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WEAKLY ω-CONTINUOS FUNCTIONS

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Abstract. The purpose of this paper is to introduce a new class of functions called weakly ω-continuous which contains the class of ω-continuous functions and to investigate their basic properties.

0. Introduction

Throughout this work a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let(X, τ) be a space and A be a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [**7**] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$ there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X, τ), denoted by τ_ω, forms a topology on X finer than τ. Let (X, τ) be a space and A be a subset of X. The closure of A, the interior of A and the relative topology on A will be denoted by $\mathrm{cl}_\tau(A)$, $\mathrm{int}_\tau(A)$ and τ_A, respectively. The ω-interior (ω-closure) of a subset A of a space (X, τ) is the interior (closure) of A in the space (X, τ_ω) and is denoted by $\mathrm{int}_{\tau_\omega}(A)(\mathrm{cl}_{\tau_\omega}(A))$.

Weak continuity due to Levine [**8**] is one of the most important weak forms of continuity in topological spaces. It is well-known that if $f : (X, \tau) \to (Y, \sigma)$ is a function from a space (X, τ) into a regular space (Y, σ), then f is continuous iff it is weakly continuous. In [**6**], Hdeib introduced the notion of ω-continuous functions and in [**3**, Theorem 3.12], Al-Zoubi showed that a function $f : (X, \tau) \to (Y, \sigma)$ from an anti-locally countable space (X, τ) into a regular space (Y, σ) is continuous iff it is ω-continuous iff for each $x \in X$ and each open set V in (Y, σ) with $f(x) \in V$, there exists an ω-open set U in (X, τ) such that $x \in U$ and $f(U) \subseteq \mathrm{cl}_\sigma(V)$.

In Section 1 of the present work we use the family of ω-open subsets to define weakly ω-continuous functions. We obtain characterizations of this type of functions and also we study its relation to other known classes of generalized continuous functions, namely the classes of ω-continuous functions, and weakly continuous functions.

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In Section 2, basic properties of weakly ω-continuous functions such as composition, product, restriction, ... etc are given.

For a nonempty set X, τ_{ind}, respectively, τ_{dis} will denote, the indiscrete, respectively, the discrete topologies on X. $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the sets of all real numbers, rational numbers, and natural numbers, respectively. By τ_u we denote the usual topology on $\mathbb{R}$. Finally, if (X,τ) and (Y,ρ) are two spaces, then $\tau \times \rho$ will denote the product topology on $X \times Y$.

Now we recall some known notions, definitions and results which will be used in the work.

Definition 0.1. A space (X,τ) is called

(a) *Locally countable* [**9**] if each point $x \in X$ has a countable open neighborhood.

(b) *Anti-locally countable* [**4**] if each non-empty open set is uncountable.

Definition 0.2. A function $f : (X,\tau) \to (Y,\sigma)$ is called

(a) *ω-continuous* [**6**] if $f^{-1}(V)$ is ω-open in (X,τ) for every open set V of (Y,σ).

(b) *ω-irresolute* [**2**] if $f^{-1}(V)$ is ω-open in (X,τ) for every ω-open set V of (Y,σ).

Lemma 0.3 ([**4**])**.** *Let A be a subset of a space(X,τ). Then*

(a) $(\tau_\omega)_\omega = \tau_\omega$.

(b) $(\tau_A)_\omega = (\tau_\omega)_A$.

Lemma 0.4 ([**1**])**.** *Let A be a subset of an anti-locally countable space (X,τ).*

(a) If $A \in \tau_\omega$, then $\text{cl}_\tau(A) = \text{cl}_{\tau_\omega}(A)$.

(b) If A is ω-closed in (X,τ), then $\text{int}(A) = \text{int}_{\tau_\omega}(A)$.

Lemma 0.5 ([**3**])**.** *Let (X,τ) and (Y,σ) be two topological spaces.*

(a) $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.

(b) If $A \subseteq X$ and $B \subseteq Y$, then $\text{cl}_{\tau_\omega}(A) \times \text{cl}_{\sigma_\omega}(B) \subseteq \text{cl}_{(\tau\times\sigma)_\omega}(A \times B)$.

1. Weakly ω-continuous functions

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is called weakly continuous [**8**] if for each $x \in X$ and each open set V in (Y,σ) containing $f(x)$, there exists an open set U in (X,τ) such that $x \in U$ and $f(U) \subseteq \text{cl}_\sigma(V)$.

Definition 1.1. A function $f : (X,\tau) \longrightarrow (Y,\sigma)$ is said to be *ω^ω-weakly continuous* (respectively, *ω-weakly continuous*, *weakly ω-continuous*) if for each $x \in X$ and for each $V \in \sigma_\omega$ (respectively, $V \in \sigma$) containing $f(x)$, there exists an ω-open subset U of X containing x such that $f(U) \subseteq \text{cl}_{\sigma_\omega}(V)$ (respectively, $f(U) \subseteq \text{cl}_\sigma(V)$, $f(U) \subseteq \text{cl}_{\sigma_\omega}(V)$).

Observe that if (X,τ) is a locally countable space, then τ_ω is the discrete topology and so every function $f : (X,\tau) \to (Y,\sigma)$ is ω^ω-weakly continuous.

The following diagram follows immediately from the definitions in which none of these implications is reversible.

$$\begin{tikzcd}
\text{continuous} \arrow[r] \arrow[dr] & \omega\text{-continuous} \arrow[r] & \text{weakly } \omega\text{-continuous} \arrow[d] & \omega^{\omega}\text{-weakly continuous} \arrow[l] \arrow[dl] \\
& \text{weakly continuous} \arrow[r] & \omega\text{-weakly continuous} &
\end{tikzcd}$$

Example 1.2. (a) Let $X = \mathbb{R}$ with the topologies $\tau = \tau_u$, $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $\rho = \{\emptyset, \mathbb{R}, \{1\}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \sigma)$ be the function defined by

$$f(x) = \begin{cases} \sqrt{2} & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{Q} \end{cases}$$

Then f is ω-weakly continuous, but it is not weakly ω-continuous. Note that

$\mathrm{cl}_{\sigma_\omega}(\mathbb{Q}) = \mathbb{Q}$ and if W is an ω-open set in $(\mathbb{R}, \tau)$, then $W \cap (\mathbb{R} - \mathbb{Q}) \neq \emptyset$. On the other hand, the function $g : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \rho)$ given by

$$g(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{Q} \end{cases}$$

is weakly continuous (ω-weakly continuous), but it is neither weakly ω-continuous nor ω^{ω}-weakly continuous.

(b) Let $X = \mathbb{R}$ with the topologies $\tau = \{U \subseteq \mathbb{R} : U \subseteq \mathbb{R} - \mathbb{Q}\} \cup \{\mathbb{R}\}$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{Q} \end{cases}$$

Then f is ω-continuous, but it is not ω^{ω}-weakly continuous. Note that if we choose $x \in \mathbb{Q}$, then $f(x) = 1 \in V = \{1\} \in \sigma_\omega$. Now if $U \in \tau_\omega$ such that $x \in U$ and $f(U) \subseteq \mathrm{cl}_{\sigma_\omega}(V) = \{1\}$, then $U \subseteq \mathbb{Q}$. But the only open set containing x is $\mathbb{R}$, therefore $\mathbb{R} - U$ is countable, a contradiction.

(c) Let $X = \mathbb{R}$ with the topologies $\tau = \tau_u$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{R} - \{0\}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{Q} \end{cases}$$

Then f is not ω-continuous since $V = \mathbb{R} - \{0\} \in \sigma$, but $f^{-1}(V) = \mathbb{Q} \notin \tau_\omega$. On the other hand, f is weakly ω-continuous since $\mathrm{cl}_{\sigma_\omega}(\mathbb{R} - \{0\}) = \mathbb{R}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then a function $f^{\omega}_{\omega} : (X, \tau_\omega) \to (Y, \sigma_\omega)$ (respectively, $f_\omega : (X, \tau_\omega) \to (Y, \sigma)$, $f^{\omega} : (X, \tau) \to (Y, \sigma_\omega)$) associated with f is defined as follows: $f^{\omega}_{\omega}(x) = f(x)$ (respectively, $f_\omega(x) = f(x)$, $f^{\omega}(x) = f(x)$) for each $x \in X$.

The proof of the following results follow immediately from the definitions and Lemma 0.3 part (a).

Remark 1.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a function.

(a) f is $\omega^{\omega}-$weakly continuous iff f^{ω}_{ω} is weakly continuous.
(b) f is ω-weakly continuous iff f_ω is weakly continuous.

(c) f_ω^ω is weakly continuous iff it is ω^ω-weakly continuous iff it is weakly ω-continuous iff it is ω-weakly continuous.
(d) If (Y, σ) is a locally countable space, then f is ω-continuous iff it is weakly ω-continuous.
(e) If (Y, σ) is an anti-locally countable space, then f is ω-weakly continuous iff it is weakly ω-continuous.

It follows from Remark 1.3 part (a) and part (b) that the basic properties of ω^ω-weakly continuous and ω-weakly continuous functions follow from the well known properties of weakly continuous functions.

Proposition 1.4. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is weakly ω-continuous iff $f^{-1}(V) \subset \text{int}_{\tau_\omega}(f^{-1}(\text{cl}_{\sigma_\omega}(V)))$ for every $V \in \sigma$.*

The easy proof is left to the reader.

2. Fundamental Properties of Weakly ω-continuous functions

In this section we obtain several fundamental properties of weakly ω-continuous functions.

The composition $g \circ f : (X, \tau) \longrightarrow (Z, \rho)$ of a continuous function $f : (X, \tau) \longrightarrow (Y, \sigma)$ and a weakly ω-continuous function $g : (Y, \sigma) \longrightarrow (Z, \rho)$ is not necessarily weakly ω-continuous as the following example shows. Thus, the composition of weakly ω-continuous functions need not be weakly ω-continuous.

Example 2.1. Let $X = \mathbb{R}$ with the topologies $\tau = \tau_u$, and $\sigma = \tau_{\text{ind}}$ and let $Y = \{0, 1\}$ with the topology $\rho = \{\emptyset, Y, \{1\}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ \sqrt{2} & \text{for } x \in \mathbb{Q} \end{cases}$$

and let $g : (\mathbb{R}, \sigma) \longrightarrow (Y, \rho)$ be the function defined by

$$g(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ 0 & \text{for } x \in \mathbb{Q} \end{cases}$$

Then f is continuous and g is weakly ω-continuous. However $g \circ f$ is not weakly ω-continuous. Note that if $x \in \mathbb{Q}$, then $(g \circ f)(x) = 1 \in V = \{1\} \in \rho$. Suppose there exists ω-open set W in $(\mathbb{R}, \tau)$ such that $x \in W$ and $(g \circ f)(W) \subset \text{cl}_{\sigma_\omega}(V) = \{1\}$. Then $W \subseteq \mathbb{Q}$, i.e. W is countable, a contradiction. Therefore $g \circ f$ is not weakly ω-continuous.

Recall that a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called θ-continuous [**5**] if for each $x \in X$ and each open set V in (Y, σ) containing $f(x)$, there exists an open set U in (X, τ) such that $x \in U$ and $f(\text{cl}_\tau(U)) \subset \text{cl}_\sigma(V)$.

Theorem 2.2. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ be two functions. Then the following statement hold*

(a) *$g \circ f$ is weakly ω-continuous if g is weakly ω-continuous and f is ω-irresolute.*
(b) *$g \circ f$ is weakly ω-continuous if f is weakly ω-continuous and g is ω-irresolute and continuous.*

(c) *$g \circ f$ is weakly ω-continuous if g^{ω} is θ-continuous and f is weakly ω-continuous.*
(d) *$g\circ f$ is weakly ω-continuous if g^{ω} is weakly continuous and f is ω-continuous.*
(e) *Let (Z,ρ) be an anti-locally countable space. Then $g \circ f$ is weakly ω-continuous if g is θ-continuous and f is weakly ω-continuous.*

The easy proof is left to the reader.
The following examples show that the conditions in Theorem 2.2 are essential.

Example 2.3. Let $X = \mathbb{R}$ with the topologies $\tau = \tau_u$ and $\eta = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and let $Y = \{1, \sqrt{2}\}$ with the topologies $\sigma = \{\emptyset, Y, \{\sqrt{2}\}\}$ and $\rho = \{\emptyset, Y, \{1\}\}$.

(a) Let $f : (\mathbb{R}, \tau) \longrightarrow (Y, \rho)$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ \sqrt{2} & \text{for } x \in \mathbb{Q} \end{cases}$$

and $g : (Y, \rho) \longrightarrow (Y, \sigma)$ be the identiy function. Clearly, (Y, ρ) is not anti-locally countable, f is weakly ω-continuous, g is θ-continuous and ω-irresolute, but not continuous. However $g \circ f$ is not weakly ω-continuous.

(b) Define $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \eta)$ and $g : (\mathbb{R}, \eta) \longrightarrow (Y, \rho)$ as follows

$$f(x) = g(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R} - \mathbb{Q} \\ \sqrt{2} & \text{for } x \in \mathbb{Q} \end{cases}$$

Then f is weakly ω-continuous since $\mathrm{cl}_{\sigma_\omega}(\mathbb{R} - \mathbb{Q}) = \mathbb{R}$ and g is continuous, but it is not ω-irresolute. However $g \circ f$ is not weakly ω-continuous.

Note that Example 2.3 shows that continuity and ω-irresoluteness are independent notions.

Lemma 2.4 ([**3**])**.** *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be an open surjective function.*
1) *If $A \subseteq X$, then $f(\mathrm{int}_{\tau_\omega}(A)) \subseteq \mathrm{int}_{\sigma_\omega} f(A)$.*
2) *If $U \in \tau_\omega$, then $f(U) \in \sigma_\omega$.*

Theorem 2.5. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be an open surjection and let $g : (Y, \sigma) \longrightarrow (Z, \rho)$ such that $g \circ f : (X, \tau) \longrightarrow (Z, \rho)$ is weakly ω-continuous. Then g is weakly ω-continuous.*

Proof. Let $y \in Y$ and let $V \in \rho$ with $g\ (y) \in V$. Choose $x \in X$ such that $f(x) = y$. Since $g \circ f$ is weakly ω-continuous, there exists $U \in \tau_\omega$ with $x \in U$ and $g(f(U)) \subset \mathrm{cl}_{\sigma_\omega}(V)$. But f is open, therefore by Lemma 2.4, $f(U) \in \sigma_\omega$ with $f(x) \in f(U)$ and the result follows. □

Theorem 2.6. *Let (X, τ) and (Y, σ) be topological spaces where (Y, σ) is locally countable. Then the projection $p_X : (X \times Y, \tau \times \sigma) \to (X, \tau)$ is ω-irresolute.*

Proof. Let $(x, y) \in X \times Y$ and let V be an ω-open subset of (X, τ) such that $p_X(x, y) = x \in V$. Choose $U \in \tau$ and a countable open subset W of (Y, σ) such that $y \in W$, $x \in U$ and $U - V$ is countable. Since $U \times W - V \times Y = (U - V) \times W$ is countable, $V \times Y \in (\tau \times \sigma)_\omega$ and so $B = p_X^{-1}(U) \cap (V \times Y) = (U \cap V) \times Y \in (\tau \times \sigma)_\omega$. Now $(x, y) \in B$ and $p_X(B) = U \cap V \subseteq V$. Therefore p_X is ω-irresolute. □

To show that the condition (Y, σ) is locally countable in Theorem 2.6 is essential we consider the following example.

Example 2.7. Consider the projection $p : (\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u) \to (\mathbb{R}, \tau_u)$ and let $A = \mathbb{R} - \mathbb{Q}$. Then A is ω-open in $(\mathbb{R}, \tau_u)$ while $p^{-1}(A) = (\mathbb{R}\text{-}\mathbb{Q}) \times \mathbb{R}$ is not ω-open in $(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u)$. Thus p is not ω-irresolute.

Corollary 2.8. *Let Δ be a countable set and let $f_\alpha : (X_\alpha, \tau_\alpha) \longrightarrow (Y_\alpha, \sigma_\alpha)$ be a function for each $\alpha \in \Delta$. If the product function $f = \prod_{\alpha\in\Delta} f_\alpha : \prod_{\alpha\in\Delta} X_\alpha \longrightarrow \prod_{\alpha\in\Delta} Y_\alpha$ is weakly ω-continuous and $(Y_\alpha, \sigma_\alpha)$ is locally countable for each $\alpha \in \Delta$, then f_α is weakly ω-continuous for each $\alpha \in \Delta$.*

Proof. For each $\beta \in \Delta$, we consider the projections $p_\beta : \prod_{\alpha\in\Delta} X_\alpha \longrightarrow X_\beta$ and $q_\beta : \prod_{\alpha\in\Delta} Y_\alpha \longrightarrow Y_\beta$. Then we have $q_\beta \circ f = f_\beta \circ p_\beta$ for each $\beta \in \Delta$. Since f is weakly ω-continuous and q_β is ω-irresolute (Theorem 2.6) for each $\beta \in \Delta$, $q_\beta \circ f$ is weakly ω-continuous and hence $f_\beta \circ p_\beta$ is weakly ω-continuous. Thus f_β is weakly ω-continuous by Theorem 2.5. $\square$

The following example shows that the converse of Corollary 2.8 is not true in general.

Example 2.9. Let $X = \mathbb{R}$ with the topology $\tau = \{U : U \subseteq \mathbb{Q}\} \cup \{\mathbb{R}\}$ and let $Y = \{0, 1, 2\}$ with the topology $\sigma = \{\emptyset, Y, \{0\}, \{1, 2\}\}$. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R} - \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{Q}. \end{cases}$$

One can easily show that f is weakly ω-continuous. However, the product function $h = f \times f : \mathbb{R} \times \mathbb{R} \longrightarrow Y \times Y$ defined by $h(x, t) = (f(x), f(t))$ for all $x, t \in \mathbb{R}$ is not weakly ω-continuous. Let $(x, t) \in (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Then $h(x, t) = (f(x), f(t)) = (1, 1)$. Take $V = \{1, 2\} \times \{1, 2\}$. Then $V \in \sigma \times \sigma$ with $h(x, t) \in V$. Suppose there exists $U \in (\tau \times \tau)_\omega$ such that $(x, t) \in U$ and $h(U) \subseteq \mathrm{cl}_{\sigma_\omega}(V) = V$. Therefore $U \subseteq (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Note that the only open set containing (x, t) is $\mathbb{R} \times \mathbb{R}$ and so $(\mathbb{R} \times \mathbb{R}) - U$ is countable. Thus

$$(\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R}) = (\mathbb{R} \times \mathbb{R}) - ((\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})) \subseteq (\mathbb{R} \times \mathbb{R}) - U,$$

a contradiction.

To see that the conditions in Corollary 2.8 are essential we consider the following examples.

Example 2.10. (a) Let $X = \mathbb{R}$ with the topologies $\tau = \tau_u, \rho = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $\mu = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \rho)$ be the function given by $f(x) = 1$ for all $x \in \mathbb{R}$ and let $g : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \mu)$ be the function defined by

$$g(x) = \begin{cases} \sqrt{2} & \text{for } x \in \mathbb{R} - \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{Q}. \end{cases}$$

One can easily show that f is weakly ω-continuous while g is not. To show that $f \times g$ is weakly ω-continuous, let $(x, y) \in \mathbb{R} \times \mathbb{R}$ and let $W \in \sigma \times \mu$ such that

$(f \times g)(x, y) \in W$. There exists a basic open set V in $(\mathbb{R} \times \mathbb{R}, \rho \times \mu)$ such that $(f \times g)(x, y) \in \{(1, 0), (1, \sqrt{2})\} \subseteq V \subseteq W$. Therefore $V \in \{\mathbb{R} \times \mathbb{R}\ , \mathbb{R} \times \mathbb{Q}\}$. To complete the proof it is enough to show that $\mathrm{cl}_{(\rho\times\mu)_\omega}(\mathbb{R} \times \mathbb{Q}) = \mathbb{R} \times \mathbb{R}$. Suppose there exists $(s, t) \in \mathbb{R} \times \mathbb{R} - \mathrm{cl}_{(\rho\times\mu)_\omega}(\mathbb{R} \times \mathbb{Q})$. Then there exist $W \in (\sigma \times \mu)_\omega$ and a basic open set U in $(\mathbb{R} \times \mathbb{R}, \rho \times \mu)$ such that $(s, t) \in W \cap U$, $W \cap (\mathbb{R} \times \mathbb{Q}) = \emptyset$ and $U - W$ is countable. Therefore $W \subseteq \mathbb{R} \times (\mathbb{R} - \mathbb{Q})$ and $U \in \{\mathbb{R} \times \mathbb{R}, (\mathbb{R} - \mathbb{Q}) \times \mathbb{R}\}$. Thus $U - (\mathbb{R} \times (\mathbb{R} - \mathbb{Q}))$ is countable, a contradiction.

(b) Let $X = \mathbb{R}$ with the topology $\tau = \tau_u$ and $Y = \{1, \sqrt{2}\}$ with the topology $\sigma = \{\emptyset, Y, \{1\}\}$. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} \sqrt{2} & \text{for } x \in \mathbb{R} - \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{Q}. \end{cases}$$

Then f is not weakly ω-continuous. Let Δ be an uncountable set and let $X_\alpha = X$ and $Y_\alpha = Y$ for all $\alpha \in \Delta$. Then the product function

$$h = \prod_{\alpha\in\Delta} f_\alpha : \prod_{\alpha\in\Delta} X_\alpha \longrightarrow \prod_{\alpha\in\Delta} Y_\alpha$$

is weakly ω-continuous where $f_\alpha = f$ for all $\alpha \in \Delta$. We show that if B is a basic open set in $\prod_{\alpha\in\Delta} Y_\alpha$, then $\mathrm{cl}_{(\sigma_p)_\omega}(B) = \prod_{\alpha\in\Delta} Y_\alpha$, where σ_p is the product topology on $\prod_{\alpha\in\Delta} Y_\alpha$. Suppose by contrary that there exists $y \in \prod_{\alpha\in\Delta} Y_\alpha - \mathrm{cl}_{(\sigma_p)_\omega}(B)$. Note that $B = \prod_{\alpha\in\Delta} B_\alpha$ where $B_\alpha = Y_\alpha$ for all but finitely many $\alpha \in \Delta$, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. Therefore

$$B_{\alpha_1} = B_{\alpha_2} = \ldots = B_{\alpha n} = \{1\}.$$

Now choose $W \in (\sigma_p)_\omega$ and a basic open set $V\ = \prod \alpha \in \Delta V_\alpha$ in $\prod_{\alpha\in\Delta} Y_\alpha$ such that $x \in W \cap V$, $W \cap B = \emptyset$, and $V - W$ is countable. Thus

$$\emptyset \neq B \cap V = \prod_{\alpha\in\Delta} (B_\alpha \cap V_\alpha) \subseteq V - W,$$

a contradiction.

Theorem 2.11. *Let $f : (X, \tau) \longrightarrow (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be a weakly ω-continuous function, where (X, τ), (Y_1, σ_1) and (Y_2, σ_2) are topological spaces. Let $f_i : (X, \tau) \longrightarrow (Y_i, \sigma_i)$ be defined as $f_i = P_i \circ f$ for $i = 1, 2$.*

(a) *If f_i is weakly ω-continuous for $i = 1, 2$, then f is weakly ω-continuous.*
(b) *If (Y_1, σ_1) and (Y_2, σ_2) are locally countable spaces and f is weakly ω-continuous, then f_i is weakly ω-continuous for $i = 1, 2$.*

Proof. (a) Let $x \in X$ and let V be an open in $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ such that $f(x) \in V$. There exist $V_1 \in \sigma_1$ and $V_2 \in \sigma_2$ such that

$$f(x) = (f_1(x), f_2(x)) \in V_1 \times V_2 \subseteq V.$$

Now

$$(P_i \circ f)(x) = P_i(f_1(x), f_2(x)) = f_i(x) \in V_i \quad \text{for } i = 1, 2$$

and so there exist $U_1, U_2 \in \tau_\omega$ such that

$$f_i(U_i) = (P_i \circ f)(U_i) \subseteq \mathrm{cl}_{\sigma_\omega}(V_i).$$

Put $U = U_1 \cap U_2$. Then $U \in \tau_\omega$ such that $x \in U$ and

$$f(U) = (f_1(U), f_2(U)) \subseteq \mathrm{cl}_{(\sigma_1)_\omega}(V_1) \times \mathrm{cl}_{(\sigma_2)_\omega}(V_2) \subseteq \mathrm{cl}_{(\sigma_1 \times \sigma_2)_\omega}(V)$$

by Lemma 0.5. Thus f is weakly ω-continuous.

(b) This follows from Theorem 2.6 and Theorem 2.2. □

To see that the condition put on (Y_1, σ_1) and (Y_2, σ_2) to be locally countable in Theorem 2.11 part (b) is essential we consider the functions f and g as given in Example 2.10 part (a). Then the function $h : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R} \times \mathbb{R}, \mu \times \rho)$ defined by $h(x) = (f(x), g(x))$ is weakly ω-continuous while g is not.

Theorem 2.12. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function with $g : (X, \tau) \longrightarrow (X \times Y, \tau \times \sigma)$ denoting the graph function of f defined by $g(x) = (x, f(x))$ for every point $x \in X$. If f is weakly ω-continuous, then g is weakly ω-continuous.*

Proof. Let $x \in X$ and let $W \in \tau \times \sigma$ with $g(x) \in W$. Then there exist $U \in \tau$ and $V \in \sigma$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is weakly ω-continuous there exists $U_1 \in \tau_\omega$ with $x \in U_1$ and $f(U_1) \subseteq \mathrm{cl}_{\sigma_\omega}(V)$. Put $U = U \cap U_1$. Then $U \in \tau_\omega$ with $x \in U$ and

$$\begin{aligned} g(U) = g(U \cap U_1) &= (U \cap U_1, f(U \cap U_1)) \subseteq U \times f(U_1) \\ &\subseteq \mathrm{cl}_{\tau_\omega}(U) \times \mathrm{cl}_{\sigma_\omega}(V) \subseteq \mathrm{cl}_{(\tau \times \sigma)_\omega}(U \times V) \subseteq \mathrm{cl}_{(\tau \times \sigma)_\omega}(W) \end{aligned}$$

by Lemma 0.5. □

The following example shows that the convese of Theorem 2.12 is not true in general.

Example 2.13. Let $X = Y = \mathbb{R}$ with the topologies $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$, and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \sigma)$ be the function defined by

$$f(x) = \begin{cases} \sqrt{2} & \text{for } x \in \mathbb{R} - \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{Q}. \end{cases}$$

Then f is not weakly ω-continuous. On the other hand, the graph function g is weakly ω-continuous since $\mathrm{cl}_{(\tau \times \sigma)_\omega}(\mathbb{R} \times \mathbb{Q}) = \mathrm{cl}_{(\tau \times \sigma)_\omega}((\mathbb{R} - \mathbb{Q}) \times \mathbb{R}) = \mathbb{R} \times \mathbb{R}$ (see Example 2.10 part (a))

The following results follow immediately from the definitions and Lemma 0.3.

Theorem 2.14. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function.*

(a) *If f is weakly ω-continuous and A a subset of X, then the restriction $f|_A : (A, \tau_A) \longrightarrow (Y, \sigma)$ is weakly ω-continuous.*
(b) *Let $x \in X$. If there exists an ω-open subset A of X containing x such that $f|_A : (A, \tau_A) \longrightarrow (Y, \sigma)$ is weakly ω-continuous at x, then f is weakly ω-continuous at x.*
(c) *If $U = \{U_\alpha : \alpha \in \Delta\}$ is an ω-open cover of X, then f is weakly ω-continuous if and only if $f|_{U_\alpha}$ is weakly ω-continuous for all $\alpha \in \Delta$.*

The following example shows that the assumption A is ω-open in Theorem 2.14 part (b) can not be replaced by the statement A is ω-closed.

Example 2.15. Let $X = \mathbb{R}$ with the topology τ_u and let $Y = \{0, 1\}$ with the topology $\sigma = \{\emptyset, Y, \{1\}\}$. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} - \mathbb{Q}, \\ 1 & \text{for } x \in \mathbb{Q}. \end{cases}$$

Then $f\,|_{\mathbb{Q}}$ is weakly ω-continuous, but f is not.

Theorem 2.16. *Let (X, τ) be an anti-locally countable space. Then (X, τ) is Hausdroff if and only if (X, τ_ω) is Hausdroff.*

Proof. We need to show the sufficiency part only. Let $x, y \in X$ with $x \neq y$. Since (X, τ_ω) is a Hausdroff space, there exist $W_x, W_y \in \tau_\omega$ such that $x \in W_x$, $y \in W_y$ and $W_x \cap W_y = \emptyset$. Choose $V_x, V_y \in \tau$ such that $x \in V_x$, $y \in V_y$, $V_x - W_x = C_x$, and $V_y - W_y = C_y$ where C_x and C_y are countable sets. Thus

$$V_x \cap V_y \subseteq (C_x \cup W_x) \cap (C_y \cup W_y) \subseteq C_x \cup C_y.$$

Since (X, τ) is anti-locally countable, then $V_x \cap V_y = \emptyset$ and the result follows. $\square$

Theorem 2.16 is no longer true if the assumption of being anti-locally countable is omitted. To see that we consider the space $(\mathbb{N}, \tau_{\text{cof}})$ where τ_{cof} is the cofinite topology. Then $(\mathbb{N}, \tau_{\text{cof}})$ is not anti-locally countable. On the other hand, $(\mathbb{N}, (\tau_{\text{cof}})_\omega) = (\mathbb{N}, \tau_{\text{dis}})$ is a Hausdroff space, but $(\mathbb{N}, \tau_{\text{cof}})$ is not.

Theorem 2.17. *Let (A, τ_A) be a subspace of a space (X, τ). If the retraction function $f : (X, \tau) \longrightarrow (A, \tau_A)$ defined by $f(x) = x$ for all $x \in A$ is weakly ω-continuous and (X, τ) is a Hausdroff space, then A is ω-closed.*

Proof. Suppose A is not ω-closed. Then, there exists $x \in \mathrm{cl}_{\tau_\omega}(A) - A$. Since f is a retraction function, $x \neq f(x)$ and so there exist two disjoint open sets U and V in (X, τ) such that $x \in U$ and $f(x) \in V$. Thus $U \cap \mathrm{cl}_{\tau_\omega}(V) \subseteq U \cap \mathrm{cl}(V) = \emptyset$. Now let W be an ω-open set in (X, τ) such that $x \in W$. Then $U \cap W$ is an ω-open set in (X, τ) containing x and so $U \cap W \cap A \neq \emptyset$. Choose $y \in U \cap W \cap A$. Then $y = f(y) \in U$ and so $f(y) \notin \mathrm{cl}_{\tau_\omega}(V)$, i.e. $f(W)$ is not a subset of $\mathrm{cl}_{\tau_\omega}(V)$. Thus f is not weakly ω-continuous at x, a contradiction. Thus A is ω-closed. $\square$

Theorem 2.18. *If (X, τ) is a connected anti-locally countable space and $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a weakly ω-continuous surjection function, then (Y, σ) is connected.*

Proof. At first we show that if V is a clopen subset of (Y, σ), then $f^{-1}(V)$ is clopen in (X, τ). Let V be a clopen subset of (Y, σ). Then by Proposition 1.4,

$$f^{-1}(V) \subset \mathrm{int}_{\tau_\omega}(f^{-1}(\mathrm{cl}_{\sigma_\omega}(V))) \subseteq \mathrm{int}_{\tau_\omega}(f^{-1}(\mathrm{cl}_{\sigma}(V))) = \mathrm{int}_{\tau_\omega}(f^{-1}(V)).$$

Thus $f^{-1}(V)$ is ω-open in (X, τ) and so, by Lemma 0.4,

$$\mathrm{cl}_{\tau}(f^{-1}(V)) = \mathrm{cl}_{\tau_\omega}(f^{-1}(V)).$$

Now we show that $f^{-1}(V)$ is ω-closed in (X, τ). Suppose by contrary that there exists $x \in \mathrm{cl}_{\tau_\omega}(f^{-1}(V)) - f^{-1}(V)$. Since f is weakly ω-continuous and $Y - V$ is an open set in (Y, σ) containing $f(x)$, there exists $U \in \tau_\omega$ such that $x \in U$ and

$$f(U) \subseteq \mathrm{cl}_{\sigma_\omega}(Y - V) = Y - V.$$

But $x \in \mathrm{cl}_{\tau_\omega}(f^{-1}(V))$ and so $U \cap f^{-1}(V) \neq \emptyset$. Therefore,

$$\emptyset \neq f(U) \cap V \subseteq V \cap (Y - V),$$

a contradiction. Thus $f^{-1}(V)$ is ω-closed in (X, τ) and so

$$\mathrm{cl}_\tau(f^{-1}(V)) = \mathrm{cl}_{\tau_\omega}(f^{-1}(V)) = f^{-1}(V),$$

i.e., $f^{-1}(V)$ is closed in (X, τ). Also by using Lemma 0.4,

$$\mathrm{int}_\tau\, f^{-1}(V) = \mathrm{int}_{\tau_\omega}(f^{-1}(V)) = f^{-1}(V),$$

i.e., $f^{-1}(V)$ is open in (X, τ).

Now suppose that (Y, σ) is not connected. Then, there exist nonempty open sets V_1 and V_2 in (Y, σ) such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_j) \neq \emptyset$ for $j = 1, 2$. Since V_j is clopen in (Y, σ), then $f^{-1}(V_j)$ is open in (X, τ) for $j = 1, 2$. This implies that (X, τ) is not connected, a contradiction. Therefore, (Y, σ) is connected. □

Theorem 2.18 is no longer true if the assumption of being anti-locally countable is omitted. To see that we consider the following example.

Example 2.19. Let $X = \mathbb{R}$ with the topology $\tau = \{U \subseteq \mathbb{R} : \mathbb{Q} \subseteq U\} \cup \{\emptyset\}$ and let $Y = \{0, 1, 2\}$ with the topology $\rho = \{\emptyset, Y, \{1\}, \{0, 2\}\}$. Let $f : (\mathbb{R}, \tau) \longrightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R} - \mathbb{Q}, \\ 2 & \text{for } x \in \mathbb{Q} - \{0\}, \\ 0 & \text{for } x = 0. \end{cases}$$

Then f is weakly ω-continuous surjection, (X, τ) is connected but not anti-locally countable, and (Y, σ) is not connected.

Recall that a space (X, τ) is called almost Lindelöf [**10**] if whenever $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ is an open cover of (X, τ) there exists a countable subset I_0 of I such that $X = \bigcup_{\alpha \in I_0} \mathrm{cl}(U_\alpha)$.

In [**7**, Theorem 4.1], Hdeib shows that a space (X, τ) is Lindelöf if and only if (X, τ_ω) is Lindelöf.

Theorem 2.20. *For any space (X, τ), the following items are equivalent*

(a) (X, τ_ω) *is almost Lindelöf.*

(b) *For every open cover $\mathcal{W} = \{W_\alpha : \alpha \in I\}$ of (X, τ) there exists a countable subset I_0 of I such that $X = \bigcup_{\alpha \in I_0} \mathrm{cl}_{\tau_\omega}(W_\alpha)$.*

Proof. We need to prove (b) implies (a). Let $\mathcal{W}$ be an open cover of (X, τ_ω). For each $x \in X$ we choose $W_x \in \mathcal{W}$ and an open set U_x in (X, τ) such that $x \in W_x$ and $U_x - W_x = C_x$ is countable. Therefore the collection $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of (X, τ) and so, by assumption, it contains a countable subfamily

$\mathcal{U}^*=\{U_{xn} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(U_{xn})$. But $\bigcup_{n\in\mathbb{N}} C_{xn}$ is a countable subset of X and we can choose a countable subfamily $\mathcal{W}^*$ of $\mathcal{W}$ such that

$$\bigcup_{n\in\mathbb{N}} C_{xn} = \bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(C_{xn}) \subseteq \cup\{W : W \in \mathcal{W}^*\}.$$

Then

$$\begin{aligned} X &= \bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(U_{xn}) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(W_{xn} \cup C_{xn}) \\ &= \left(\bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(W_{xn})\right) \cup \left(\bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(C_{xn})\right) \\ &\subseteq \left(\bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(W_{xn})\right) \cup \left(\bigcup_{W\in\mathcal{W}^*} W\right) \\ &\subseteq \left(\bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\tau_\omega}(W_{xn})\right) \cup \left(\bigcup_{W\in\mathcal{W}^*} \mathrm{cl}_{\tau_\omega}(W)\right). \end{aligned}$$

Thus (X, τ_ω) is almost Lindelöf.

It is clear that if (X, τ_ω) is almost Lindelöf, then (X, τ) is almost Lindelöf. To see that the converse is not true, in general; we consider the space (X, τ) where $X = \mathbb{R}$ and $\tau = \{U : \mathbb{Q} \subseteq U\} \cup \{\emptyset\}$. Then (X, τ) is almost Lindelöf since $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$. On the other hand, $\tau_\omega = \tau_{\mathrm{disc}}$ and so (X, τ_ω) is not almost Lindelöf. □

Corollary 2.21. *Let (X, τ) be an anti-locally countable space. Then (X, τ) is almost Lindelöf if and only if (X, τ_ω) is almost Lindelöf.*

Theorem 2.22. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a weakly ω-continuous function from a Lindelöf space (X, τ) onto a space (Y, σ). Then (Y, σ_ω) is almost Lindelöf.*

Proof. Let $\mathcal{V}$ be an open cover of (Y, σ). For each $x \in X$ choose $V_x \in \mathcal{V}$ such that $f(x) \in V_x$. Since f is weakly ω-continuous, there exists an ω-open set U_x in (X, τ) such that $x \in U_x$ and $f(U_x) \subseteq \mathrm{cl}_{\sigma_\omega}(V_x)$. Therefore the collection $\mathcal{U} = \{U_x : x \in X\}$ is an ω-open cover of the Lindelöf space (X, τ), and so it contains a countable subfamily $\mathcal{U}^* = \{U_{xn} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} U_{xn}$.

Thus

$$Y = f(X) = f\left(\bigcup_{n\in\mathbb{N}} U_{xn}\right) = \bigcup_{n\in\mathbb{N}} f(U_{xn}) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\sigma_\omega}(V_{xn}).$$

Therefore (Y, σ_ω) is almost Lindelöf by Theorem 2.20. □

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