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SOME SIMPLE EXTENSIONS OF EULERIAN LATTICES

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ABSTRACT. Let L be a lattice. If K is a sublattice of L, then L is called an extension of K. Lattice extension concept was elaborately studied by G. Grätzer and E. T. Schmidt in their papers [**6**], [**7**], [**9**], [**10**]. A lattice L is said to be simple if it has no non-trivial congruences. A finite graded poset P is said to be Eulerian if its Möbius function assumes the value $\mu(x, y) = (-1)^{l(x,y)}$ for all $x \leq y$ in P, where $l(x, y) = \rho(y) - \rho(x)$ and ρ is the rank function on P. In this paper, we exhibit various possible Eulerian extensions which are simple for any given Eulerian lattice L and we prove that there exists a congruence-preserving extension of an Eulerian lattice. The cubic extension of a lattice was defined by G. Grätzer and E. T. Schmidt in [**11**]. We show that the cubic extension becomes a congruence-preserving extension when the lattice is Eulerian.

1. Eulerian Lattices

Introduction

The subject of combinatorial theory has its origin in the work of G. C. Rota. In the 1960's, G. C. Rota introduced the concept of posets and lattices within combinatorics in his seminal paper [**17**]. In G. C. Rota's work one can find a connection between combinatorics and Möbius functions.

This led L. Solomon to introduce Möbius algebra of a poset [**19**] which, in turn, was studied by C. Greene [**12**] who showed that it could be used to derive many apparently unrelated properties of Möbius functions. Though, classically the origin of Eulerian posets could be found in the work of B. Grünbaum [**13**] and V. Klee [**14**] in 1964, it was first explicitly defined by R. P. Stanley in the paper [**20**] in 1982. In the book [**21**] by R. P. Stanley, we find many characterizations of Eulerian lattices.

Thereafter, several authors have made contributions to the field of Eulerian lattices, for example, Bayer and Billera [**1**], V. K. Santhi [**18**] and A. Vethamanickam [**23**].

In particular, a lot of basic results and properties of Eulerian posets were elaborately first studied by V. K. Santhi in her thesis [**18**]. Also, she dealt with the product of two Eulerian posets and construction of an Eulerian poset from Eulerian

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posets of smaller ranks. In her thesis, we can find so many results in lower Eulerian and semi Eulerian posets. A. Vethamanickam's subsequent work on Eulerian lattices which resulted in many findings inspired us for further study. His work on Eulerian lattices, strongly uniform Eulerian lattices and pleasant Eulerian posets are of great inspiration to us.

In this section, we give the basic definition and examples of Eulerian lattices.

Definition 1.1. A finite graded poset P is said to be Eulerian if its Möbius function assumes the value $\mu(x,y) = (-1)^{l(x,y)}$ for all $x \leq y$ in P, where $l(x,y) = \rho(y) - \rho(x)$ and ρ is the rank function on P.

An equivalent definition for an Eulerian poset is as follows:

Lemma 1.2 ([**15**]). *A finite graded poset P is Eulerian if and only if all intervals $[x,y]$ of length $l \geq 1$ in P contain an equal number of elements of odd and even rank.*

Example. Every Boolean algebra of rank n is Eulerian and the lattice C_4 of Figure 1 is an example for a non-modular Eulerian lattice. Also, every C_n is Eulerian for $n \geq 4$.

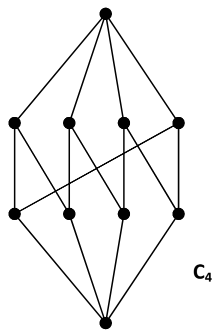

Figure 1

We note that any interval of an Eulerian lattice is Eulerian and an Eulerian lattice cannot contain a three element chain as an interval. We give the following basic definitions which are in [**9**] and [**10**].

Definition 1.3. An equivalence relation θ on a Lattice L is said to be a congruence relation on L if it is compatible with both meet and join, that is, if for all $a, b, c, d \in L$, $a \equiv b\,(\theta)$ and $c \equiv d\,(\theta)$ imply that $a \vee c \equiv b \vee d\,(\theta)$ and $a \wedge c \equiv b \wedge d\,(\theta)$.

Definition 1.4. A lattice L is said to be simple if it has no non-trivial congruences.

Definition 1.5 ([**9**])**.** Let L be a lattice. If K is a sublattice of L, we call L an extension of K. If L is an extension of K, θ is a congruence of K and ϕ is a congruence of L, then ϕ is said to be an extension of θ to L if and only if the restriction of ϕ to K equals θ.

Definition 1.6. A sublattice K of L is said to have a congruence extension property if and only if every congruence of K has an extension to L and if K has zero and it is also the zero of L then L is called a 0-extension of K. If L properly contains K then L is called a proper-extension of K. That is, $L \setminus K \neq \phi$.

L is said to be a congruence-preserving extension of K if and only if every congruence of K has exactly one extension to L. In this case, the congruence lattice of K is isomorphic to the congruence lattice of L, that is, $\operatorname{Con} K \cong \operatorname{Con} L$.

For the undefined terms in this section we refer to [**5**] and [**21**].

2. Simple extensions of Eulerian lattices

An algebra has a number of related structures, namely, the automorphism group, the congruence lattice, the subalgebra lattice, the endomorphism semigroups and so on. The congruence lattice and the automorphism group are two among the related structures of a finite lattice. We wish to state two famous characterization theorems. First one is due to R. P. Dilworth.

Theorem 2.1. [**6**] *Let D be a finite distributive lattice. Then there exists a finite lattice K such that the congruence lattice of K is isomorphic to D.*

The other result was due to G. Grätzer and E. T. Schmidt which is found in [**6**] and which is stronger than the result of R. P. Dilworth.

Theorem 2.2. *Every finite distributive lattice D can be represented as the congruence lattice of a finite sectionally complemented lattice L.*

For the finite groups, the characterization theorem was first published by G. Birkhoff [**3**] and R. Frucht [**2**]. It is: "Let G be a finite group. Then there exists a finite lattice K such that the automorphism group of K is isomorphic to G. The lattice K can be chosen as a simple, sectionally complemented lattice of length three." Since 1990, the emphasis has shifted from representation theorems to extension theorems typified by the following important theorem of M. Tischendorf [**22**].

"Every finite lattice has congruence-preserving embedding into a finite atomistic lattice."

Using the result of M. Tischendorf and their above-mentioned characterization theorems, G. Grätzer and E. T. Schmidt proved the following theorems which appeared in [**9**] and [**10**].

Theorem 2.3 ([**9**])**.** *Every finite lattice K has a congruence-preserving embedding into a finite sectionally complemented lattice L.*

Theorem 2.4 ([**10**])**.** *Every lattice K has a congruence-preserving embedding into a regular lattice L.*

These results inspired us to work in the areas of lattice extension property and congruence-preserving extension property.

In this section, we exhibit various possible Eulerian extensions which are simple, for any given Eulerian lattice L.

The following two results are appeared in [**21**].

Lemma 2.5. *Let $\overline{P}$ and $\overline{Q}$ be Eulerian posets. Then $\overline{R} = \overline{P} \times \overline{Q}$ is also an Eulerian poset.*

Lemma 2.6. *Let $\overline{P}$ and $\overline{Q}$ be Eulerian posets and $P = \overline{P} \setminus \{1\}$ and $Q = \overline{Q} \setminus \{1\}$ and let $R = P \times Q$. Then $\overline{R} = R \cup \{(1,1)\}$ is Eulerian.*

2.1. The Simple Extension $S_g(L_1, L_2)$

Let L_1 and L_2 be two Eulerian lattices of ranks $d_1 + 1$ and $d_2 + 1$ respectively. We denote the least elements of both L_1 and L_2 by 0.

Define $S_g(L_1, L_2) = (\overline{L_1} \times \overline{L_2}) \cup \{(1,1)\}$, where $\overline{L_1} = L_1 \setminus \{1\}$ and $\overline{L_2} = L_2 \setminus \{1\}$. By Lemma 2.6, $S_g(L_1, L_2)$ is Eulerian and of rank $d_1 + d_2 + 2$. We define meet and join in $S_g(L_1, L_2)$ as follows:

$$(a_1,\, a_2) \wedge (b_1,\, b_2) = (a_1 \wedge b_1,\, a_2 \wedge b_2)$$

$$(a_1,\, a_2) \vee (b_1,\, b_2) = \begin{cases} (a_1 \vee b_1,\, a_2 \vee b_2), & \text{if } (a_1 \vee b_1, a_2 \vee b_2) \text{ exists} \\ (1,1), & \text{if either } a_1 \vee b_1 = 1 \text{ or } a_2 \vee b_2 = 1 \end{cases}$$

Let us define a mapping $f :\ L_1 \ \rightarrow\ S_g(L_1, L_2)$ by

$$f(x) = \begin{cases} (x, 0) & \text{if } x \neq 1 \\ (1, 1) & \text{if } x = 1 \end{cases}$$

We can easily prove that f is a one-one homomorphism which implies that L_1 is isomorphic to a sublattice of $S_g(L_1, L_2)$. Therefore, $S_g(L_1, L_2)$ is an Eulerian extension of L_1.

Theorem 2.7. *$S_g(L_1, L_2)$ is simple.*

Proof. Let $\theta \neq \omega$ be a congruence of $S_g(L_1, L_2)$.
The atoms in $S_g(L_1, L_2)$ are of the form either $(0,\, x)$, where x is an atom in $\overline{L_2}$ or of the form $(a,\, 0)$, where a is an atom in $\overline{L_1}$.

Since θ is a congruence of $S_g(L_1, L_2)$, we can find an atom $(0, a)$ in $S_g(L_1, L_2)$ such that $(0,\, 0) \equiv (0,\, a)\,(\theta)$

Since L_2 is co-atomic[**21**], we can find a co-atom e such that $e \ngeq a$, for the atom a.

Suppose $c_1, c_2, \cdots c_i$ are the $i\,(i \geq d_1 + 1)$ distinct co-atoms in L_1 then $(c_1,\, e)$, $(c_2,\, e), (c_3,\, e), \cdots, (c_i,\, e)$ are i co-atoms in $S_g(L_1, L_2)$.

Now, $(0,\, 0) \vee (c_1,\, e) \equiv (0,\, a) \vee (c_1,\, e)(\theta)$ implies that $(c_1,\, e) \equiv (1,\, 1)(\theta)$.

Similarly,

$$\begin{aligned}(c_2,\, e) &\equiv (1,\, 1)(\theta)\\ (c_3,\, e) &\equiv (1,\, 1)(\theta)\\ &\cdots\\ (c_i,\, e) &\equiv (1,\, 1)(\theta).\end{aligned}$$

Since $c_1, c_2, c_3, \cdots, c_i$ are i co-atoms in L_1 and L_1 is atomic [**21**], we can find i distinct atoms $b_1, b_2, b_3, \cdots, b_i$ in L_1 which are respectively not comparable with these co-atoms.

Now, we have $(c_1,\, e) \wedge (b_1,\, 0) \equiv (1,\, 1) \wedge (b_1,\, 0)(\theta)$ which implies that $(0,\, 0) \equiv (b_1,\, 0)(\theta)$.

Similarly, we can find

$$\begin{aligned}(0,\, 0) &\equiv (b_2,\, 0)(\theta)\\ (0,\, 0) &\equiv (b_3,\, 0)(\theta)\\ &\cdots\\ (0,\, 0) &\equiv (b_i,\, 0)(\theta)\end{aligned}$$

Taking join of these equations, we get,

$$(0,\, 0) \equiv (1,\, 1)(\theta)$$

Therefore, $\theta = S_g(L_1, L_2) \times S_g(L_1, L_2)$.

Therefore, the only congruences of $S_g(L_1, L_2)$ are the trivial ones. So, $S_g(L_1, L_2)$ is simple. Hence, $S_g(L_1, L_2)$ is a simple Eulerian extension of an Eulerian lattice L_1. $\square$

In particular, if we take $L_1 = B_n$ and $L_2 = L$ then we have the following corollary.

Corollary 2.8. *If L is an Eulerian lattice and B_n is a Boolean algebra of rank n then $S_n(L) = (\overline{B_n} \times \overline{L}) \cup \{(1,1)\}$ is a simple Eulerian extension of L.*

2.2. The Extension $D(L)$

In this section, we give one more simple extension of a given Eulerian lattice.

Let L be an Eulerian lattice and let $\overline{L} = L \setminus \{0,1\}$. Define $D(L) = (\overline{L} \,\dot\cup\, \overline{L}) \cup \{0,1\}$, where the symbol $\dot\cup$ stands for a disjoint union. Since L is Eulerian $D(L)$ is Eulerian.

Define a mapping $\psi : L \to D(L)$ by $\psi(a) = a$, for any $a \in L$. This mapping is an one-one homomorphism and so is isomorphic to a sublattice of $D(L)$.

Therefore, $D(L)$ is an Eulerian extension of L.

Theorem 2.9. *$D(L)$ is simple.*

Proof. Suppose θ is a proper congruence relation on $D(L)$. Then we can find an atom $a \in D(L)$ such that $0 \equiv a\,(\theta)$. Suppose a is one of the atoms of one copy $\overline{L}$ in $D(L)$, we can find two atoms b and c which are not comparable with a in the other copy $\overline{L}$ in $D(L)$.

Therefore, $0 \vee b \equiv a \vee b\,(\theta)$ which implies that

$$b \equiv 1\,(\theta). \tag{1}$$

Similarly, $0 \vee c \equiv a \vee c\,(\theta)$ which implies that

$$c \equiv 1\,(\theta). \tag{2}$$

From (1) and (2) we get, $b \wedge c \equiv 1 \wedge 1\,(\theta)$. That is, $0 \equiv 1\,(\theta)$.

So $\theta = \tau$.

Therefore, $D(L)$ is simple. Hence $D(L)$ is a simple Eulerian extension of the Eulerian lattice L. □

We can extend the above theorem to the following lattice. Define $D_n(L) = \dot{\bigcup}_{r=1}^{n} \overline{L_r} \cup \{0, 1\}$, where each L_r is an Eulerian lattice of the same rank. By using the above theorem, we can easily prove the following corollary.

Corollary 2.10. *$D_n(L)$ is a simple Eulerian extension of each L_r, $r = 1, 2, \ldots \ldots, n$.*

In fact, we even have the following lemma.

Lemma 2.11. *A disjoint union of any two atomistic lattices is simple.*

Remark 2.12. Lemma 2.11 shows that any atomistic lattice can be embedded into a simple atomistic lattice. Hence we conclude that any atomistic lattice has a simple extension which is also atomistic.

3. Congruence-Preserving Extension

In this section, we prove that every Eulerian lattice has a congruence-preserving Eulerian extension. To show the congruence-preserving Eulerian extension for any Eulerian lattice we follow the Cubic extension S which was defined in [**11**].

Let K be an Eulerian lattice. Define $S = \prod\big(K/\phi \;/\; \phi \in M(\operatorname{Con} K)\big)$, where $M(\operatorname{Con} K)$ is the set of all meet-irreducible elements of $\operatorname{Con} K$.

For $a \in K$, $D(a) = \Big\langle a/\phi \;/\; \phi \in M(\operatorname{Con} K)\Big\rangle$. The mapping $\psi : a \to D(a)$ is an natural embedding from K to S. For a congruence θ of K, let $\theta\psi$ denote the corresponding congruence of $K\psi$. By identifying a with $D(a)$, for $a \in K$, we can view S as an extension of K. S is called the Cubic extension of K.

Theorem 3.1. *Let K be an Eulerian lattice and S be the cubic extension of K. Then S is a congruence-preserving Eulerian extension of K.*

Proof. Every Eulerian lattice is either simple or a direct product of simple Eulerian lattices [**23**].

Since K is Eulerian, $K \cong K_1 \times K_2 \times \cdots \times K_n$, where K_i's are simple Eulerian lattices. Therefore, $\operatorname{Con} K \cong \prod_{i=1}^{n} \operatorname{Con} K_i$ [**16**].

Since K_i is simple, $\operatorname{Con} K$ is a direct product of two element chains and thus $\operatorname{Con} K$ is Boolean. Since $\operatorname{Con} K$ is Boolean, its meet irreducible elements are just the co-atoms of $\mathrm{Con}\, K$. Since $\mathrm{Con}\,(K/\phi) \cong [\phi,\ \tau]$ [**16**], $\mathrm{Con}\,(K/\phi)$ is a two element chain, when $\phi \in M(\mathrm{Con}\, K)$.

Since $S = \prod\big(K/\phi\ /\ \phi \in M(\mathrm{Con}\, K)\big)$, $\mathrm{Con}\, S \cong \prod_{\phi}(\mathrm{Con}\,(K/\phi))$. Since each $\mathrm{Con}\,(K/\phi)$ is a two element chain, $\mathrm{Con}\, S$ is a product of two element chains. Therefore, $\mathrm{Con}\, S$ is Boolean.

We have to prove that every congruence of K has exactly one extension to S. That is, to prove that $\mathrm{Con}\, K \cong \mathrm{Con}\, S$. Since $\mathrm{Con}\, K$ and $\mathrm{Con}\, S$ are Boolean, it is enough to prove that they have the same number of atoms (co-atoms). Since

$$\mathrm{Con}\, S \ \cong \prod_{\phi \in M(\mathrm{Con}\, K)} \mathrm{Con}\,(K/\phi), \qquad \mathrm{Con}\, S \ \cong \prod_{\phi \in M(\mathrm{Con}\, K)} [\phi,\ \tau].$$

A meet irreducible congruence in $\mathrm{Con}\, K$ contributes to a two element chain in the product defining $\mathrm{Con}\, S$. The atoms of $\mathrm{Con}\, S$ are of the form $(0, 0, , \cdots, 1, 0, 0)$, where 1 comes in exactly one place. Therefore, there are as many atoms in $\mathrm{Con}\, S$ as there are co-atoms (meet-irreducible congruences) in $\mathrm{Con}\, K$. Since both are Boolean, we get, $\mathrm{Con}\, K \cong \mathrm{Con}\, S$. Thus, S is a congruence-preserving extension of the Eulerian lattice K.

Next we claim that S is Eulerian: Since a homomorphic image of an Eulerian lattice is Eulerian [**23**], K/ϕ being a homomorphic image of K, it is Eulerian for each $\phi \in M(\mathrm{Con}\, K)$. Hence, S being a finite product of such Eulerian lattices is Eulerian. So, we conclude that S is a congruence-preserving Eulerian extension of the Eulerian lattice K. Hence the theorem. $\square$

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