3.6}$$

Since $|x_0 - x_1| = \varepsilon$ and $|x_1 - x_2| = 0$, (3.6) becomes

$$\frac{1}{10} \leq \varphi(\varepsilon, 0).$$

Using the properties of φ, this implies

$$\frac{1}{10} \leq \varphi(\varepsilon, \varepsilon) < \varepsilon$$

which obviously does not hold for any $\varepsilon > 0$, so f cannot be a Presić-Rus operator. □

As shown by this simple example, there are operators (not necessarily continuous) and corresponding difference equations which cannot be approached by means of the Presić type results mentioned in the introductory section, but to which Theorem 2 can be applied.

Therefore, for example, in view of the study in [**5**] based on the theorem of L. Ćirić and S. Presić, Theorem 2 proposed in the present paper appears to have potential applicability in the study of nonlinear difference equations, targeting special classes of operators that cannot be approached by means of other known Presić type theorems.

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