Acta Math. Univ. Comenianae
Vol. LXXIX, 2(2010), pp. 281–292

MATRIX SUMMABILITY AND KOROVKIN TYPE APPROXIMATION THEOREM ON MODULAR SPACES

S. KARAKUŞ AND K. DEMIRCI

ABSTRACT. In this paper, using a matrix summability method we obtain a Korovkin type approximation theorem for a sequence of positive linear operators defined on a modular space.

1. Introduction

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations [**9**], [**10**], [**11**]. Most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, some sharp exceptions such as the interpolation operator of Hermite-Fejer (see [**7**]). These operators do not converge at points of simple discontinuity. For such misbehavior, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence (see [**8**]). Using a matrix summability method some approximation results were studied in [**1, 2, 18, 19, 21**]. In this paper, using a matrix summability method we give a theorem of the Korovkin type for a sequence of positive linear operators defined on a modular space.

We now recall some basic definitions and notations used in the paper.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I)$ we denote the space of all real-valued measurable functions on I provided with equality *a.e.* As usual, let $C(I)$ denote the space of all continuous real-valued functions, and $C^{\infty}(I)$ denote the space of all infinitely differentiable functions on I. In this case, we say that a functional $\rho : X(I) \to [0, +\infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in I,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X(I)$,

Received May 5, 2010; revised July 13,2010.
2000 *Mathematics Subject Classification.* Primary 40A30, 41A36, 47G10, 46E30.
Key words and phrases. Positive linear operators; modular space; matrix summability; Korovkin theorem.

(iii) $\rho(\alpha f+\beta g)\leq\rho(f)+\rho(g)$ for every $f,g\in X(I)$ and for any $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

A modular ρ is said to be N-*quasi convex* if there exists a constant $N\geq 1$ such that the inequality

$$\rho(\alpha f+\beta g)\leq N\alpha\rho(Nf)+N\beta\rho(Ng)$$

holds for every $f,g\in X(I)$, $\alpha,\beta\geq 0$ with $\alpha+\beta=1$. In particular, if $N=1$, then ρ is called *convex*.

A modular ρ is said to be N-*quasi semiconvex* if there exists a constant $N\geq 1$ such that the inequality

$$\rho(af)\leq Na\rho(Nf)$$

holds for every $f\in X(I)$ and $a\in(0,1]$.

It is clear that every N-quasi convex modular is N-quasi semiconvex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et. al. [**6**].

We now consider some appropriate vector subspaces of $X(I)$ by means of a modular ρ as follows

$$L^{\rho}(I):=\left\{f\in X(I):\lim_{\lambda\to 0^{+}}\rho(\lambda f)=0\right\}$$

and

$$E^{\rho}(I):=\left\{f\in L^{\rho}(I):\rho(\lambda f)<+\infty\ \text{ for all }\lambda>0\right\}.$$

Here, $L^{\rho}(I)$ is called the *modular space* generated by ρ and $E^{\rho}(I)$ is called the space of the finite elements of $L^{\rho}(I)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\left\{f\in X(I):\rho(\lambda f)<+\infty\ \text{ for some }\lambda>0\right\}$$

coincides with $L^{\rho}(I)$. The notions about modulars were introduced in [**17**] and widely discussed in [**6**] (see also [**12, 16**]).

Now we recall the convergence methods in modular spaces.

Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_n\}$ is *modularly convergent* to a function $f\in L^{\rho}(I)$ iff

$$\lim_{n}\rho(\lambda_0(f_n-f))=0\quad\text{for some }\lambda_0>0. \tag{1.1}$$

Also, $\{f_n\}$ is F-*norm convergent* (or *strongly convergent*) to f iff

$$\lim_{n}\rho(\lambda(f_n-f))=0\quad\text{for every }\lambda>0. \tag{1.2}$$

It is known from [**16**] that (1.1) and (1.2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e., there exists a constant $M>0$ such that $\rho(2f)\leq M\rho(f)$ for every $f\in X(I)$.

In this paper, we will need the following assumptions on a modular ρ:

- if $\rho(f)\leq\rho(g)$ for $|f|\leq|g|$, then ρ is called *monotone*,
- if the characteristic function χ_I of the interval I belongs to $L^{\rho}(I)$, ρ is called *finite*,

- if ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists $\delta > 0$ such that $\rho(\lambda \chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $|B| < \delta$, then ρ is called *absolutely finite*,
- if $\chi_I \in E^\rho(I)$, then ρ is called *strongly finite,*
- ρ is called *absolutely continuous* provided that there exists $\alpha > 0$ such that, for every $f \in X(I)$ with $\rho(f) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I with $|B| < \delta$.

Observe now that (see [**5**]) if a modular ρ is monotone and finite, then we have $C(I) \subset L^\rho(I)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I) \subset E^\rho(I)$. Some important relations between the above properties may be found in [**4, 6, 14, 17**].

2. Korovkin Type Theorems

Let $\mathcal{A} := (A^n)_{n\geq 1}$, $A^n = \left(a_{kj}^n\right)_{k,j\in\mathbb{N}}$ be a sequence of infinite non-negative real matrices. For a sequence of real numbers, $x = (x_j)_{j\in\mathbb{N}}$, the double sequence

$$\mathcal{A}x := \{(Ax)_k^n : k, n \in \mathbb{N}\}$$

defined by $(Ax)_k^n := \sum_{j=1}^{\infty} a_{kj}^n x_j$ is called the $\mathcal{A}$-transform of x whenever the series converges for all k and n. A sequence x is said to be $\mathcal{A}$-summable to L if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n x_j = L$$

uniformly in n ([**3**], [**20**]).

If $A^n = A$ for a matrix A, then $\mathcal{A}$-summability is the ordinary matrix summability by A. If $a_{kj}^n = \frac{1}{k}$ for $n \leq j \leq k+n$, $(n = 1, 2, \ldots)$ and $a_{kj}^n = 0$ otherwise, then $\mathcal{A}$-summability reduces to almost convergence [**13**].

Let ρ be a monotone and finite modular on $X(I)$. Assume that D is a set satisfying $C^\infty(I) \subset D \subset L^\rho(I)$. We can construct such a subset D since ρ is monotone and finite (see [**5**]). Assume further that $\mathbb{T} := \{T_n\}$ is a sequence of positive linear operators from D into $X(I)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ containing $C^\infty(I)$ such that

$$\limsup_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \rho(\lambda(T_j h)) \leq P\rho(\lambda h), \qquad \text{uniformly in } n. \tag{2.1}$$

The inequality holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. Throughout the paper we use the test functions e_i defined by

$$e_i(x) = x^i \quad (i = 0, 1, 2, \ldots).$$

Theorem 2.1. *Let $\mathcal{A} = (A^n)_{n\geq 1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty \tag{2.2}$$

and let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_j\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (2.1)*. Suppose that*

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(\lambda\left(T_j e_i - e_i\right)\right) = 0, \qquad \textit{uniformly in } n \tag{2.3}$$

for every $\lambda > 0$ and $i = 0, 1, 2$. Now, let f be any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Then, we have

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(\lambda_0\left(T_j f - f\right)\right) = 0, \qquad \textit{uniformly in } n$$

for some $\lambda_0 > 0$.

Proof. We first claim that

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(\mu\left(T_j g - g\right)\right) = 0, \qquad \text{uniformly in } n \tag{2.4}$$

for every $g \in C(I)$ and every $\mu > 0$. To see this assume that g belongs to $C(I)$ and μ is any positive number. Then, there exists a constant $M > 0$ such that $|g(x)| \leq M$ for every $x \in I$. Given $\varepsilon > 0$, we can choose $\delta > 0$ such that $|y - x| < \delta$ implies $|g(y) - g(x)| < \varepsilon$ where $y, x \in I$. It is easy to see that for all $y, x \in I$

$$|g(y) - g(x)| < \varepsilon + \frac{2M}{\delta^2}(y - x)^2 .$$

Since T_j is a positive linear operator, we get

$$\begin{aligned}
&|T_j(g;x) - g(x)| \\
&\qquad = |T_j(g(\cdot) - g(x);x) + g(x)(T_j(e_0(\cdot);x) - e_0(x))| \\
&\qquad \leq T_j(|g(\cdot) - g(x)|;x) + |g(x)|\,|T_j(e_0(\cdot);x) - e_0(x)| \\
&\qquad \leq T_j\left(\varepsilon + \frac{2M}{\delta^2}(\cdot - x)^2;x\right) + M\,|T_j(e_0(\cdot);x) - e_0(x)| \\
&\qquad \leq \varepsilon T_j(e_0(\cdot);x) + \frac{2M}{\delta^2}T_j\left((\cdot - x)^2;x\right) + M\,|T_j(e_0(\cdot);x) - e_0(x)| \\
&\qquad \leq \varepsilon + (\varepsilon + M)\,|T_j(e_0(\cdot);x) - e_0(x)| \\
&\qquad\quad + \frac{2M}{\delta^2}\left[T_j(e_2(\cdot);x) - 2e_1(x)T_j(e_1(\cdot);x) + e_2(x)T_j(e_0(\cdot);x)\right]
\end{aligned}$$

$$\leq \varepsilon + (\varepsilon + M) \left| T_j \left(e_0 \left(\cdot \right); x \right) - e_0 \left(x \right) \right| + \frac{2M}{\delta^2} \left| T_j \left(e_2 \left(\cdot \right); x \right) - e_2 \left(x \right) \right|$$
$$+ \frac{4M \left| e_1 \left(x \right) \right|}{\delta^2} \left| T_j \left(e_1 \left(\cdot \right); x \right) - e_1 \left(x \right) \right| + \frac{2M e_2 \left(x \right)}{\delta^2} \left| T_j \left(e_0 \left(\cdot \right); x \right) - e_0 \left(x \right) \right|$$
$$\leq \varepsilon + \left(\varepsilon + M + \frac{2Mc^2}{\delta^2} \right) \left| T_j \left(e_0 \left(\cdot \right); x \right) - e_0 \left(x \right) \right| + \frac{4Mc}{\delta^2} \left| T_j \left(e_1 \left(\cdot \right); x \right) - e_1 \left(x \right) \right|$$
$$+ \frac{2M}{\delta^2} \left| T_j \left(e_2 \left(\cdot \right); x \right) - e_2 \left(x \right) \right|$$

where $c := \max \{|a|, |b|\}$. So, the last inequality gives, for any $\mu > 0$ that

$$\mu \left| T_j \left(g; x \right) - g \left(x \right) \right| \leq \mu\varepsilon + \mu K \left| T_j \left(e_0 \left(\cdot \right); x \right) - e_0 \left(x \right) \right|$$
$$+ \mu K \left| T_j \left(e_1 \left(\cdot \right); x \right) - e_1 \left(x \right) \right| + \mu K \left| T_j \left(e_2 \left(\cdot \right); x \right) - e_2 \left(x \right) \right|$$

where $K := \max \left\{ \varepsilon + M + \frac{2Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2} \right\}$. Applying the modular ρ in the both-sides of the above inequality, since ρ is monotone, we have

$$\rho(\mu(T_j(g; \cdot) - g(\cdot)))$$
$$\leq \rho \left(\mu\varepsilon + \mu K \left| T_j e_0 - e_0 \right| + \mu K \left| T_j e_1 - e_1 \right| + \mu K \left| T_j e_2 - e_2 \right| \right).$$

So, we may write that

$$\rho \left(\mu \left(T_j \left(g; \cdot \right) - g \left(\cdot \right) \right) \right) \leq \rho \left(4\mu\varepsilon \right) + \rho \left(4\mu K \left(T_j e_0 - e_0 \right) \right)$$
$$+ \rho \left(4\mu K \left(T_j e_1 - e_1 \right) \right) + \rho \left(4\mu K \left(T_j e_2 - e_2 \right) \right).$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \leq 1$

$$\rho \left(\mu \left(T_j \left(g; \cdot \right) - g \left(\cdot \right) \right) \right) \leq N\varepsilon\rho \left(4\mu N \right) + \rho \left(4\mu K \left(T_j e_0 - e_0 \right) \right)$$
$$+ \rho \left(4\mu K \left(T_j e_1 - e_1 \right) \right) + \rho \left(4\mu K \left(T_j e_2 - e_2 \right) \right).$$

Hence

$$\sum_{j=1}^{\infty} a_{kj}^n \rho \left(\mu \left(T_j \left(g; \cdot \right) - g \left(\cdot \right) \right) \right) \leq N\varepsilon\rho \left(4\mu N \right) \sum_{j=1}^{\infty} a_{kj}^n + \sum_{j=1}^{\infty} a_{kj}^n \rho \left(4\mu K \left(T_j e_0 - e_0 \right) \right)$$
$$(2.5) \qquad + \sum_{j=1}^{\infty} a_{kj}^n \rho \left(4\mu K \left(T_j e_1 – e_1 \right) \right) + \sum_{j=1}^{\infty} a_{kj}^n \rho \left(4\mu K \left(T_j e_2 – e_2 \right) \right)$$

By taking limit superior as $k \to \infty$ in the both-sides of (2.5), by using (2.3), we get

$$\lim_{k \to \infty} \sum_{j=1}^{\infty} a_{kj}^n \rho \left(\mu \left(T_j \left(g; \cdot \right) - g \left(\cdot \right) \right) \right) = 0 \qquad \text{uniformly in } n$$

which proves our claim (2.4). Now let $f \in L^\rho(I)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty(I)$. Since $|I| < \infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I)$ (see [**4**]). Using these properties of the modular ρ, it is known from [**6, 14**] that the space $C^{\infty}(I)$ is modularly dense in $L^{\rho}(I)$, i.e., there exists a sequence $\{g_k\} \subset C^{\infty}(I)$ such that

$$\lim_{k} \rho\left(3\lambda_0\left(g_k - f\right)\right) = 0 \qquad \text{for some } \lambda_0 > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0\left(g_k - f\right)\right) < \varepsilon \qquad \text{for every } k \geq k_0. \tag{2.6}$$

On the other hand, by the linearity and positivity of the operators T_j, we may write that

$$\lambda_0 \left|T_j f - f\right| \leq \lambda_0 \left|T_j(f - g_{k_0})\right| + \lambda_0 \left|T_j g_{k_0} - g_{k_0}\right| + \lambda_0 \left|g_{k_0} - f\right|.$$

Applying the modular ρ in the both-sides of the above inequality, since ρ is monotone, we have

$$\begin{aligned}\rho\left(\lambda_0\left(T_j f - f\right)\right) \leq{}& \rho\left(3\lambda_0\left(T_j f - g_{k_0}\right)\right) + \rho\left(3\lambda_0\left(T_j g_{k_0} - g_{k_0}\right)\right)\\ &+ \rho\left(3\lambda_0\left(g_{k_0} - f\right)\right).\end{aligned} \tag{2.7}$$

Then, it follows from (2.6) and (2.7) that

$$\rho\left(\lambda_0\left(T_j f - f\right)\right) \leq \rho\left(3\lambda_0\left(T_j f - g_{k_0}\right)\right) + \rho\left(3\lambda_0\left(T_j g_{k_0} - g_{k_0}\right)\right) + \varepsilon.$$

Hence, using the facts that $g_{k_0} \in C^{\infty}(I)$ and $f - g_{k_0} \in X_{\mathbb{T}}$, we have

$$\begin{aligned}&\sum_{j=1}^{\infty} a_{kj}^n \rho\left(\lambda_0\left(T_j f - f\right)\right)\\ &\leq \sum_{j=1}^{\infty} a_{kj}^n \rho\left(3\lambda_0\left(T_j f - g_{k_0}\right)\right) + \sum_{j=1}^{\infty} a_{kj}^n \rho\left(3\lambda_0\left(T_j g_{k_0} - g_{k_0}\right)\right) + \varepsilon \sum_{j=1}^{\infty} a_{kj}^n.\end{aligned} \tag{2.8}$$

From (2.2), there exists a constant $B > 0$ such that $\sup\limits_{n,k} \sum\limits_{j=1}^{\infty} a_{kj}^n < B$. So, taking limit superior as $k \to \infty$ in the both-sides of (2.8), from (2.1) and (2.2) we obtain that

$$\begin{aligned}&\limsup_{k} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(\lambda_0\left(T_j f - f\right)\right)\\ &\leq \varepsilon \limsup_{k} \sum_{j=1}^{\infty} a_{kj}^n + P\rho\left(3\lambda_0(f - g_{k_0})\right) + \limsup_{k} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(3\lambda_0\left(T_j g_{k_0} - g_{k_0}\right)\right),\end{aligned}$$

which gives

$$\begin{aligned}&\limsup_{k} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(\lambda_0\left(T_j f - f\right)\right)\\ &\leq \varepsilon\left(B + P\right) + \limsup_{k} \sum_{j=1}^{\infty} a_{kj}^n \rho\left(3\lambda_0\left(T_j g_{k_0} - g_{k_0}\right)\right).\end{aligned} \tag{2.9}$$

By (2.4), since

$$\lim_{k}\sum_{j=1}^{\infty}a_{kj}^{n}\rho\left(3\lambda_{0}\left(T_{j}g_{k_{0}}-g_{k_{0}}\right)\right)=0,\qquad \text{uniformly in } n$$

we get

$$\limsup_{k}\sum_{j=1}^{\infty}a_{kj}^{n}\rho\left(3\lambda_{0}\left(T_{j}g_{k_{0}}-g_{k_{0}}\right)\right)=0,\qquad \text{uniformly in } n. \tag{2.10}$$

Combining (2.9) with (2.10), we conclude that

$$\limsup_{k}\sum_{j=1}^{\infty}a_{kj}^{n}\rho\left(\lambda_{0}\left(T_{j}(f;x)-f(x)\right)\right)\leq\varepsilon\left(B+P\right).$$

Since $\varepsilon>0$ was arbitrary, we find

$$\limsup_{k}\sum_{j=1}^{\infty}a_{kj}^{n}\rho\left(\lambda_{0}\left(T_{j}f-f\right)\right)=0\qquad \text{uniformly in } n.$$

Furthermore, since $\sum\limits_{j=1}^{\infty}a_{kj}^{n}\rho\left(\lambda_{0}\left(T_{j}(f;x)-f(x)\right)\right)$ is non-negative for all $k,n\in\mathbb{N}$, we can easily show that

$$\lim_{k}\sum_{j=1}^{\infty}a_{kj}^{n}\rho\left(\lambda_{0}\left(T_{j}f-f\right)\right)=0,\qquad \text{uniformly in } n$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get the following result from Theorem 2.1 at once.

Theorem 2.2. *Let $\mathcal{A}=\left(A^{n}\right)_{n\geq1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k}\sum_{j=1}^{\infty}a_{kj}^{n}<\infty,$$

and $\mathbb{T}:=\{T_n\}$, ρ be the same as in Theorem 2.1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:

(*a*) $\lim\limits_{k}\sum\limits_{j=1}^{\infty}a_{kj}^{n}\rho\left(\lambda\left(T_{j}e_{i}-e_{i}\right)\right)=0$ *uniformly in n for every $\lambda>0$ and $i=0,1,2,$*

(*b*) $\lim\limits_{k}\sum\limits_{j=1}^{\infty}a_{kj}^{n}\rho\left(\lambda\left(T_{j}f-f\right)\right)=0$ *uniformly in n for every $\lambda>0$ provided that f is any function belonging to $L^{\rho}(I)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$.*

If $A^n=I$, identity matrix, then the condition (2.1) reduces to

$$\limsup_{j}\rho\left(\lambda\left(T_{j}h\right)\right)\leq P\rho\left(\lambda h\right) \tag{2.11}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. In this case, the next results which were obtained by Bardaro and Mantellini [**5**] immediately follows from our Theorems 2.1 and 2.2.

Corollary 2.3. Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_j\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (2.11). If $\{T_j e_i\}$ is strongly convergent to e_i for each $i = 0, 1, 2$, then $\{T_j f\}$ is modularly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.

Corollary 2.4. $\mathbb{T} := \{T_j\}$ and ρ be the same as in Corollary 2.3. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:

(a) $\{T_j e_i\}$ is strongly convergent to e_i for each $i = 0, 1, 2$,

(b) $\{T_j f\}$ is strongly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.

3. Application

Take $I = [0,1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} on $X(I)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \varphi(|f(x)|)\,\mathrm{d}x \quad \text{for} \quad f \in X(I).$$

In this case, ρ^{φ} is a convex modular on $X(I)$, which satisfies all assumptions listed in Section 1 (see [**5**]). Consider the Orlicz space generated by φ as follows

$$L^{\rho}_{\varphi}(I) := \{f \in X(I) : \rho^{\varphi}(\lambda f) < +\infty \quad \text{for some } \lambda > 0\}.$$

Then, consider the following classical Bernstein-Kantorovich operator $\mathbb{U} := \{U_n\}$ on the space $L^{\rho}_{\varphi}(I)$ (see [**5**]) which is defined by

$$U_j(f; x) := \sum_{k=0}^{j} \binom{j}{k} x^k (1-x)^{j-k} (j+1) \int_{k/(j+1)}^{(k+1)/(j+1)} f(t)\,\mathrm{d}t \quad \text{for } x \in I.$$

Observe that the operators U_j map the Orlicz space $L^{\rho}_{\varphi}(I)$ into itself. Moreover, the property (2.11) is satisfied with the choice of $X_{\mathbf{U}} := L^{\rho}_{\varphi}(I)$. Then, by Corollary 2.3, we know that, for every function $f \in L^{\rho}_{\varphi}(I)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}(I)$, $\{U_j f\}$ is modularly convergent to f.

Assume that $\mathcal{A} := (A^n)_{n \geq 1} = \left(a^n_{kj}\right)_{k,j \in \mathbb{N}}$ is a sequence of infinite matrices defined by $a^n_{kj} = \frac{1}{k}$ if $n \leq j \leq k + n$, $(n = 1, 2, \ldots)$, and $a^n_{kj} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence. Define $s = (s_n)$ of the form

$$\underset{\to n_1 \text{ terms} \leftarrow}{0101\ldots0101}; \underset{\to n_2 \text{ terms} \leftarrow}{001001\ldots001}; \underset{\to n_2 \text{ terms} \leftarrow}{00010001\ldots0001}; \ldots \tag{3.1}$$

where n_1 is a multiple of 2, n_2 is a multiple of 6, n_3 is a multiple of $1, 2, \ldots$ and n_k is a multiple of $k\,(k+1)$. So s is almost convergent to zero (see [**15**]). However, the sequence $\{s_n\}$ is not convergent to zero. Then, using the operators U_j, we define the sequence of positive linear operators $\mathbb{V} := \{V_n\}$ on $L_{\varphi}^{\rho}(I)$ as follows:

$$V_j(f;x) = (1+s_j)\, U_j(f;x) \quad \text{for } f \in L_{\varphi}^{\rho}(I)\,,\ x \in [0,1] \text{ and } j \in \mathbb{N}, \tag{3.2}$$

where $s = \{s_j\}$ is the same as in (3.1). By [**5**, Lemma 5.1], for every $h \in X_{\mathbb{V}} := L_{\varphi}^{\rho}(I)$, all $\lambda > 0$ and for an absolute positive constant P, we get

$$\begin{aligned}\rho^{\varphi}(\lambda V_j h) &= \rho^{\varphi}(\lambda\,(1+s_j)\,U_j h) \le \rho^{\varphi}(2\lambda U_j h) + \rho^{\varphi}(2\lambda s_j U_j h)\\ &= \rho^{\varphi}(2\lambda U_j h) + s_j \rho^{\varphi}(2\lambda U_j h) = (1+s_j)\,\rho^{\varphi}(2\lambda U_j h) \le (1+s_j)\, P \rho^{\varphi}(2\lambda h)\,.\end{aligned}$$

Then, we get

$$\limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \rho^{\varphi}(\lambda V_j h) \right) \le P\rho^{\varphi}(2\lambda h)\,.$$

So, the condition (2.1) works for our operators V_n given by (3.2) with the choice of $X_{\mathbb{V}} = X_{\mathbb{U}} = L_{\varphi}^{\rho}(I)$.

Now, we show that condition (2.3) in the Theorem 2.1 holds.

First observe that

$$\begin{aligned} V_j(e_0;x) &= 1 + s_j,\\ V_j(e_1;x) &= (1+s_j)\left(\frac{jx}{j+1} + \frac{1}{2\,(j+1)} \right),\\ V_j(e_2;x) &= (1+s_j)\left(\frac{j\,(j-1)\,x^2}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3\,(j+1)^2} \right). \end{aligned}$$

So, for any $\lambda > 0$, we can see, that

$$\lambda\,|V_j(e_0;x) - e_0(x)| = \lambda\,|1 + s_j - 1| = \lambda s_j,$$

which implies

$$\rho^{\varphi}(\lambda\,(V_j e_0 - e_0)) = \rho^{\varphi}(\lambda s_j) = \int_0^1 \varphi(\lambda s_j)\,\mathrm{d}x = \varphi(\lambda s_j) = s_j \varphi(\lambda)$$

because of the definition of (s_j). Since (s_j) is almost convergent to zero, we get

$$\limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \rho^{\varphi}(\lambda\,(V_j e_0 - e_0)) \right) = 0 \quad \text{for every } \lambda > 0,$$

which guarantees that (2.3) holds true for $i = 0$. Also, since

$$\begin{aligned}\lambda\,|V_j(e_1;x) - e_1(x)| &= \lambda \left| x\left(\frac{j}{j+1} + \frac{js_j}{j+1} - 1 \right) + \frac{1}{2(j+1)} + \frac{s_j}{2(j+1)} \right|\\ &\le \lambda\,|x| \left(\left| \frac{j}{j+1} - 1 \right| + \frac{js_j}{j+1} \right) + \frac{1}{2(j+1)} + \frac{s_j}{2(j+1)}\end{aligned}$$

$$\leq \lambda \left\{ \frac{1}{(j+1)} + \frac{2js_j}{2(j+1)} + \frac{s_j}{2(j+1)} + \frac{1}{2(j+1)} \right\}$$
$$\leq \lambda \left\{ \frac{3}{2(j+1)} + s_j \left(\frac{2j+1}{2(j+1)} \right) \right\},$$

we may write that

$$\rho^{\varphi}\left(\lambda\left(V_j e_1 - e_1\right)\right) \leq \rho^{\varphi}\left(\lambda\left\{ s_j \left(\frac{2j+1}{2(j+1)}\right) + \frac{3}{2(j+1)} \right\}\right)$$
$$\leq s_j \rho^{\varphi}\left(\lambda\left(\frac{2j+1}{j+1}\right)\right) + \rho^{\varphi}\left(\frac{3\lambda}{j+1}\right)$$

by the definitions of (s_j) and ρ^{φ}. Since $\left(\frac{2j+1}{j+1}\right)$ is convergent, it is bounded. So there exists a constant $M > 0$ such that $\left(\frac{2j+1}{j+1}\right) \leq M$ for every $j \in \mathbb{N}$. Then using the monotonicity of ρ^{φ}, we have

$$\rho^{\varphi}\left(\lambda\left(\frac{2j+1}{j+1}\right)\right) \leq \rho^{\varphi}\left(\lambda M\right)$$

for any $\lambda > 0$, which implies

$$\rho^{\varphi}\left(\lambda\left(V_j e_1 - e_1\right)\right) \leq s_j \rho^{\varphi}\left(\lambda M\right) + \rho^{\varphi}\left(\frac{3\lambda}{j+1}\right) = s_j \varphi\left(\lambda M\right) + \varphi\left(\frac{3\lambda}{j+1}\right).$$

Since φ is continuous, we have $\lim_j \varphi\left(\frac{3\lambda}{j+1}\right) = \varphi\left(\lim_j \frac{3\lambda}{j+1}\right) = \varphi(0) = 0$. So, we get $\varphi\left(\frac{3\lambda}{j+1}\right)$ is almost convergent to zero. Using s and $\varphi\left(\frac{3\lambda}{j+1}\right)$ are almost convergent to zero, we obtain

$$\limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \rho^{\varphi}\left(\lambda\left(V_j e_1 - e_1\right)\right) \right) \leq \limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \left[s_j \varphi\left(\lambda M\right) + \varphi\left(\frac{3\lambda}{j+1}\right) \right] \right)$$
$$= \varphi\left(\lambda M\right) \limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} s_j \right) + \limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \varphi\left(\frac{3\lambda}{j+1}\right) \right) = 0.$$

Finally, since

$$\lambda\left|V_j(e_2;x) - e_2(x)\right| = \lambda \left| x^2 \frac{j(j-1)}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2} \right.$$
$$\left. + s_j \frac{j(j-1)x^2}{(j+1)^2} + s_j \frac{2jx}{(j+1)^2} + s_j \frac{1}{3(j+1)^2} - x^2 \right|$$
$$\leq \lambda x^2 \left| \frac{j(j-1)}{(j+1)^2} - 1 \right| + x^2 s_j \frac{j(j-1)}{(j+1)^2}$$
$$+ |x| \left(\frac{2j}{(j+1)^2} + s_j \frac{2j}{(j+1)^2} \right) + \frac{1}{3(j+1)^2} + s_j \frac{1}{3(j+1)^2}$$

$$\leq \lambda \left\{ \frac{3j+1}{(j+1)^2} + s_j \frac{j(j-1)}{(j+1)^2} + \frac{2j}{(j+1)^2} + s_j \frac{2j}{(j+1)^2} \right.$$
$$\left. + \frac{1}{3(j+1)^2} + s_j \frac{1}{3(j+1)^2} \right\}$$
$$\leq \lambda \left\{ \frac{15j+4}{3(j+1)^2} + s_j \left(\frac{3j^2+3j+1}{3(j+1)^2} \right) \right\}.$$

Since $\left(\frac{3j^2+3j+1}{3(j+1)^2}\right)$ is convergent, it is bounded. So there exists a constant $K>0$ such that $\left|\frac{3j^2+3j+1}{3(j+1)^2}\right| \leq K$ for every $j \in \mathbb{N}$. Then using the monotonicity of ρ^{φ} and the definition of (s_j), we have

$$\rho^{\varphi}\left(\lambda\left(V_j e_2 - e_2\right)\right) \leq \rho^{\varphi}\left(2\lambda\left(\frac{15j+4}{3(j+1)^2}\right)\right) + \rho^{\varphi}\left(2\lambda s_j\left(\frac{3j^2+3j+1}{3(j+1)^2}\right)\right)$$
$$\leq \rho^{\varphi}\left(\lambda\left(\frac{30j+8}{3(j+1)^2}\right)\right) + \rho^{\varphi}\left(2\lambda s_j K\right),$$

where which yields

$$\rho^{\varphi}\left(\lambda\left(V_j e_2 - e_2\right)\right) \leq \varphi\left(\lambda\left(\frac{30j+8}{3(j+1)^2}\right)\right) + s_j \varphi\left(2\lambda K\right) \tag{3.3}$$

Since φ is continuous, we have $\lim_j \varphi\left(\lambda \frac{30j+8}{3(j+1)^2}\right) = \varphi\left(\lambda \lim_j \frac{30j+8}{3(j+1)^2}\right) = \varphi(0) = 0$. So, we get $\varphi\left(\lambda \frac{30j+8}{3(j+1)^2}\right)$ is almost convergent to zero. Using s and $\varphi\left(\lambda \frac{30j+8}{3(j+1)^2}\right)$ are almost convergent to zero, it follows from (3.3) that

$$\limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \rho^{\varphi}\left(\lambda\left(V_j e_2 - e_2\right)\right) \right) = 0 \quad \text{uniformly in } n \text{ for every } \lambda > 0.$$

Our claim (2.3) holds true for each $i = 0, 1, 2$ and for any $\lambda > 0$. So, we can say that our sequence $\mathbb{V} := \{V_j\}$ defined by (3.2) satisfy all assumptions of Theorem 2.1. Therefore, we conclude that

$$\limsup_k \left(\sup_n \frac{1}{k} \sum_{j=n}^{n+k} \rho^{\varphi}\left(\lambda_0\left(V_j f - f\right)\right) \right) = 0 \quad \text{uniformly in } n \text{ for some } \lambda_0 > 0$$

holds for every $f \in L_{\varphi}^{\rho}(I)$ such that $f - g \in X_{\mathbb{V}} = L_{\varphi}^{\rho}(I)$ for every $g \in C^{\infty}(I)$.

However, since (s_j) is not convergent to zero, it is clear that $\{V_j f\}$ is not modularly convergent to f. So, Corollary 2.3 does not work for the sequence $\mathbb{V} := \{V_j\}$.

References

1. Atlıhan Ö. G. and Orhan C., *Matrix summability and positive linear operators*, Positivity **11** (2007), 387–389.
2. ______, *Summation process of positive linear operators*, Computers & Mathematics with Applications, **56** (2008), 1188–1195.
3. Bell H. T., *Order summability and almost convergence*, Proc. Am. Math. Soc. **38** (1973), 548–553.
4. Bardaro C. and Mantellini I., *Approximation properties in abstract modular spaces for a class of general sampling-type operators*, Appl. Anal. **85** (2006), 383–413.
5. ______, *Korovkin's theorem in modular spaces*, Commentationes Math. **47** (2007), 239–253.
6. Bardaro C., Musielak J. and Vinti G., *Nonlinear Integral Operators and Applications*, de Gruyter Series in Nonlinear Analysis and Appl. Vol. **9**, Walter de Gruyter Publ., Berlin, 2003.
7. Bojanic R. and Cheng F., *Estimates for the rate of approximation of functions of bounded variation by Hermite-Fejer polynomials*, Proceedings of the conference of Canadian Math. Soc. **3** (1983), 5–17.
8. Bojanic R. and Khan M. K., *Summability of Hermite-Fejer interpolation for functions of bounded variation*, J. Nat. Sci. Math. **32(1)** (1992), 5–10.
9. Devore R. A., *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics, Springer-Verlag, **293**, Berlin 1972.
10. Gadžiev A. D., *The convergence problem for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of P. P. Korovkin*, Soviet Math. Dokl. **15(5)** (1974), 1433–1436.
11. Korovkin P. P., *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, 1960.
12. Kozlowski W. M., *Modular Function Spaces*, Pure Appl. Math., **122**, Marcel Dekker, Inc., New York, 1988.
13. Lorentz G. G., *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.
14. Mantellini I., *Generalized sampling operators in modular spaces*, Commentationes Math., **38** (1998), 77–92.
15. Miller H. I. and Orhan C., *On almost convergent and statistically convergent subsequences*, Acta Math. Hungar **93(1-2)** (2001), 135–151.
16. Musielak J., *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. **1034**, Springer-Verlag, Berlin, 1983.
17. Musielak J., *Nonlinear approximation in some modular function spaces I*, Math. Japon. **38** (1993), 83–90.
18. Nishishiraho T., *Quantitative theorems on linear approximation processes of convolution operators in Banach spaces*, Tôhoku Math. J. **33** (1981), 109–126.
19. ______, *Convergence of positive linear approximation processes*, Tôhoku Math. J. **35** (1983), 441–458.
20. Stieglitz M., *Eine verallgemeinerung des begriffs der fastkonvergenz*, Math. Japon. **18** (1973) 53–70.
21. Swetits J. J., *On summability and positive linear operators*, J. Approx. Theory **25** (1979), 186–188.

S. Karakuş, Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000, Sinop, Turkey, *e-mail*: `skarakus@sinop.edu.tr`

K. Demirci, Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000, Sinop, Turkey, *e-mail*: `kamild@sinop.edu.tr`