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# ON SOME PROPERTIES OF A FUNCTION CONNECTING WITH AN INFINITE SERIES

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ABSTRACT. An attempt has been made in this paper to investigate some set theoretic properties of a function suitably defined on the space of all sequences of non-negative real numbers endowed with Fréchet metric.

## 0. INTRODUCTION

Inspiration for this paper arises from the papers [**1**], [**2**] where the authors proved several interesting theorems in relation to Borel and Baire classifications of functions defined by the exponent of convergence of the family of all non-decreasing sequences of real numbers, the first term of which is at least $\gamma$ where $\gamma$ is a positive real number, endowed with Fréchet metric. Our approach in this paper is somewhat different. Instead of taking the family of all non-decreasing sequences $x = \{\xi_k\}_{k=1}^{\infty}$ of real numbers with $\xi_1 > 0$, we consider the set of all sequences of non-negative real numbers with Fréchet metric and after defining a function suitably different from [**1**], [**2**] we study the behaviour of the function from various aspects.

Let $X$ be the set of all real sequences $\{x_n\}$ with Fréchet metric $d(x, y)$ given by

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

where $x = \{x_k\}$, $y = \{y_k\} \in X$.

The metric space $(X, d)$ is complete. Let $S$ denote the set of all sequences $\{x_n\}$ of non-negative real numbers with Fréchet metric. The convergence in this space is considered to be the point-wise convergence.

Let $x \in S$ and $r > 0$. We denote by $B(x, r)$, the open sphere with $x$ as the center and $r$ as the radius. It follows easily that if $x_n = y_n$ for $n = 1, 2, 3, \ldots, N$, then $y \in B(x, \frac{1}{2^N})$. If $x$, $y$ etc. are points of $S$, we shall represent them generally by $x = \{x_k\}$, $y = \{y_k\}$ etc. Also $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}$ denotes the set of real numbers. On the space $S$ we shall define a real function

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$\phi : S \to [1, \infty)$ as follows

$$\phi(x) = \inf \left\{ p > 1 : \sum_{n=1}^{\infty} p^{-x_n} < \infty \right\}, \quad \text{for } \; x = \{x_n\} \in S.$$

It may happen that $\phi(x) = +\infty$. We shall study some properties of $\phi : S \to [1, \infty)$. The interval $[1, \infty)$ is considered with usual topology.

**Proposition 0.1.** *Let $\{a_n\} \in S$, $a_n > 0$ be such that $\sum_n^{\infty} 1/a_n < \infty$ and $\sup a_n^{1/x_n} > 0$, where $\{x_n\} \in S$, $x_n > 0$. Then there exists $a > 0$ such that $\sum_{n=1}^{\infty} a^{-x_n} < \infty$.*

*Proof.* Take $a = \sup a_n^{1/x_n}$. Then $a > 0$. Since $\sup a_n^{1/x_n} = a$, we have $a_n^{1/x_n} \leq a$, for all $n \in \mathbb{N}$. Therefore $\sum_{n=1}^{\infty} a^{-x_n} \leq \sum_{n=1}^{\infty} 1/a_n < \infty$. Hence the result. $\square$

In support of the proposition we present an example.

**Example.** Let $x_n = \log n$, $n > 1$ and $a_n = n^2$. Then $a_n^{1/x_n} = (n^2)^{1/\log n} = (e^{2\log n})^{1/\log n} = e^2$, for each $n > 1$. Take $a = e^2$.

**Proposition 0.2.** *$(S, d)$ is complete and has the power of continuum.*

*Proof.* Let $\{x_n^{(r)}\}_r \in S$ be any sequence converging to $x = \{x_n\}$. Since the convergence in $S$ is the point-wise convergence in the sense of Fréchet metric, it follows that $x \in S$ and $S$ becomes a closed set. Let $x = \{x_n\} \in S$. Then we have a sequence $x^{(r)} = \{x_n^{(r)}\}_n \in S$ such that $\lim_{r\to\infty} x^{(r)} = x$ where

$$x_k^{(r)} = x_k, \qquad \text{for } \; k = 1, 2, \dots r$$
$$\text{and } \; x_k^{(r)} = x_k + 1, \qquad \text{for } \; k > r; \quad r \in \mathbb{N}.$$

So, $S$ becomes a perfect set and therefore $S$ has the cardinal number $c$ where $c$ is the power of continuum and hence (S,d) is complete. $\square$

## 1. Some set theoretic properties of the function $\phi$

**Theorem 1.1.** *The function $\phi : S \to (1, \infty)$ is onto but not one-to-one.*

*Proof.* Let $A = \{a_n\}_{n=1}^{\infty}$ be a monotonic increasing sequence with $a_n \to \infty$ as $n \to \infty$. It is well known ([**4**, p. 40]) that there exists a unique $\lambda = \lambda(A)$, $0 \leq \lambda(A) \leq \infty$ such that

$$\sum_{n=1}^{\infty} a_n^{-\sigma} = +\infty, \qquad \text{for each } \; \sigma \in \mathbb{R}, \;\; \sigma > 0, \;\; \sigma < \lambda$$
$$\text{and} \qquad \sum_{n=1}^{\infty} a_n^{-\sigma} < +\infty, \qquad \text{for each } \; \sigma \in \mathbb{R}, \;\; \sigma > 0, \;\; \sigma > \lambda,$$

i.e.

$$\lambda(A) = \inf\{\sigma > 0 : \sum_{n=1}^{\infty} a_n^{-\sigma} < +\infty\}.$$

Now, we can choose such a sequence $A = \{a_n\}_{n=1}^{\infty}$ with $\lambda(A) = +\infty$.

We know [**5**] that the function $\lambda : (0,1] \to [0,\infty)$ is onto. Then for $1 < a < \infty$, there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that

$$a = \inf\left\{\sigma > 0 : \sum_{k=1}^{\infty} a_{n_k}^{-\sigma} < +\infty\right\}.$$

Now, we show that there exists $y \in S$ such that $\phi(y) = a$.

Let $t = \frac{a}{\log a}$ and choose $y = \{y_n\}_{n=1}^{\infty} \in S$ such that $y_k = \log a_{n_k}^t$. Now, for any real number $b > a$,

$$\sum_{k=1}^{\infty}(b)^{-\log a_{n_k}^t} = \sum_{k=1}^{\infty}(e)^{(-\log b)\log a_{n_k}^t} = \sum_{k=1}^{\infty} a_{n_k}^{-t\log b} < +\infty,$$

since $t\log b > a$.

Again if $c$ is a real number such that $1 < c < a$, then

$$\sum_{k=1}^{\infty}(c)^{-\log a_{n_k}^t} = \sum_{k=1}^{\infty}(e)^{(-\log c)\log a_{n_k}^t} = \sum_{k=1}^{\infty} a_{n_k}^{-t\log c} = +\infty,$$

since $t\log c < a$.

Therefore

$$\inf\left\{p > 1 : \sum_{k=1}^{\infty} p^{-\log a_{n_k}^t} < \infty\right\} = a,$$

i.e. $\phi(y) = a$.

We now show that $\phi$ is not one-to-one.

Let $a \in (1,\infty)$. Then there exists $x = \{x_n\}_{n=1}^{\infty} \in S$ such that $\phi(x) = a$, i.e.

$$a = \inf\left\{p > 1 : \sum_{n=1}^{\infty} p^{-x_n} < \infty\right\}.$$

Let $y_n = x_{n+1}$, for $n = 1, 2, 3, \ldots$. Then $y = \{y_n\}_{n=1}^{\infty} \in S$. Clearly

$$\inf\left\{p > 1 : \sum_{n=1}^{\infty} p^{-y_n} < \infty\right\} = a,$$

i.e. $\phi(y) = a$. So $\phi(x) = \phi(y)$ when $x \neq y$. Therefore, $\phi$ is not one-to-one. $\square$

**Theorem 1.2.** *The sets $H^t = \{x \in S : \phi(x) < t\}$ and $H_t = \{x \in S : \phi(x) > t\}$ belong to the third additive Borel class for every $t \in (-\infty,\infty)$.*

*Proof.* If $t \leq 1$, then $H^t = \phi$ and the theorem is true.

Let $t > 1$. Then,

$$\begin{aligned} H^t &= \{x \in S\colon \phi(x) < t\} \\ &= \{x = \{x_i\}_{i=1}^{\infty} \in S\colon \sum_{i=1}^{\infty} (a)^{-x_i} < \infty\}, \quad \text{for some } a > 1 \text{ and } 1 < a < t, \\ &= \{x \in S\colon \sum_{i=1}^{\infty} \left(t - \frac{1}{k}\right)^{-x_i} < \infty\}, \end{aligned}$$

for $k \geq k_0$ and $k_0$ is the least positive integer such that $a = t - 1/k > 1$.

We consider $F(k) = \{x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} a^{-x_i} < \infty\}$, for some $a > 1$ and $1 < a < t$, $k = k_0, k_0+1, k_0+2, \ldots$. Then

$$F(k) = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{x : a^{-x_{q+m}} + a^{-x_{q+m+1}} + \ldots + a^{-x_{q+m+n}} \leq \frac{1}{p}\right\}.$$

Set

$$F(k,p,q,m,n) = \left\{x : a^{-x_{q+m}} + a^{-x_{q+m+1}} + \ldots + a^{-x_{q+m+n}} \leq \frac{1}{p}\right\}.$$

Let $x^{(r)} = \{x_n^r\}_{n=1}^{\infty} \in F(k,p,q,m,n)$ and $\lim_{r\to\infty} x^{(r)} = x$. So $\lim_{r\to\infty} a^{-x_n{}^{(r)}} = a^{-x_n}$ for each $n = q+m, q+m+1, q+m+2, \ldots, q+m+n$, whence $x \in F(k,p,q,m,n)$. Consequently, each of the set $F(k,p,q,m,n)$ is closed. This proves that $H^t$ is an $F_{\sigma\delta\sigma}$ set. Hence, the set $\{x \in S : \phi(x) < t\}$ belongs to the third additive Borel class.

We now investigate the set $H_t$.
If $t < 1$, then $H_t = S$ and the theorem is true.
If $t \geq 1$, then

$$\begin{aligned} H_t &= \{x \in S : \phi(x) > t\} \\ &= \bigcup_{k=1}^{\infty} \left\{x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} \left(t + \frac{1}{k}\right)^{-x_i} = \infty\right\}. \end{aligned}$$

Consider the set $G(k) = \{x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} (a)^{-x_i} = \infty\}$, where $a = t+1/k$, $k = 1, 2, 3, \ldots$. Then,

$$G(k) = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \left\{x \in S\colon \sum_{i=1}^{q+m} (a)^{-x_i} \geq p\right\}, \qquad k = 1, 2, \ldots.$$

It is clear that each of the sets $G(k,p,q,m) = \{x \in S : \sum_{i=1}^{q+m} (a)^{-x_i} \geq p\}$ is closed. Therefore, the set

$$\{x \in S : \phi(x) > t\} = \bigcup_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} G(k,p,q,m)$$

is an $F_{\sigma\delta\sigma}$ set, i.e. $H_t$ belongs to the third additive Borel class. □

**Theorem 1.3.** *The set $H^t = \{x \in S : \phi(x) < t\}$ is of first category for every $t \in (-\infty, \infty)$.*

*Proof.* It follows from the previous theorem that

$$H^t = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} F(k,p,q,m,n) = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} F(k,p),$$

where

$$F(k,p) = \left\{x \in S : \underset{q=1}{\overset{\infty}{\exists}} \ \underset{m=1}{\overset{\infty}{\forall}} \ \underset{n=1}{\overset{\infty}{\forall}} \left\{a^{-x_{q+m}} + a^{-x_{q+m+1}} + \ldots + a^{-x_{q+m+n}} \leq \frac{1}{p}\right\}\right\}.$$

In order to show that each of the set $F(k,p)$ is of first category in $S$, it is sufficient to show that $F(k,p)$ is an $F_\sigma$ set and its complement is dense in $S$.

Let $\varepsilon > 0$. Let $u = \{u_n\}_n$ and $B(u,\varepsilon)$ be an open sphere with $u$ as the center and $\varepsilon$ as the radius. Let $r$ be the smallest positive integer such that $\sum_{i=r+1}^{\infty} 1/2^i < \varepsilon$. Define a sequence $x = \{x_n\}$ in $S$ as follows: $x_i = u_i$ for $i = 1, 2, \ldots r$.

If $x_r \leq r+1$, take $x_h = \dfrac{1}{h}$, for $h = r+1, r+2, \ldots$

If $x_r > r+1$, set $x_j = u_r$, for $j = r+1, r+2, \ldots, l-1$, where $l$ is the smallest positive integer for which $l \geq x_r$ and $x_h = \dfrac{1}{h}$, $h = l, l+1, l+2, \ldots$.

Therefore, we can find an integer $q$ such that $x_i = 1/i$ for $i = q, q+1, q+2, \ldots$. Clearly $x = \{x_n\}_n \in B(u,\varepsilon)$. For every integer $q$, there exist integers $m$ and $n$ such that

$$a^{-1/(q+m+1)} + a^{-1/(q+m+2)} + \ldots\ldots + a^{-1/(q+m+n)} = \sum_{\alpha=q+m+1}^{q+m+n} a^{-1/\alpha} > \frac{1}{p}$$

since the series $\sum_{n=1}^{\propto} a^{-1/n}$ is divergent. Thus, the complement of $F(k,p)$ is dense in $S$. Also each of the set $F(k,p,q,m,n)$ is closed and hence

$$F(k,p) = \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} F(k,p,q,m,n)$$

is an $F_\sigma$ set. Then $F(k,p)$ is of first category in $S$. But

$$\begin{aligned} F(k) &= \{x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} a^{-x_i} < \infty\} \quad \text{for some} \quad a > 1, \quad 1 < a < t \\ &= \{x \in S : \phi(x) < t\} = H^t \end{aligned}$$

Hence, $H^t = \displaystyle\bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} F(k,p)$ is of first category in $S$. □

**Theorem 1.4.** *The set* $\{x \in S : \phi(x) = \infty\}$ *is residual in* $S$.

*Proof.* By Theorem 1.3, the set

$$\{x \in S : \phi(x) < \infty\} = \bigcup_{n=1}^{\infty} \{x \in S : \phi(x) < n\}$$

is of first category in $S$ and also the space $S$ is complete. Hence, the set $\{x \in S : \phi(x) = \infty\}$ is residual in $S$. □

**Theorem 1.5.** *The function* $\phi$ *is discontinuous everywhere in* $S$.

*Proof.* Let $x = \{x_k\} \in S$. We choose a sequence $y = \{y_k\} \in S$ such that $\phi(x) \neq \phi(y)$. Let $\delta > 0$. It is sufficient to show that there exists a sequence $z = \{z_k\}$ in the neighborhood $B(x,\delta)$ such that $\phi(z) = \phi(y)$. For $\delta > 0$, let $l$

be the smallest positive integer such that $\sum_{i=l+1}^{\infty} 1/2^i < \delta$. Now, we consider the sequence $\{z_k\}_{k=1}^{\infty}$ as follows:

$$z_k = \begin{cases} x_k, & \text{for } k \le l \\ y_k, & \text{for } k > l \end{cases}$$

It is clear that $z \in B(x, \delta)$ and

$$\begin{aligned}
\phi(z) &= \inf\left\{p > 1 : \sum_{k=1}^{\infty} p^{-z_k} < \infty\right\} \\
&= \inf\left\{p > 1 : \left(\sum_{k=1}^{l} p^{-x_k} + \sum_{k=l+1}^{\infty} p^{-y_k}\right) < \infty\right\} \\
&= \inf\left\{p > 1 : \sum_{k=1}^{\infty} p^{-y_k} + \left(\sum_{k=1}^{l} p^{-x_k} - \sum_{k=1}^{l} p^{-y_k}\right) < \infty\right\} \\
&= \inf\left\{p > 1 : \sum_{k=1}^{\infty} p^{-y_k} < \infty\right\}, \\
&= \phi(y)
\end{aligned}$$

Hence $\phi$ is discontinuous everywhere in $S$. ☐

**Corollary 1.6.** *$\phi$ does not belong to the first Baire class.*

We now investigate the connected property of $\phi : S \to (1, \infty)$. Here we show that for any arbitrary subset of $(1, \infty)$, there exists a connected pre-image in $S$ under $\phi$. For this purpose we introduce the following lemma.

**Lemma 1.7.** *For $a \in (1, \infty)$, we consider the set*

$$\begin{aligned}
D_a^i = \{y(t) = \{y_k\} \in S : y_k = t \cdot x_k, \quad &\textit{for } \ k \le i, \qquad \textit{and} \\
y_k = x_k, \quad &\textit{for } \ k > i, \quad 0 < t \le 1\}
\end{aligned}$$

*where $i \in \mathbb{N}$ and $\phi(x) = a$, for some $x = \{x_k\}_{k=1}^{\infty} \in S$. Then $D_a = \bigcup_{i \in \mathbb{N}} D_a^i$ is connected and $\phi(D_a) = a$.*

*Proof.* Since $\{x_n\} \in D_a$, $D_a$ is nonempty. It is clear that $\phi(D_a) = a$. Now our goal is to show that $D_a$ is connected. For this purpose we define a function $f : (0, 1] \to S$ by

$$f(t) = y(t), \text{ for } t \in (0, 1] \text{ and } y(t) \in D_a^i.$$

It is clear that $f$ is continuous in $t$ on $(0, 1]$. So, $f(0, 1] = D_a^i$ is a connected set in $S$. Again $f(1) = \{x_n\} \in D_a^i$ for each $i \in \mathbb{N}$ and hence $\bigcap_{i \in \mathbb{N}} D_a^i \ne \phi$. Thus $\bigcup_{i \in \mathbb{N}} D_a^i = D_a$ is connected. ☐

**Theorem 1.8.** *Let $B$ be an arbitrary nontrivial subset of $(1, \infty)$. Then there exists a connected set $D \subseteq S$ such that $\phi(D) = B$.*

*Proof.* Let $a \in B$. Since $\phi$ is onto, there exists $x = \{x_n\} \in S$ such that $\phi(x) = a$. Define the set $D_a = \bigcup_{i \in \mathbb{N}} D_a^i$, where

$$D_a^i = \{y(t) = \{y_k\} \in S : y_k = t \cdot x_k, \quad \text{for} \quad k \leq i, \qquad \text{and} \\ y_k = x_k, \quad \text{for} \quad k > i, \quad 0 < t \leq 1\}$$

where $i \in \mathbb{N}$. Let $D = \bigcup_{a \in B} D_a$. Then by the previous lemma, $\phi(D_a) = a$. Therefore $\phi(D) = B$. We are to show that $D$ is connected. Let $a_1, a_2 \in B$ be such that $a_1 \neq a_2$. Then there exist $x^{(1)} = \{x_n^{(1)}\}_{n=1}^{\infty}$ and $x^{(2)} = \{x_n^{(2)}\}_{n=1}^{\infty} \in S$ such that $\phi(x^{(1)}) = a_1$ and $\phi(x^{(2)}) = a_2$. Let $y = \{y_n\} \in D_{a_1}$ and $\varepsilon > 0$. Since $\{y_n\} \in D_{a_1}$, there exists $i \in \mathbb{N}$ such that

$$y_n = \begin{cases} t \cdot x_n^{(1)}, & \text{for} \quad n \leq i, \\ x_n^{(1)}, & \text{for} \quad k > i, \quad 0 < t \leq 1 \quad \text{and} \quad i \in \mathbb{N}. \end{cases}$$

We choose $j \in \mathbb{N}$ such that $\sum_{k=j+1}^{\infty} 1/2^k < \varepsilon$. We construct a sequence $z = \{z_k\} \in S$ as follows

$$z_k = \begin{cases} y_k, & \text{for} \quad k \leq j, \\ x_k^{(2)}, & \text{for} \quad k > j; \quad k \in \mathbb{N}. \end{cases}$$

Then $z \in D_{a_2}$ and $d(y, z) < \varepsilon$. This shows that every $\varepsilon$-ball of $y$ contains a member of $D_{a_2}$. So $y \in \overline{D_{a_2}}$, where the symbol 'bar' indicates the closure of the set. Hence $D_{a_1} \subseteq \overline{D_{a_2}}$. Similarly $D_{a_2} \subseteq \overline{D_{a_1}}$. Therefore, $D_{a_1}$ and $D_{a_2}$ are not separated. This implies that no two of the sets $\{D_{a_i}, a_i \in B\}$ are separated. Thus $D$ is connected. This completes the proof. □

**Corollary 1.9.** *The function $\phi : S \to (1, \infty)$ is not Darboux.*

## References

1. Kostyrko P., *Note on the exponent of convergence*, Acta. Fac. Rer. Nat. Univ. Com **34** (1979), 29–58.
2. Kostyrko P. and Šalát T., *On the exponent of convergence*, Rend. Circ. Mat. Palermo Ser. II **XXXI** (1982), 187–194.
3. Goffman, C. and Pedric G., *First course in Functional Analysis*, Prentice Hall of India Private Limited, New Delhi 1974.
4. Pólya G. and Szegö G., *Aufgaben und Lehrsätze aus der Analysis I* (Russian Translation), Nauka, Moskva 1978.
5. Šalát T., *On exponent of convergence of subsequences*, Czechoslovak Mathematical Journal **34** (109), 1984.

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