Acta Math. Univ. Comenianae
Vol. LXXIX, 1(2010), pp. 89–104

FIFTH-ORDER NUMERICAL METHODS FOR HEAT EQUATION SUBJECT TO A BOUNDARY INTEGRAL SPECIFICATION

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Abstract. In this paper a fifth-order numerical scheme is developed and implemented for the solution of the homogeneous heat equation $u_t = \alpha u_{xx}$ with a nonlocal boundary condition as well as for the inhomogeneous heat equation $u_t = u_{xx}+s(x,t)$ with a nonlocal boundary condition. The results obtained show that the numerical method based on the proposed technique is fifth-order accurate as well as L-acceptable. In the development of this method second-order spatial derivatives are approximated by fifth-order finite-difference approximations which give a system of first order, linear, ordinary differential equations whose solution satisfies a recurrence relation which leads to the development of algorithm. The algorithm is tested on various heat equations and no oscillations are observed in the experiments. This method is based on a partial fraction technique which is useful in parallel processing and it does not require complex arithmetic.

1. Introduction

In this paper we have considered the heat equation in one-dimension with a nonlocal boundary condition. Much attention has been paid to the development, analysis and implementation of accurate methods for the numerical solution of this typical problem in the literature.

Many physical phenomena are modeled by nonclassical boundary value problems with nonlocal boundary conditions. These can be classified into two types: (i) boundary-value problems with nonlocal initial conditions or (ii) boundary-value problems with nonlocal boundary conditions. The present work focuses on the second group of nonlocal boundary-value problems. A number of sequential procedures (other than the new scheme) have been suggested in the literature: see, for instantce [**3, 7, 9, 10, 11**].

In the present paper the method of lines, the semi discretization approach, will be used to transform the model partial differential equation (PDE) into a system of first-order, linear, ordinary differential equations (ODEs), the solution of which satisfies a recurrence relation involving a matrix exponential function. A suitable

Received December 24, 2008; revised June 8, 2009.
2000 *Mathematics Subject Classification.* Primary 65M06, 65N40.
Key words and phrases. heat equation; non-local boundary condition; fifth-order numerical methods; method of lines; parallel algorithm.

rational approximation will be used to approximate such exponential functions which will lead to an L_0-acceptable algorithm and may be parallelized through the partial fraction splitting technique.

2. Homogeneous heat equation with nonlocal boundary condition

This section considers the problem of obtaining numerical approximation to the concentration $u = u(x,t)$ which satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t \leq T, \tag{1}$$

subject to the initial condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq 1, \tag{2}$$

the boundary condition

$$u(0,t) = g(t), \qquad 0 < t \leq T \tag{3}$$

and the nonlocal boundary condition

$$\int_0^1 u(x,t)\mathrm{d}x = M(t), \qquad 0 < t \leq T, \quad 0 < x < 1 \tag{4}$$

where f, g and M are known functions and are assumed to be sufficiently smooth to produce a smooth classical solution of u. α is any positive constant and T is a given positive constant.

2.1. Discretization and recurrence relation

Choosing a positive odd integer $N \geq 7$ and dividing the interval $[0,1]$ into $N+1$ subintervals each of width h, so that $h = 1/(N+1)$, and the time variable t into time steps each of length l gives rectangular mesh points with co-ordinates $(x_m, t_n) = (mh, nl)$ where $(m = 0, 1, 2, \ldots, N+1$ and $n = 0, 1, 2, \ldots)$ covering the region $R = [0 < x < 1] \times [t > 0]$ and its boundary ∂R consisting of lines $x = 0$, $x = 1$ and $t = 0$.

Assuming that $u(x,t)$ is at least seven times continuously differentiable with respect to variable x and that these derivatives are uniformly bounded, the space derivative in (1) may be approximated to the fifth-order accuracy at some general point (x,t) of the mesh by using the seven-point formula

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{h^2}\{&au(x-2h,t) + bu(x-h,t) + cu(x,t) + du(x+h,t) \\ &+ eu(x+2h,t) + fu(x+3h,t) + gu(x+4h,t)\}\end{aligned} \tag{5}$$

After expanding the terms $u(x-2h,t)$, $u(x-h,t)$, $u(x+h,t)$, $u(x+2h,t)$, $u(x+3h,t)$ and $u(x+4h,t)$ about (x,t) by using the Taylor's expansion and then equating the different powers of h on both sides we have

$$a = \frac{-13}{180}, \quad b = \frac{19}{15}, \quad c = \frac{-7}{3}, \quad d = \frac{10}{9}, \quad e = \frac{1}{12}, \quad f = \frac{-1}{15}, \quad g = \frac{1}{90}.$$

So by substituting the values of a, b, c, d, e, f and g in (5)

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{180h^2}\{-13u(x-2h,t)+228u(x-h,t)-420u(x,t)\\ &\quad +200u(x+h,t)+15u(x+2h,t)-12u(x+3h,t)\\ &\quad +2u(x+4h,t)\}-\frac{h^5}{90}\frac{\partial^7 u(x,t)}{\partial x^7}+O(h^6).\end{aligned} \tag{6}$$

Equation (6) is valid only for $(x,t)=(x_m,t_n)$ with $m=2,3,\ldots,N-3$. To achieve the same accuracy at the points (x_i,t_n), for $i=1,N-2,N-1,N$, special formulae (as derived above)are used in this paper which approximate $\partial^2 u(x,t)/\partial x^2$ not only to fifth-order but also with the dominant error term $(-1/90)h^5\partial^7 u(x,t)/\partial x^7$ for $x=x_1,x_{N-2},x_{N-1},x_N$ and $t=t_n$. These approximations are

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{180h^2}\{124u(x-h,t)-56u(x,t)-528u(x+h,t)\\ &\quad +925u(x+2h,t)-740u(x+3h,t)+366u(x+4h,t)\\ &\quad -104u(x+5h,t)+13u(x+6h,t)\}-\frac{1}{90}h^5\frac{\partial^7 u(x,t)}{\partial x^7}+O(h^6)\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{180h^2}\{-2u(x-4h,t)+16u(x-3h,t)-69u(x-2h,t)\\ &\quad +340u(x-h,t)-560u(x,t)+312u(x+h,t)-41u(x+2h,t)\\ &\quad +4u(x+3h,t)\}-\frac{1}{90}h^5\frac{\partial^7 u(x,t)}{\partial x^7}+O(h^6),\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{180h^2}\{-4u(x-5h,t)+30u(x-4h,t)-96u(x-3h,t)\\ &\quad +155u(x-2h,t)-60u(x-2h,t)-336u(x,t)+200u(x+h,t)\\ &\quad -9u(x+72h,t)\}-\frac{1}{90}h^5\frac{\partial^7 u(x,t)}{\partial x^7}+O(h^6),\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{180h^2}\{9u(x-6h,t)-76u(x-5h,t)+282u(x-4h,t)\\ &\quad -600u(x-3h,t)+785u(x-2h,t)-444u(x-h,t)-84u(x,t)\\ &\quad +128u(x+h,t)\}-\frac{1}{90}h^5\frac{\partial^7 u(x,t)}{\partial x^7}+O(h^6),\end{aligned} \tag{10}$$

for the mesh points (x_1,t_n), (x_{N-2},t_n), (x_{N-1},t_n) and (x_N,t_n), respectively.

2.2. Treatment of the non-local boundary condition

The integral in (4) is approximated by using Simpson's $\frac{1}{3}$ quadrature rule as

$$\int_0^b u(x,t)dx \approx \frac{h}{3}[u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} u(2i-1,t) + 2\sum_{i=1}^{\frac{N+1}{2}-1} u(2i,t) + u(N+1,t)] + O(h^5) \tag{11}$$

Here (11) serves as the boundary condition at zero and near zero.

Applying (1) together with (6)–(10) to all interior mesh points of the grid at time level $t = t_n$ produces a system of ordinary differential equations of the form

$$\frac{d\mathbf{U}(t)}{dt} = \mathbf{A}\mathbf{U}(t) + \mathbf{v}(t), \qquad t > 0 \tag{12}$$

with initial distribution

$$\mathbf{U}(0) = \mathbf{f} \tag{13}$$

in which $\mathbf{U}(t) = [U_1(t), U_2(t), U_3(t), \ldots, U_N(t)]^T$, $\mathbf{f} = [f(x_1), f(x_2), f(x_3), \ldots, f(x_N)]^T$, T denoting transpose and matrix $\mathbf{A}$ is given by

(14)

$$\mathbf{A} = \frac{\alpha}{180h^2}\begin{bmatrix} -56 & -528 & 925 & -740 & 366 & -104 & 13 & & & 0 \\ 228 & -420 & 200 & 15 & -12 & 2 & & & & \\ -13 & 228 & -420 & 200 & 15 & -12 & 2 & & & \\ & -13 & 228 & -420 & 200 & 15 & -12 & 2 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -13 & 228 & -420 & 200 & 15 & -12 & 2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \ldots & \alpha_{N-1} & \alpha_N \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \ldots & \beta_{N-1} & \beta_N \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \ldots & \gamma_{N-1} & \gamma_N \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \ldots & \delta_{N-1} & \delta_N \end{bmatrix}_{N\times N}$$

where

$\alpha_i = \begin{cases} -8 \text{ when } i \text{ is odd} \\ -4 \text{ when } i \text{ is even} \end{cases}$ and $\alpha_{N-5} = -17$, $\alpha_{N-4} = 220$, $\alpha_{N-3} = -424$, $\alpha_{N-2} = 192$, $\alpha_{N-1} = 11$, $\alpha_N = -20$,

$\beta_i = \begin{cases} -16 \text{ when } i \text{ is odd} \\ -8 \text{ when } i \text{ is even} \end{cases}$ and $\beta_{N-6} = -18$, $\beta_{N-5} = 8$, $\beta_{N-4} = -85$, $\beta_{N-3} = 332$, $\beta_{N-2} = -576$, $\beta_{N-1} = 304$, $\beta_N = -57$,

$\gamma_i = \begin{cases} 36 \text{ when } i \text{ is odd} \\ 18 \text{ when } i \text{ is even} \end{cases}$ and $\gamma_{N-6} = 32$, $\gamma_{N-5} = 48$, $\gamma_{N-4} = -60$, $\gamma_{N-3} = 173$, $\gamma_{N-2} = 96$, $\gamma_{N-1} = -318$, $\gamma_N = 236$,

$\delta_i = \begin{cases} -512 \text{ when } i \text{ is odd} \\ -256 \text{ when } i \text{ is even} \end{cases}$ and $\delta_{N-6} = -503$, $\delta_{N-5} = -332$, $\delta_{N-4} = -230$, $\delta_{N-3} = -856$, $\delta_{N-2} = 273$, $\delta_{N-1} = -700$, $\delta_N = -596$, and the column vector

$$\mathbf{v}(t) = \frac{\alpha}{180h^2}\Big[124g(t), -13g(t), 0, 0, \ldots, -2g(t) - \frac{6}{h}M(t), -4g(t) + \frac{12}{h}M(t) + 9g(t) + \frac{-27}{h}M(t), -128g(t) + \frac{384}{h}M(t)\Big]^T. \tag{15}$$

The solution of the system (12) subject to (13) becomes

$$\mathbf{U}(t) = \exp(t\mathbf{A})f + \int_0^t \exp[(t-s)\mathbf{A}]\mathbf{v}(s)\mathrm{d}s, \qquad t \geq 0 \tag{16}$$

where $\mathbf{v}(s)$ is due to the boundary condition and the source term. The solution given by (16) satisfies the recurrence relation

$$\mathbf{U}(t+l) = \exp(lA)\mathbf{U}(t) + \int_t^{t+l} \exp[(t+l-s)A]\mathbf{v}(s)\mathrm{d}s, \qquad t = 0, l, 2l, \ldots \tag{17}$$

Eigenvalues of matrix $\mathbf{A}$ are calculated using MATLAB for $N = 9, 19, 39, 79, 99$ and 199 and it is observed that these are distinct negative the real ones or with negative real parts.

To approximate the matrix exponential in (17) we use the rational approximation [**12**] for real scalar θ which is of the form

$$E_M(\theta) = \frac{\sum_{K=0}^{M-1} b_K\theta^K}{\sum_{K=0}^{M} a_K(-\theta)^K} \tag{18}$$

where M is a positive integer and $a_0 = 1,\ a_M,\ b_{M-1} \neq 0$ and $a_K \geq 0$ for all $K = 1, 2, 3, \ldots, M$. Matching $E_M(\theta)$ with the first $M+1$ terms of the Maclaurin's expansion of $\exp(\theta)$ leads to the following relations in the parameters

$$a_M = (-1)^{M-1}\sum_{K=0}^{M-1}(-1)^K \frac{a_K}{(M-K)!} \tag{19}$$

and

$$b_K = \sum_{i=0}^{K}(-1)^i \frac{a_i}{(K-i)!}, \qquad K = 0, 1, 2, \ldots, M-1 \tag{20}$$

The magnitude of the coefficient of the error term is

$$\mu_M = \sum_{K=0}^{M-1} \frac{(M-K)(-1)^{K+1}a_K}{(M-K+1)!} \tag{21}$$

In this method we are concerned with $E_5(\theta)$, so, for $M = 5$ we have

$$E_5(\theta) = \frac{1 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4}{1 - a_1\theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - a_5\theta^5} = \frac{p(\theta)}{q(\theta)} \quad (say), \tag{22}$$

where

$$b_1 = (1 - a_1), \qquad b_2 = \left(\frac{1}{2} - a_1 + a_2\right), \quad b_3 = \left(\frac{1}{6} - \frac{a_1}{2} + a_2 - a_3\right),$$

$$b_4 = \left(\frac{1}{24} - \frac{a_1}{6} + \frac{a_2}{2} - a_3 + a_4\right), \qquad a_5 = \left(\frac{1}{120} - \frac{a_1}{24} + \frac{a_2}{6} - \frac{a_3}{2} + a_4\right),$$

choosing the values of parameters a_1, a_2, a_3, a_4 and a_5 as $91/20, 481/120, 107/80$, $691/3600$ and $1/100$ respectively.

The quadrature term appearing in recurrence relation (17) is approximated as

$$\begin{aligned}\int_t^{t+l} \exp((t+l-s)\mathbf{A})\mathbf{v}(s)\mathrm{d}s = \mathbf{W}_1\mathbf{v}(s_1) + \mathbf{W}_2\mathbf{v}(s_2) + \mathbf{W}_3\mathbf{v}(s_3)\\ + \mathbf{W}_4\mathbf{v}(s_4) + \mathbf{W}_5\mathbf{v}(s_5)\end{aligned} \tag{23}$$

where $s_1 \neq s_2 \neq s_3 \neq s_4 \neq s_5$ and $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$, $\mathbf{W}_5$ are matrices. It can be shown that

$$\sum_{j=1}^{5} s_j^{k-1}\mathbf{W}_j = M_k, \qquad k = 1,2,3,4,5 \tag{24}$$

where

$$M_k = \mathbf{A}^{-1}\{t^{k-1}E - (t+l)^{k-1}I + (k-1)M_{k-1}\}, \qquad k = 1,2,3,4,5. \tag{25}$$

Taking $s_1 = t$, $s_2 = t + \frac{l}{4}$, $s_3 = t + \frac{l}{2}$, $s_4 = t + \frac{3l}{4}$, $s_5 = t + l$ and solving the system (25) for $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$, $\mathbf{W}_5$, we have

$$\begin{aligned}\mathbf{W}_1 = -\frac{1}{3l^4}(\mathbf{A}^{-1})^5\{768I + 288l\mathbf{A} + 44l^2\mathbf{A}^2 + 3l^3\mathbf{A}^3\\ - (768I - 480l\mathbf{A} + 140l^2\mathbf{A}^2 - 25l^3\mathbf{A}^3 + 3l^4\mathbf{A}^4)E\},\end{aligned} \tag{26}$$

$$\begin{aligned}\mathbf{W}_2 = \frac{16}{3l^4}(\mathbf{A}^{-1})^5\{192I + 84l\mathbf{A} + 14l^2\mathbf{A}^2 + l^3\mathbf{A}^3\\ - (192I - 108l\mathbf{A} + 26l^2\mathbf{A}^2 - 3l^3\mathbf{A}^3)E\},\end{aligned} \tag{27}$$

$$\begin{aligned}\mathbf{W}_3 = -\frac{4}{l^4}(\mathbf{A}^{-1})^5\{384I + 192l\mathbf{A} + 38l^2\mathbf{A}^2 + 3l^3\mathbf{A}^3\\ + (384I + 192l\mathbf{A} - 38l^2\mathbf{A}^2 + 3l^3\mathbf{A}^3)E\},\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{W}_4 = \frac{16}{3l^4}(\mathbf{A}^{-1})^5\{192I + 108l\mathbf{A} + 26l^2\mathbf{A}^2 + 3l^3\mathbf{A}^3\\ - (192I - 84l\mathbf{A} + 14l^2\mathbf{A}^2 - l^3\mathbf{A}^3)E\},\end{aligned} \tag{29}$$

$$\begin{aligned}\mathbf{W}_5 = -\frac{1}{3l^4}(\mathbf{A}^{-1})^5\{768I + 480l\mathbf{A} + 140l^2\mathbf{A}^2 + 25l^3\mathbf{A}^3 + 3l^4\mathbf{A}^4\\ - (768I - 288l\mathbf{A} + 44l^2\mathbf{A}^2 - 3l^3\mathbf{A}^3)E\},\end{aligned} \tag{30}$$

Using $E = PQ$, due to (22) where

$$P = \left(I - a_1l\mathbf{A} + a_2l^2\mathbf{A}^2 - a_3l^3\mathbf{A}^3 + a_4l^4\mathbf{A}^4 - a_5l^5\mathbf{A}^5\right)^{-1} \tag{31}$$

and

$$Q = I + b_1 l\mathbf{A} + b_2 l^2\mathbf{A}^2 + b_3 l^3\mathbf{A}^3 + b_4 l^4\mathbf{A}^4, \tag{32}$$

in (26)–(30) gives

$$\begin{aligned}
\mathbf{W}_1 = {} & \frac{l}{360}\{28I + (668 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4)l\mathbf{A} \\
& + (-21 + 100a_1 - 260a_2 + 1920a_4)l^2\mathbf{A}^2 \\
& + (18 - 75a_1 + 240a_2 - 540a_3 + 720a_4)l^3\mathbf{A}^3\}P, \\
\mathbf{W}_2 = {} & \frac{2l}{45}\{8I + (-154 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4)l\mathbf{A} \\
& + (1 + 10a_1 - 100a_2 + 480a_3 - 1200a_4)l^2\mathbf{A}^2 \\
& + (-1 + 5a_1 - 20a_2 + 60a_3 - 120a_4)l^3\mathbf{A}^3\}P, \\
\mathbf{W}_3 = {} & \frac{l}{30}\{4I + (322 - 1540a_1 + 5760a_2 - 15360a_3 + 23040a_4)l\mathbf{A} \\
& + (23 - 130a_1 + 580a_2 - 1920a_3 + 3840a_4)l^2\mathbf{A}^2 \\
& + (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4)l^3\mathbf{A}^3\}P, \\
\mathbf{W}_4 = {} & \frac{2l}{45}\{8I + (-158 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4)l\mathbf{A} \\
& + (-21 + 110a_1 - 460a_2 + 1440a_3 - 2640a_4)l^2\mathbf{A}^2 \\
& + (-3 + 15a_1 - 60a_2 + 180a_3 - 360a_4)l^3\mathbf{A}^3\}P, \\
\mathbf{W}_5 = {} & \frac{l}{360}\{28I + (640 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4)l\mathbf{A} \\
& + (125 - 640a_1 + 2620a_2 - 7680a_3 + 13440a_4)l^2\mathbf{A}^2 \\
& + (25 - 125a_1 + 500a_2 - 1500a_3 + 2640a_4)l^3\mathbf{A}^3 \\
& + (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4)l^4\mathbf{A}^4\}P.
\end{aligned}$$

2.3. Development of Algorithm

Assuming that r_1, r_2, r_3, r_4, r_5, $(r_i \neq 0)$ are distinct real zeros of $q(\theta)$, the denominator of $E_5(\theta)$, then

$$\begin{aligned}
P^{-1} &= \prod_{i=1}^{5}\left(I - \frac{l}{r_i}\mathbf{A}\right) \\
E &= \sum_{j=1}^{5} p_j \left(I - \frac{l}{r_j}\mathbf{A}\right)^{-1}
\end{aligned} \tag{33}$$

where

$$p_j = \frac{1}{\prod\limits_{i=1,\, i\neq j}^{5}\left(1 - \dfrac{r_j}{r_i}\right)}\{1 + b_1 r_j + b_2 r_j^2 + b_3 r_j^3 + b_4 r_j^4\},$$

$j = 1, 2, 3, 4, 5$ and

$$\mathbf{W}_k = \frac{m_k l}{360} \sum_{j=1}^{5} p_{5k+j} \left(I - \frac{l}{r_1} \mathbf{A} \right)^{-1}, \qquad k = 1, 2, 3, 4, 5 \tag{34}$$

in which $m_1 = 1$, $m_2 = 16$, $m_3 = 12$, $m_4 = 16$, $m_5 = 1$, and for $j = 1, 2, 3, 4, 5$

$$\begin{aligned}
p_{5+j} = & \frac{1}{\prod\limits_{i=1,\, i\neq j}^{5} \left(1 - \frac{r_j}{r_i}\right)} \\
& \times \{28 + (668 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4)r_j \\
& \;+ (-21 + 100a_1 - 260a_2 + 1920a_4)r_j^2 \\
& \;+ (18 - 75a_1 + 240a_2 - 540a_3 + 720a_4)r_j^3\} \\
p_{10+j} = & \frac{1}{\prod\limits_{i=1,\, i\neq j}^{5} \left(1 - \frac{r_j}{r_i}\right)} \\
& \times \{8 + (-154 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4)r_j \\
& \;+ (1 + 10a_1 - 100a_2 + 480a_3 - 1200a_4)r_j^2 \\
& \;+ (-1 + 5a_1 - 20a_2 + 60a_3 - 120a_4)r_j^3\} \\
p_{15+j} = & \frac{1}{\prod\limits_{i=1,\, i\neq j}^{5} \left(1 - \frac{r_j}{r_i}\right)} \\
& \times \{4 + (322 - 1540a_1 + 5760a_2 - 15360a_3 + 23040a_4)r_j \\
& \;+ (23 - 130a_1 + 580a_2 - 1920a_3 + 3840a_4)r_j^2 \\
& \;+ (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4)r_j^3\} \\
p_{20+j} = & \times \frac{1}{\prod\limits_{i=1,\, i\neq j}^{5} \left(1 - \frac{r_j}{r_i}\right)} \\
& \{8 + (-158 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4)r_j \\
& \;+ (-21 + 110a_1 - 460a_2 + 1440a_3 - 2640a_4)r_j^2 \\
& \;+ (-3 + 15a_1 - 60a_2 + 180a_3 - 360a_4)r_j^3\} \\
p_{25+j} = & \frac{1}{\prod\limits_{i=1,\, i\neq j}^{5} \left(1 - \frac{r_j}{r_i}\right)} \\
& \times \{28 + (640 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4)r_j \\
& \;+ (125 - 640a_1 + 2620a_2 - 7680a_3 + 13440a_4)r_j^2 \\
& \;+ (25 - 125a_1 + 500a_2 - 1500a_3 + 2640a_4)r_j^3 \\
& \;+ (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4)r_j^4\}
\end{aligned}$$

Equation (17) becomes

$$\mathbf{U}(t+l) = \sum_{i=1}^{5} \mathbf{A}_i^{-1} \left\{ p_i \mathbf{U}(t) + \frac{l}{360} \left\{ p_{i+5}\mathbf{v}(t) + 16p_{i+10}\mathbf{v}\left(t+\frac{l}{4}\right) \right.\right.$$
$$\left.\left. +12p_{i+15}\mathbf{v}\left(t+\frac{l}{2}\right) + 16p_{i+20}\mathbf{v}\left(t+\frac{3l}{4}\right) + p_{i+25}\mathbf{v}(t+l) \right\}\right\} \tag{35}$$

where

$$A_i = I - \frac{l}{r_i}\mathbf{A}, \qquad i = 1,2,3,4,5.$$

Hence

$$\mathbf{U}(t+l) = \sum_{i=1}^{5} \mathbf{y}_i(t) \tag{36}$$

where $\mathbf{y}_i$, $(i = 1,2,3,4,5)$ are the solutions of the systems

$$\mathbf{A}_i \mathbf{y}_i = \; p_i \mathbf{U}(t) + \frac{l}{360} \left\{ p_{i+5}\mathbf{v}(t) + 16p_{i+10}\mathbf{v}\left(t+\frac{l}{4}\right) + 12p_{i+15}\mathbf{v}\left(t+\frac{l}{2}\right) \right.$$
$$\left. +16p_{i+20}\mathbf{v}\left(t+\frac{3l}{4}\right) + p_{i+25}\mathbf{v}(t+l) \right\} \tag{37}$$

3. Inhomogeneous heat equation with non-local boundary condition

This section considers the problem of obtaining numerical approximation to the concentration $u = u(x,t)$ which satisfies the inhomogeneous partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + s(x,t), \;\; 0 < x < 1, \qquad 0 < t \le T, \tag{38}$$

subject to the initial condition (2), the boundary condition

$$u(1,t) = g(t), \qquad 0 < t \le T \tag{39}$$

(which is different from (3)) and the nonlocal boundary condition

$$\int_0^b u(x,t)\mathrm{d}x = M(t), \qquad 0 < t \le T, \qquad 0 < b < 1 \tag{40}$$

where f, g, M and s are known functions, assumed to be sufficiently smooth to produce a smooth solution of u, and T is a given positive constant.

The existence, uniqueness and continuous dependence on data of the solution to this problem have been studied by [**4, 5, 6**].

Applying (31) and (6)–(10) to all the interior mesh points of the grid at time level $t = t_n$ and using the boundary conditions (39) and (40) produce a system of ordinary differential equations of the form (12) with initial distribution (13) with matrix $\mathbf{A}$ given by

$$\mathbf{A} = \frac{1}{180h^2}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \dots & & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \dots & & \\ -13 & 228 & -420 & 200 & 15 & -12 & 2 & & \\ & -13 & 228 & -420 & 200 & 15 & -12 & 2 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & -13 & 228 & -420 & 200 & 15 & -12 & 2 \\ & & & -13 & 228 & -420 & 200 & 15 & -12 \\ & & -2 & 16 & -69 & 340 & -560 & 312 & -41 \\ & & -4 & 30 & -96 & 155 & 60 & -336 & 200 \\ 0 & & 9 & -76 & 282 & -600 & 785 & -444 & -84 \end{bmatrix}_{N\times N} \tag{41}$$

where

$\alpha_1 = -552,\ \alpha_2 = -776,\ \alpha_3 = 429,\ \alpha_4 = -988,\ \alpha_5 = -130,\ \alpha_6 = -352,$

$$\alpha_7 = -483 \text{ and } \alpha_i = \begin{cases} -248 & \text{when } i \text{ is even} \\ -496 & \text{when } i \text{ is odd} \\ -124 & \text{when } i = J \end{cases}$$

also

$\beta_1 = 280,\ \ \beta_2 = -394,\ \ \beta_3 = 252,\ \ \beta_4 = 41,\ \ \beta_5 = 40,\ \ \beta_6 = 28$ and

$$\beta_i = \begin{cases} 52 & \text{when } i \text{ is even} \\ 26 & \text{when } i \text{ is odd} \\ 13 & \text{when } i = J \end{cases}$$

where $J = \dfrac{b}{h}$ and the column vector

$$\mathbf{v}(t) = \frac{1}{180h^2}\left[\frac{372}{h}M(t), \frac{-39}{h}M(t), 0, 0, \dots, 2g(t), 4g(t), -9g(t), 128g(t)\right]^T \tag{42}$$

4. Numerical Experiments

In this section the numerical methods described in this paper are applied to several problems from the literature and results obtained are compared with exact solutions as well as with the results existing in the literature.

Example 1. Consider (1)–(4) with

$$f(x) = \cos\left(\frac{\pi}{2}x\right), \quad 0 < x < 1, \qquad\qquad g(t) = \exp\left(-\frac{\pi^2}{4}t\right), \quad 0 < t < 1,$$

$$M(t) = \frac{2}{\pi}\exp\left(-\frac{\pi^2}{4}t\right), \quad 0 < t \le 1.$$

The theoretical solution of the problem (see [**9**]) is

$$u(x,t) = \exp\left(-\frac{\pi^2}{4}t\right)\cos\left(\frac{\pi}{2}x\right)$$

The relative errors $\frac{|\mathbf{U}_{\text{approx}}-\mathbf{U}_{\text{exact}}|}{\mathbf{U}_{\text{exact}}}$ for the results of $u(0.5, 0.1)$ at $h = l = 0.05$, 0.025, 0.01, 0.005, 0.0025, 0.001, using the new scheme discussed in this paper as well as using the implicit method [**3**], Galerkin technique [**7**], Keller-Box method [**10**], the Rung Kutta Chebyshev (RKC) scheme [**11**] and the Saulyev's explicit scheme [**9**] are shown in Table 1. CPU time for different values of h are also given in Table 2.

Table 1. Relative errors of various spatial lengths in Example 1.

h	Implicit	Galerkin	Keller-Box	RKC	Saulyev I	**Fifth-order**
0.0500	9.1×10^{-03}	9.9×10^{-02}	9.4×10^{-02}	9.8×10^{-02}	9.6×10^{-03}	$\mathbf{1.6 \times 10^{-07}}$
0.0250	2.3×10^{-03}	3.0×10^{-02}	2.4×10^{-02}	3.7×10^{-02}	2.5×10^{-03}	$\mathbf{1.1 \times 10^{-08}}$
0.0100	3.8×10^{-04}	4.9×10^{-03}	4.1×10^{-03}	6.1×10^{-03}	3.9×10^{-04}	$\mathbf{3.6 \times 10^{-10}}$
0.0050	9.4×10^{-05}	1.2×10^{-03}	1.0×10^{-03}	1.5×10^{-03}	9.6×10^{-05}	$\mathbf{9.0 \times 10^{-11}}$
0.0025	2.3×10^{-05}	3.1×10^{-04}	2.5×10^{-04}	3.5×10^{-04}	2.5×10^{-05}	$\mathbf{6.5 \times 10^{-12}}$
0.0010	4.1×10^{-06}	5.0×10^{-05}	4.0×10^{-05}	6.0×10^{-05}	4.3×10^{-06}	$\mathbf{9.8 \times 10^{-11}}$

Table 2. CPU time in Example 1 for fifth-order scheme.

h	CPU Time (in seconds)
0.0500	0.001
0.0250	0.032
0.0125	0.145
0.0100	0.234
0.0050	1.781
0.0025	19.750

Example 2. Consider (38) with (2), (39) and (40)

$$f(x) = \sin(\pi x), \quad 0 < x < 1, \qquad g(t) = 0, \quad 0 < t < 1,$$

$$b = 0.75, \qquad M(t) = \frac{2+\sqrt{2}}{2\pi} \exp(-\pi^2 t), \quad 0 < t \le 1,$$

$$s(x,t) = 0, \quad 0 < t \le 1, \quad 0 < x < 1.$$

The theoretical solution of this problem (see [**8**]) is

$$u(x,t) = \exp(-\pi^2 t) \sin(\pi x).$$

The results for Example 2 are given in Table 3. Calculations are performed for different values of $h = 0.01$, 0.005, 0.0025, 0.001 and $l = 0.01$, 0.005, 0.0025, 0.001 for comparison purpose.

Table 3. Maximum errors in Example 2 at $t = 1$.

h	Numerical Methods	$l = 0.01$	$l = 0.005$	$l = 0.0025$	$l = 0.001$
0.01	The implicit scheme	9.0×10^{-04}	3.0×10^{-04}	2.0×10^{-04}	1.0×10^{-04}
	The pade scheme	6.0×10^{-05}	2.0×10^{-05}	1.0×10^{-05}	1.0×10^{-05}
	Fifth-order scheme	$\mathbf{4.9 \times 10^{-11}}$	$\mathbf{4.6 \times 10^{-11}}$	$\mathbf{3.2 \times 10^{-11}}$	$\mathbf{3.0 \times 10^{-11}}$
0.005	The implicit scheme	8.0×10^{-04}	4.0×10^{-04}	5.0×10^{-04}	4.0×10^{-04}
	The pade scheme	5.0×10^{-05}	2.0×10^{-05}	2.0×10^{-05}	3.0×10^{-05}
	Fifth-order scheme	$\mathbf{1.1 \times 10^{-12}}$	$\mathbf{5.0 \times 10^{-14}}$	$\mathbf{1.3 \times 10^{-14}}$	$\mathbf{1.2 \times 10^{-14}}$
0.0025	The implicit scheme	7.0×10^{-04}	5.0×10^{-04}	4.0×10^{-04}	2.0×10^{-04}
	The pade scheme	6.0×10^{-05}	2.0×10^{-05}	3.0×10^{-05}	1.0×10^{-05}
	Fifth-order scheme	$\mathbf{1.1 \times 10^{-12}}$	$\mathbf{3.0 \times 10^{-14}}$	$\mathbf{2.1 \times 10^{-15}}$	$\mathbf{7.6 \times 10^{-15}}$
0.001	The implicit scheme	3.0×10^{-04}	2.0×10^{-04}	3.0×10^{-05}	1.0×10^{-05}
	The pade scheme	1.0×10^{-05}	3.0×10^{-05}	5.0×10^{-06}	2.0×10^{-06}
	Fifth-order scheme	$\mathbf{2.6 \times 10^{-14}}$	$\mathbf{3.5 \times 10^{-15}}$	$\mathbf{8.7 \times 10^{-16}}$	$\mathbf{2.1 \times 10^{-17}}$

Table 4. Maximum errors in Example 3 at $t = 1$.

h	Numerical Methods	$l = 0.01$	$l = 0.005$	$l = 0.0025$	$l = 0.001$
0.01	The implicit scheme	8.0×10^{-04}	2.0×10^{-04}	6.0×10^{-04}	2.0×10^{-04}
	The pade scheme	6.0×10^{-05}	1.0×10^{-05}	5.0×10^{-05}	1.0×10^{-05}
	Fifth-order scheme	$\mathbf{3.7 \times 10^{-11}}$	$\mathbf{3.0 \times 10^{-11}}$	$\mathbf{3.3 \times 10^{-11}}$	$\mathbf{4.3 \times 10^{-11}}$
0.005	The implicit scheme	7.0×10^{-04}	2.0×10^{-04}	5.0×10^{-04}	3.0×10^{-04}
	The pade scheme	5.0×10^{-05}	2.0×10^{-05}	4.0×10^{-05}	3.0×10^{-05}
	Fifth-order scheme	$\mathbf{9.6 \times 10^{-12}}$	$\mathbf{6.3 \times 10^{-12}}$	$\mathbf{1.7 \times 10^{-12}}$	$\mathbf{1.2 \times 10^{-12}}$
0.0025	The implicit scheme	6.0×10^{-04}	3.0×10^{-04}	5.0×10^{-04}	3.0×10^{-04}
	The pade scheme	4.0×10^{-05}	2.0×10^{-05}	3.0×10^{-05}	2.0×10^{-05}
	Fifth-order scheme	$\mathbf{2.5 \times 10^{-12}}$	$\mathbf{6.2 \times 10^{-12}}$	$\mathbf{2.3 \times 10^{-12}}$	$\mathbf{1.1 \times 10^{-12}}$
0.001	The implicit scheme	3.0×10^{-04}	5.0×10^{-04}	7.0×10^{-05}	9.0×10^{-05}
	The pade scheme	1.0×10^{-05}	3.0×10^{-05}	4.0×10^{-06}	5.0×10^{-06}
	Fifth-order scheme	$\mathbf{2.3 \times 10^{-13}}$	$\mathbf{9.6 \times 10^{-13}}$	$\mathbf{7.5 \times 10^{-13}}$	$\mathbf{1.3 \times 10^{-13}}$

Example 3. Again consider (38) with (2), (39) and (40)

$$f(x) = exp(x), \quad 0 < x < 1 \qquad g(t) = \frac{e}{1+t^2}, \quad 0 < t < 1$$

$$b = 0.3 \qquad M(t) = \frac{\mathrm{e}^{0.3} - 1}{1+t^2}, \quad 0 < t \leq 1$$

$$s(x,t) = \frac{-(1+t^2)\exp(x)}{(1+t^2)^2}, \; 0 < t \leq 1, \; 0 < x < 1.$$

and with the theoretical solution (see [**1, 8**])

$$u(x,t) = \frac{\exp(x)}{1+t^2}$$

The results regarding this example are given in Table 4 and Table 5. In Table 4, the relative errors of the results of $u(x,t)$ with $h = 0.01$, 0.005, 0.0025, 0.001, and $l = 0.01$, 0.005, 0.0025, 0.001, using the fifth-order scheme discussed in this paper as well as using the implicit finite-difference scheme of [**2**] and the Pade scheme of [**8**], are shown. While absolute errors for $l = 0.001$ and $h = 0.01$ for different values of t are given in Table 5 and are compared with the results of [**1**].

Table 5. Results in Example 3 at $l = 0.001$ and $h = 0.01$.

x	t	Exact solution	Approximate solution		Absolute relative error	
			Wavelet Galerkin	**Fifth-order scheme**	Wavelet Galerkin	**Fifth-order scheme**
0.25	0.025	1.2832234020614	1.283193	**1.2832234020325**	2.4×10^{-05}	$\mathbf{2.3\times10^{-11}}$
	0.050	1.2808233582920	1.280765	**1.2808233582754**	4.6×10^{-05}	$\mathbf{1.3\times10^{-11}}$
	0.100	1.2713122937502	1.271207	**1.2713122937272**	8.3×10^{-05}	$\mathbf{1.8\times10^{-11}}$
0.50	0.025	1.6476914635354	1.647651	**1.6476914635227**	2.5×10^{-05}	$\mathbf{7.7\times10^{-12}}$
	0.050	1.6446097364344	1.644534	**1.6446097463106**	4.6×10^{-05}	$\mathbf{1.4\times10^{-11}}$
	0.100	1.6323972977229	1.632272	**1.6323972976851**	7.7×10^{-05}	$\mathbf{1.4\times10^{-11}}$
0.75	0.025	2.1156777180389	2.115633	**2.1156777180254**	2.1×10^{-05}	$\mathbf{2.3\times10^{-11}}$
	0.050	2.1117207148256	2.111648	**2.1117207148029**	3.5×10^{-05}	$\mathbf{1.1\times10^{-11}}$
	0.100	2.0960396304086	2.095934	**2.0969396203748**	5.0×10^{-05}	$\mathbf{1.5\times10^{-11}}$

Example 4. Now consider (38) with (2), (39) and (40)

$$f(x) = \ln(x+2.5),\quad 0<x<1,\qquad g(t) = \ln(2t+3.5),\quad 0<t<1,\qquad b = 0.5,$$

$$M(t) = (2t+3)\ln(2t+3) - (2t+2.5)\ln(2t+2.5) - 0.5,\quad 0<t\le 1,$$

$$s(x,t) = \frac{2(x+2t+3)}{(x+2t+2.5)^2},\quad 0<t\le 1,\ 0<x<1$$

The theoretical solution of this problem (see [**1**]) is

$$u(x,t) = \ln(x+2t+2.5).$$

The results regarding Example 4 are given in Table 6. Calculations are performed for $x = 0.25, 0.50, 0.75$ and $t = 0.025, 0.05, 0.10$. The results for fifth-order scheme are compared with the exact ones and with the results of [**1**]. The CPU time for $h = 0.025$ and $l = 0.001$ for different values of t are given in Table 7.

Table 6. Results in Example (4) at $l = 0.001$ and $h = 0.01$.

x	t	Exact solution	Approximate solution		Absolute relative error	
			Wavelet Galerkin	**Fifth-order scheme**	Wavelet Galerkin	**Fifth-order scheme**
0.25	0.025	1.0296194171812	1.029611	**1.0296194171774**	8.5×10^{-06}	$\mathbf{3.7 \times 10^{-12}}$
	0.050	1.0473189942806	1.047302	**1.0473189942765**	1.6×10^{-05}	$\mathbf{3.9 \times 10^{-12}}$
	0.100	1.0818051703517	1.081772	**1.0818051703473**	3.0×10^{-05}	$\mathbf{4.1 \times 10^{-12}}$
0.50	0.025	1.1151415906193	1.115134	**1.1151415906119**	6.5×10^{-06}	$\mathbf{6.7 \times 10^{-12}}$
	0.050	1.1314021114911	1.131388	**1.1314021114780**	1.2×10^{-05}	$\mathbf{1.2 \times 10^{-11}}$
	0.100	1.1631508098057	1.163125	**1.1631508097864**	2.2×10^{-05}	$\mathbf{1.7 \times 10^{-11}}$
0.75	0.025	1.1939224684724	1.193917	**1.1939224684648**	4.5×10^{-06}	$\mathbf{6.4 \times 10^{-12}}$
	0.050	1.2089603458370	1.208951	**1.2089603458241**	7.9×10^{-06}	$\mathbf{1.1 \times 10^{-11}}$
	0.100	1.2383742310433	1.238358	**1.2383742310243**	1.3×10^{-05}	$\mathbf{1.5 \times 10^{-11}}$

Table 7. CPU time in Example 4 when $h = 0.025$ and $l = 0.001$.

t	CPU Time (in seconds)
0.025	0.203
0.05	0.359
0.10	0.735
0.25	1.782
0.50	3.313
1.0	6.985

Example 5. Once again consider (38) with (2), (39) and (40)

$$f(x) = \pi^2 \cos x, \quad 0 < x < 1, \qquad g(t) = (\pi^2 + t)\cos(1.0), \quad 0 < t < 1, \qquad b = 0.75$$
$$M(t) = (\pi^2 + t)\sin(0.75), \quad 0 < t \le 1,$$
$$s(x,t) = (1 + \pi^2 + t)\cos x, \quad 0 < t \le 1,\ 0 < x < 1.$$

The theoretical solution of the problem (see [**1**]) is

$$u(x,t) = (\pi^2 + t)\cos x.$$

Experiments are performed for $x = 0.25$, 0.50, 0.75 and $t = 0.025$, 0.05, 0.10. and the results are given in Table 8 where these are compared with the results of [**1**].

Table 8. Results in Example 5 at $l = 0.001$ and $h = 0.001$.

x	t	Exact solution	Approximate solution		Absolute relative error	
			Wavelet Galerkin	**Fifth-order scheme**	Wavelet Galerkin	**Fifth-order scheme**
0.25	0.025	9.5870051121283	9.587025	**9.5870051120588**	2.1×10^{-06}	$\mathbf{7.2 \times 10^{-12}}$
	0.050	9.6112279226711	9.611272	**9.6112279226230**	4.6×10^{-06}	$\mathbf{5.0 \times 10^{-12}}$
	0.100	9.6596735437566	9.659767	**9.6596735437372**	9.8×10^{-06}	$\mathbf{2.0 \times 10^{-12}}$
0.50	0.025	8.6833322791998	8.683344	**8.6833322791363**	1.8×10^{-06}	$\mathbf{7.3 \times 10^{-12}}$
	0.050	8.7052718432470	8.705298	**8.7052718431376**	3.0×10^{-06}	$\mathbf{1.3 \times 10^{-11}}$
	0.100	8.7491509713415	8.749208	**8.7491509712009**	6.5×10^{-06}	$\mathbf{1.6 \times 10^{-11}}$
0.75	0.025	7.2397719021870	7.239781	**7.2397719021386**	1.2×10^{-06}	$\mathbf{6.7 \times 10^{-12}}$
	0.050	7.2580641239088	7.258080	**7.2580641238226**	2.2×10^{-06}	$\mathbf{1.2 \times 10^{-11}}$
	0.100	7.2946485673525	7.294680	**7.2946485672273**	4.3×10^{-06}	$\mathbf{1.7 \times 10^{-11}}$

5. Conclusion

The results obtained by using the fifth-order scheme are highly accurate when compared with those results which have already existed in the literature. It is noted that the method is fifth-order accurate except for very small values of l and h when errors accumulate due to a large number of arithmetic operations.

So it is clear that the new scheme is the best candidate for the solution of model problems. This technique can be coded easily on serial or parallel computers.

It is worth mentioning that the use of real arithmetic and multiprocessor architecture can save remarkable CPU time rather than methods based on complex arithmetic.

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