Acta Math. Univ. Comenianae
Vol. LXXIX, 1(2010), pp. 129–134

ON THE OSTROWSKI TYPE INTEGRAL INEQUALITY

M. Z. SARIKAYA

Abstract. In this note, we establish an inequality of Ostrowski-type involving functions of two independent variables newly by using certain integral inequalities.

1. Introduction

In [**3**], Ujević proved the following double integral inequality:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and suppose that $\gamma \le f^{''}(t) \le \Gamma$ for all $t \in (a,b)$. Then we have the double inequality*

$$\frac{3S-\Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{3S-\gamma}{24}(b-a)^2 \tag{1.1}$$

where $S = \dfrac{f'(b)-f'(a)}{b-a}$.

In a recent paper [**2**], Liu et al. have proved the following two sharp inequalities of perturbed Ostrowski-type

Theorem 2. *Under the assumptations of Theorem 1, we have*

$$\begin{aligned} &\frac{\Gamma[(x-a)^3-(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^2 (S-\Gamma) \\ &\qquad \le \frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\qquad \le \frac{\gamma[(x-a)^3-(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^2 (S-\gamma), \end{aligned} \tag{1.2}$$

for all $x \in [a,b]$,

where $S = \dfrac{f'(b)-f'(a)}{b-a}$. *If* γ, Γ *are given by*

$$\gamma = \min_{t\in[a,b]} f''(t), \qquad \Gamma = \max_{t\in[a,b]} f''(t)$$

then the inequality given by (2) *is sharp in the usual sense.*

Received March 9, 2009; revised August 11, 2009.
2000 *Mathematics Subject Classification.* Primary 26D07, 26D15.
Key words and phrases. Ostrowski's inequality.

In [**1**], Cheng has proved the following integral inequality

Theorem 3. *Let* $I \subset \mathbb{R}$ *be an open interval,* $a, b \in I$, $a < b$. $f : I \to \mathbb{R}$ *is a differentiable function such that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \le f'(x) \le \Gamma$, $x \in [a, b]$. *Then we have*

$$\left|\frac{1}{2}f(x) - \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le \frac{(x-a)^2+(b-x)^2}{8(b-a)}(\Gamma-\gamma), \tag{1.3}$$

for all $x \in [a, b]$.

The main purpose of this paper is to establish new inequality similar to the inequalities (1.1)–(1.3) involving functions of two independent variables.

2. Main Result

Theorem 4. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be an absolutely continuous fuction such that the partial derivative of order* 2 *exists and supposes that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \le \frac{\partial^2 f(t,s)}{\partial t \partial s} \le \Gamma$ *for all* $(t, s) \in [a, b] \times [c, d]$. *Then, we have*

$$\begin{aligned}
&\left|\frac{1}{4}f(x,y) + \frac{1}{4}H(x,y) - \frac{1}{2(b-a)}\int_a^b f(t,y)\mathrm{d}t - \frac{1}{2(d-c)}\int_c^d f(x,s)\mathrm{d}s\right.\\
&\qquad - \frac{1}{2(b-a)(d-c)}\int_a^b [(y-c)f(t,c)+(d-y)f(t,d)]\mathrm{d}t\\
&\qquad - \frac{1}{2(b-a)(d-c)}\int_c^d [(x-a)f(a,s)+(b-x)f(b,s)]\mathrm{d}s\\
&\qquad \left. + \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\mathrm{d}s\,\mathrm{d}t\right|\\
&\quad \le \frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32(b-a)(d-c)}(\Gamma-\gamma),
\end{aligned} \tag{2.1}$$

for all $(x, y) \in [a, b] \times [c, d]$ *where*

$$\begin{aligned}
&H(x,y)\\
&= \frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)}\\
&+ \frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a} + \frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.
\end{aligned}$$

Proof. We define the functions: $p : [a, b] \times [a, b] \to \mathbb{R}$, $q : [c, d] \times [c, d] \to \mathbb{R}$ given by

$$p(x,t) = \begin{cases} t - \dfrac{a+x}{2}, & t \in [a,x] \\ t - \dfrac{b+x}{2}, & t \in (x,b] \end{cases}$$

and

$$q(y,s) = \begin{cases} s - \dfrac{c+y}{2}, & s \in [c,y] \\ s - \dfrac{d+y}{2}, & s \in (y,d]. \end{cases}$$

By definitions of $p(x, t)$ and $q(y, s)$, we have

$$\begin{aligned} &\int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\mathrm{d}t \\ &= \int_a^x \int_c^y \left(t - \frac{a+x}{2}\right)\left(s - \frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t \\ &\quad + \int_a^x \int_y^d \left(t - \frac{a+x}{2}\right)\left(s - \frac{d+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t \\ &\quad + \int_x^b \int_c^y \left(t - \frac{b+x}{2}\right)\left(s - \frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t \\ &\quad + \int_x^b \int_y^d \left(t - \frac{b+x}{2}\right)\left(s - \frac{d+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{2.2}$$

Integrating by parts twice, we can state:

$$\begin{aligned} &\int_a^x \int_c^y \left(t - \frac{a+x}{2}\right)\left(s - \frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t \\ &= \frac{(x-a)(y-c)}{4}[f(x,y) + f(a,y) + f(x,c) + f(a,c)] \\ &\quad - \frac{y-c}{2}\int_a^x [f(t,y) + f(t,c)]\mathrm{d}t - \frac{x-a}{2}\int_c^y [f(x,s) + f(a,s)]\mathrm{d}s \\ &\quad + \int_a^x \int_c^y f(t,s)\mathrm{d}s\mathrm{d}t. \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&\int_a^x\int_y^d \left(t-\frac{a+x}{2}\right)\left(s-\frac{d+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&\quad=\frac{(x-a)(d-y)}{4}[f(x,y)+f(x,d)+f(a,y)+f(a,d)]\\
&\qquad-\frac{d-y}{2}\int_a^x[f(t,d)+f(t,y)]\mathrm{d}t-\frac{x-a}{2}\int_y^d[f(x,s)+f(a,s)]\mathrm{d}s\\
&\qquad+\int_a^x\int_y^d f(t,s)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}\tag{2.4}
$$

$$
\begin{aligned}
&\int_x^b\int_c^y \left(t-\frac{b+x}{2}\right)\left(s-\frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&\quad=\frac{(b-x)(y-c)}{4}[f(x,y)+f(b,y)+f(x,c)+f(b,c)]\\
&\qquad-\frac{y-c}{2}\int_x^b[f(t,c)+f(t,y)]\mathrm{d}t-\frac{b-x}{2}\int_c^y[f(x,s)+f(b,s)]\mathrm{d}s\\
&\qquad+\int_x^b\int_c^y f(t,s)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}\tag{2.5}
$$

$$
\begin{aligned}
&\int_x^b\int_y^d (t-\frac{b+x}{2})(s-\frac{d+y}{2})\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&\quad=\frac{(b-x)(d-y)}{4}[f(x,y)+f(x,d)+f(b,y)+f(b,d)]\\
&\qquad-\frac{d-y}{2}\int_x^b[f(t,d)+f(t,y)]\mathrm{d}t-\frac{b-x}{2}\int_y^d[f(x,s)+f(b,s)]\mathrm{d}s\\
&\qquad+\int_x^b\int_y^d f(t,s)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}\tag{2.6}
$$

Adding (2.3)–(2.6) and rewriting, we easily deduce

$$
\begin{aligned}
\int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t = \frac{1}{4}\{&(b-a)(d-c)f(x,y) \\
&+ [(x-a)f(a,y) + (b-x)f(b,y)](d-c) \\
&+ [(y-c)f(x,c) + (d-y)f(x,d)](b-a) \\
&+ [(y-c)f(a,c) + (d-y)f(a,d)](x-a) \\
&+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x)\} \\
&- \frac{d-c}{2}\int_a^b f(t,y)\mathrm{d}t - \frac{b-a}{2}\int_c^d f(x,s)\mathrm{d}s \\
&- \frac{1}{2}\int_a^b [(y-c)f(t,c) + (d-y)f(t,d)]\mathrm{d}t \\
&- \frac{1}{2}\int_c^d [(x-a)f(a,s) + (b-x)f(b,s)]\mathrm{d}s \\
&+ \int_a^b \int_c^d f(t,s)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
\tag{2.7}
$$

We also have

$$
\int_a^b \int_c^d p(x,t)q(y,s)\mathrm{d}s\,\mathrm{d}t = 0. \tag{2.8}
$$

Let $M = \frac{\Gamma+\gamma}{2}$. From (2.7) and (2.8), it follows that

$$
\begin{aligned}
&\int_a^b \int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right]\mathrm{d}s\,\mathrm{d}t \\
&\qquad = \int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
\tag{2.9}
$$

On the other hand, we get

$$
\begin{aligned}
&\left|\int_a^b \int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right]\mathrm{d}s\mathrm{d}t\right| \\
&\qquad \le \max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right| \int_a^b \int_c^d |p(x,t)q(y,s)|\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
\tag{2.10}
$$

We also have

$$\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right|\le\frac{\Gamma-\gamma}{2} \tag{2.11}$$

and

$$\int_a^b\int_c^d |p(x,t)q(y,s)|\,\mathrm{d}s\,\mathrm{d}t=\frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{16}. \tag{2.12}$$

From (2.10) to (2.12), we easily get

$$\begin{aligned}&\left|\int_a^b\int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right]\mathrm{d}s\,\mathrm{d}t\right|\\ &\qquad\le\frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32}(\Gamma-\gamma).\end{aligned} \tag{2.13}$$

From (2.9) and (2.13), we see that (2.1) holds. □

References

1. Cheng X. L., *Improvement of some Ostrowski-Grüss type inequalities,* Computers Math. Applic., **42** (2001), 109–114.

2. Liu W-J., Xue Q-L. and Wang S-F., *Several new perturbed Ostrowski-like type inequalities*, J. Inequal. Pure and App.Math.(JIPAM), **8(4)** (2007), Article:110.

3. Ujević N., *Some double integral inequalities and applications,* Appl. math. E-Notes, **7** (2007), 93–101.

M. Z. Sarikaya, Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyon-Turkey, *e-mail*: `sarikaya@aku.edu.tr`